Eventual Linear Ranking Functions

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Context

- Program termination
- Ranking function
- SLC (*single-path linear constraint*) rational loop a.k.a. simple loop
- Eventual linear ranking function
- Might sound highly specific but compositional via Ramsey's theorem

Plan

SLC (*single-path linear constraint*) rational loop

$$\textbf{while } (B\,\mathbf{x} \le \mathbf{b}) \textbf{ do } A\begin{pmatrix}\mathbf{x}\\ \mathbf{x}'\end{pmatrix} \le \mathbf{c}$$

where:

- $\mathbf{x} = (x_1, \ldots, x_n)^T$ input values $\in \mathbb{Q}$
- $\mathbf{x}' = (x'_1, \ldots, x'_n)^T$ output values $\in \mathbb{Q}$ after one iteration
- $B \in \mathbb{Z}^{p\times n}$, $\mathbf{b} \in \mathbb{Z}^p$, $A \in \mathbb{Z}^{q\times 2n}$ et $\mathbf{c} \in \mathbb{Z}^q$

Notation:

$$p(\mathbf{x}) \leftarrow B\,\mathbf{x} \le \mathbf{b}, A\begin{pmatrix}\mathbf{x}\\ \mathbf{x}'\end{pmatrix} \le \mathbf{c},\ p(\mathbf{x}')$$

Example

- The loop **while** $(x \geq 0)$ **do** $x' \leq x - 1$ is denoted

$$p(x) \leftarrow x \geq 0, x' \leq x - 1, p(x')$$

- Operationally:
  $p(9) \Rightarrow 9 \geq 0 \wedge x' \leq 9 - 1, \mathit{choice} : x' = 7$
  $p(7) \Rightarrow 7 \geq 0 \wedge x' \leq 7 - 1, \mathit{choice} : x' = \ldots$
  $\ldots$
  $p(0) \Rightarrow 0 \geq 0 \wedge x' \leq 0 - 1, \mathit{choice} : x' = -3/2$
  $p(-3/2) \Rightarrow -3/2 \geq 0, \mathit{stop}!$
- The loop always terminates because its argument is 1-decreasing and bounded by below

Linear ranking function

Let C be the SLC loop $p(\mathbf{x}) \leftarrow c(\mathbf{x}, \mathbf{x}'),\ p(\mathbf{x}')$ where p is an n-ary relation

Definition

A *linear ranking function* ρ for C is a linear function from $\mathbb{Q}^n$ to $\mathbb{Q}$ st:

$$\forall \mathbf{x}, \mathbf{x}' \left[c(\mathbf{x}, \mathbf{x}') \Rightarrow \begin{cases} \rho(\mathbf{x}) & \geq & 1 + \rho(\mathbf{x}') \\ \rho(\mathbf{x}) & \geq & 0 \end{cases} \right]$$

Example

For $p(x) \leftarrow x \geq 0, x' \leq x - 1, p(x')$

- $\rho(x) = x$ is a linear ranking function:
  - $\rho(x) - \rho(x') = x - x' \geq 1$
  - $\rho(x) \geq 0$

Easily checked. Detection? Podelski & Rybalchenko (2004)

Farkas' Lemma

Given a satisfiable linear system

$$S := \begin{cases} a_{1,1}x_1 & + & \cdots & + & a_{1,n}x_n & + & b_1 & \geq & 0 \\ \dots \\ a_{m,1}x_1 & + & \cdots & + & a_{m,n}x_n & + & b_m & \geq & 0 \end{cases}$$

any inequality implied by S is a positive combination of its inequalities:

$$\forall x_1, \dots, x_n \, [S \Rightarrow c_1 x_1 + \cdots + c_n x_n + d \geq 0]$$

iff

$$\exists \lambda_1 \geq 0, \dots, \lambda_n \geq 0$$

$$\begin{cases} c_1 & = & \sum_{i=1}^{m} \lambda_i a_{i,1} \\ \dots \\ c_n & = & \sum_{i=1}^{m} \lambda_i a_{i,n} \\ d & \geq & \sum_{i=1}^{m} \lambda_i b_i \end{cases}$$

For the SLC loop C_1:

$$p(x, y) \leftarrow x \geq 0, y' \leq y - 1, x' \leq x + y, y \leq -1, p(x', y')$$

a linear ranking function $\rho(x, y) = ax + by$ exists iff:

$$\exists a, b\ \forall x, y, x', y' \quad \left\{ \begin{array}{lcl} x & \geq & 0 \\ y' & \leq & y - 1 \\ x' & \leq & x + y \\ y & \leq & -1 \end{array} \right. \Rightarrow \left\{ \begin{array}{lcl} ax + by & \geq & 1 + ax' + by' \\ ax + by & \geq & 0 \end{array} \right.$$

Solvable by quantifier elimination, hence decidable but expensive!
By considering a and b as parameters, apply the Farkas' Lemma

Decreasing of the ranking function:

$$\forall x, y, x', y' \quad \begin{cases} x & \geq & 0 \\ y' & \leq & y - 1 \\ x' & \leq & x + y \\ y & \leq & -1 \end{cases} \Rightarrow \quad ax + by \quad \geq \quad 1 + ax' + by'$$

Positivity of the ranking function:

$$\forall x, y, x', y' \quad \begin{cases} x & \geq & 0 \\ y' & \leq & y - 1 \\ x' & \leq & x + y \\ y & \leq & -1 \end{cases} \Rightarrow \quad ax + by \quad \geq \quad 0$$

Preparing for the Farkas' Lemma

$$
\begin{array}{lcccccccc}
\lambda_1 : & 1x & +0y & +0x' & +0y' & +0 & \geq & 0 \\
\lambda_2 : & 1x & +1y & -1x' & +0y' & +0 & \geq & 0 \\
\lambda_3 : & 0x & +1y & +0x' & -1y' & -1 & \geq & 0 \\
\lambda_4 : & 0x & -1y & +0x' & +0y' & -1 & \geq & 0 \\
\Rightarrow & & & & & & & \\
 & ax & +by & -ax' & -by' & -1 & \geq & 0
\end{array}
$$

$$
\begin{array}{lcccccccc}
\lambda'_1 : & 1x & +0y & +0x' & +0y' & +0 & \geq & 0 \\
\lambda'_2 : & 1x & +1y & -1x' & +0y' & +0 & \geq & 0 \\
\lambda'_3 : & 0x & +1y & +0x' & -1y' & -1 & \geq & 0 \\
\lambda'_4 : & 0x & -1y & +0x' & +0y' & -1 & \geq & 0 \\
\Rightarrow & & & & & & & \\
 & ax & +by & +0x' & +0y' & +0 & \geq & 0
\end{array}
$$

Applying the Farkas' Lemma

$$
\exists\lambda_1 \geq 0, \ldots, \lambda_4 \geq 0
\begin{cases}
a & = & \lambda_1 + \lambda_2 \\
b & = & \lambda_2 + \lambda_3 - \lambda_4 \\
-a & = & -\lambda_2 \\
-b & = & -\lambda_3 \\
-1 & \geq & -\lambda_3 - \lambda_4
\end{cases}
$$

$$
\exists\lambda'_1 \geq 0, \ldots, \lambda'_4 \geq 0
\begin{cases}
a & = & \lambda'_1 + \lambda'_2 \\
b & = & \lambda'_2 + \lambda'_3 - \lambda'_4 \\
0 & = & -\lambda'_2 \\
0 & = & -\lambda'_3 \\
0 & \geq & -\lambda'_3 - \lambda'_4
\end{cases}
$$

Conjuncting

$$\exists a, b\ \exists \lambda_1 \geq 0, \ldots, \lambda_4 \geq 0, \lambda'_1 \geq 0, \ldots, \lambda'_4 \geq 0$$
$$\begin{cases} a & = & \lambda_1 + \lambda_2 \\ b & = & \lambda_2 + \lambda_3 - \lambda_4 \\ -a & = & -\lambda_2 \\ -b & = & -\lambda_3 \\ -1 & \geq & -\lambda_3 - \lambda_4 \\ a & = & \lambda'_1 + \lambda'_2 \\ b & = & \lambda'_2 + \lambda'_3 - \lambda'_4 \\ 0 & = & -\lambda'_2 \\ 0 & = & -\lambda'_3 \\ 0 & \geq & -\lambda'_3 - \lambda'_4 \end{cases}$$

By solving linear inequalities:

$b = 0 = \lambda_1 = \lambda_3 = \lambda'_2 = \lambda'_3 = \lambda'_4, a = 1 = \lambda_2 = \lambda_4 = \lambda'_1$

Summarizing

The rational SLC loop C_1 admits $\rho(x,y) = x$ as a linear ranking function because:

$$\forall x, y, x', y'$$
$$\begin{cases} x & \geq & 0 \\ y' & \leq & y - 1 \\ x' & \leq & x + y \\ y & \leq & -1 \end{cases} \Rightarrow \begin{cases} x & \geq & 1 + x' \\ x & \geq & 0 \end{cases}$$

Eventual Linear Ranking Functions

Let C be the SLC loop $p(\mathbf{x}) \leftarrow c(\mathbf{x}, \mathbf{x}'),\ p(\mathbf{x}')$ where p is an n-ary relation
Let $f(\mathbf{x})$ be a linear function st:

$$\forall \mathbf{x}, \mathbf{x}'\ c(\mathbf{x}, \mathbf{x}') \Rightarrow f(\mathbf{x}') \geq 1 + f(\mathbf{x})$$

Definition

An eventual linear ranking function ρ pour (C, f) is a linear function from $\mathbb{Q}^n$ to $\mathbb{Q}$ st:

$$\exists k\, \forall \mathbf{x}, \mathbf{x}' \begin{cases} c(\mathbf{x}, \mathbf{x}') \\ f(\mathbf{x}) \geq k \end{cases} \Rightarrow \begin{cases} \rho(\mathbf{x}) & \geq & 1 + \rho(\mathbf{x}') \\ \rho(\mathbf{x}) & \geq & 0 \end{cases}$$

- The SLC loop C_2

$$p(x,y) \leftarrow x \geq 0, x' \leq x + y, y' \leq y - 1, p(x', y')$$

  does not admit a linear ranking function
- The function $f(x,y) = -y$ is strictly increasing for C_2
- The loop admits an eventual linear function $\rho(x,y) = ax + by$ iff:

$$\exists a, b, k \; \forall x, y, x', y' \quad \begin{cases} x & \geq & 0 \\ x' & \leq & x + y \\ y' & \leq & y - 1 \\ -y & \geq & k \end{cases} \Rightarrow \begin{cases} ax + by & \geq & 1 + ax' + by' \\ ax + by & \geq & 0 \end{cases}$$

Farkas' Lemma

$\mathrm{DEC}(a, b, k)$:

$$\exists \lambda_1 \geq 0, \ldots, \lambda_4 \geq 0$$
$$\begin{cases} a & = & \lambda_1 + \lambda_2 \\ b & = & \lambda_2 + \lambda_3 - \lambda_4 \\ -a & = & -\lambda_2 \\ -b & = & -\lambda_3 \\ -1 & \geq & -\lambda_3 - k\lambda_4 \end{cases}$$

$\mathrm{POS}(a, b, k)$:

$$\exists \lambda'_1 \geq 0, \ldots, \lambda'_4 \geq 0$$
$$\begin{cases} a & = & \lambda'_1 + \lambda'_2 \\ b & = & \lambda'_2 + \lambda'_3 - \lambda'_4 \\ 0 & = & -\lambda'_2 \\ 0 & = & -\lambda'_3 \\ 0 & \geq & -\lambda'_3 - k\lambda'_4 \end{cases}$$

$\mathrm{DEC}_1(a, b)$:

$$\exists \lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0, \lambda_4 = 0$$
$$\begin{cases} a & = & \lambda_1 + \lambda_2 \\ b & = & \lambda_2 + \lambda_3 \\ -a & = & -\lambda_2 \\ -b & = & -\lambda_3 \\ -1 & \geq & -\lambda_3 \end{cases}$$

$\mathrm{DEC}_2(a, b)$ with $P = k\lambda_4$:

$$\exists \lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0, \lambda_4 > 0, P$$
$$\begin{cases} a & = & \lambda_1 + \lambda_2 \\ b & = & \lambda_2 + \lambda_3 - \lambda_4 \\ -a & = & -\lambda_2 \\ -b & = & -\lambda_3 \\ -1 & \geq & -\lambda_3 - P \end{cases}$$

Idem for *POS*

Lemma

$\exists k\, \mathrm{DEC}(a, b, k)$ *is equivalent to* $\mathrm{DEC}_1(a, b) \vee \mathrm{DEC}_2(a, b)$
$\exists k\, \mathrm{POS}(a, b, k)$ *is equivalent to* $\mathrm{POS}_1(a, b) \vee \mathrm{POS}_2(a, b)$

Hence:

Given an strictly increasing function, an eventual linear ranking exists iff one of theses formulas is satisfiable:

$$
\begin{array}{c}
\mathrm{DEC}_1(a, b) \wedge \mathrm{POS}_1(a, b) \\
\mathrm{DEC}_1(a, b) \wedge \mathrm{POS}_2(a, b) \\
\mathrm{DEC}_2(a, b) \wedge \mathrm{POS}_1(a, b) \\
\mathrm{DEC}_2(a, b) \wedge \mathrm{POS}_2(a, b)
\end{array}
$$

Decidable and polynomial
For our example, $\mathrm{DEC}_2(a, b) \wedge \mathrm{POS}_1(a, b)$ is satisfiable

$$\exists \lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0, \lambda_4 > 0, P, \lambda'_1 \geq 0, \lambda'_2 \geq 0, \lambda'_3 \geq 0$$
$$\begin{cases} a & = & \lambda_1 + \lambda_2 \\ b & = & \lambda_2 + \lambda_3 - \lambda_4 \\ -a & = & -\lambda_2 \\ -b & = & -\lambda_3 \\ -1 & \geq & -\lambda_3 - P \\ a & = & \lambda'_1 + \lambda'_2 \\ b & = & \lambda'_2 + \lambda'_3 \\ 0 & = & -\lambda'_2 \\ 0 & = & -\lambda'_3 \\ 0 & \geq & -\lambda'_3 \end{cases}$$

$$b = 0 = \lambda_1 = \lambda_3 = \lambda'_2 = \lambda'_3, \ a = 1 = \lambda'_1 = \lambda_2 = \lambda_4 = P, P/\lambda_4 = k = 1$$

Summarizing

Given the increasing linear function $f(x, y) = -y$, the rational SLC loop C_2 admits $\rho(x, y) = x$ from $k = 1$ as an eventual linear ranking because:

$$\exists k \geq 1 \; \forall x, y, x', y' \\ \begin{cases} x & \geq & 0 \\ x' & \leq & x + y \\ y' & \leq & y - 1 \\ -y & \geq & 1 \end{cases} \Rightarrow \begin{cases} x & \geq & 1 + x' \\ x & \geq & 0 \end{cases}$$

A Correct, Complete, and Polynomial Algorithm

Algorithm 1 Existence of an eventual linear ranking function, given a linear increasing function

Require: $p(\mathbf{x}) \rightarrow c(\mathbf{x}, \mathbf{x}'), p(\mathbf{x}')$ an an-ary SLC loop and $f(\mathbf{x})$ a linear function st: $\forall \mathbf{x}, \mathbf{x}' \; c(\mathbf{x}, \mathbf{x}') \Rightarrow f(\mathbf{x}') \geq 1 + f(\mathbf{x})$

Ensure: decide the existence of an eventual linear ranking function $\rho(\mathbf{x}) = \mathbf{ax} = \sum_{i=1}^{n} a_i x_i$ from k

$\mathrm{DEC}(\mathbf{a}, k)$:= Farkas for the decreasing of ρ
$\mathrm{DEC}_1(\mathbf{a}), \mathrm{DEC}_2(\mathbf{a})$:=linearization of $\mathrm{DEC}(\mathbf{a}, k)$
$\mathrm{POS}(\mathbf{a}, k)$:= Farkas pour the positivity of ρ
$\mathrm{POS}_1(\mathbf{a}), \mathrm{POS}_2(\mathbf{a})$:=linearization of $\mathrm{POS}(\mathbf{a}, k)$
if $\bigvee_{1 \leq i,j \leq 2} \mathrm{DEC}_i(\mathbf{a}) \wedge \mathrm{POS}_j(\mathbf{a})$ is satisfiable **then**
    return true
else
    return false
end if

Fully Automated Detection

- Given an SLC loop C, does there exist an increasing function such that C admits an eventual linear ranking function?
- The space of increasing functions can be obtained via the Farkas' Lemma and existentially quantified variables elimination

Definition

Let $C = \big(p(\mathbf{x}) \leftarrow c(\mathbf{x}, \mathbf{x}'), p(\mathbf{x}')\big)$ be an SLC loop. We denote by INC the set of vectors $\mathbf{b}$ such that $f(\mathbf{x}) = \langle \mathbf{b}, \mathbf{x} \rangle$ is increasing for C.

Example

- No linear ranking function for C_3:
  $p(x, y) \leftarrow x \geq 0, x' \leq x + y, y' \leq -y - 1, p(x', y')$
- $\mathrm{INC} = \{ (b_1, b_2) \in \mathbb{Q} \times \mathbb{Q} \mid b_1 \leq -2, b_1 - 2b_2 = 0 \}$ = the space of functions of the form $f(x, y) = b_1 x + b_2 y$ which are increasing for C_3
- Defining $\rho(x, y) = a_1 x + a_2 y$ and considering b_1 and b_2 as parameters, ρ is an eventual linear ranking function iff:

$$\exists a_1, a_2, k \,.\, \forall x, y, x', y' : \begin{Bmatrix} x \geq 0, & x' \leq x + y \\ b_1 x + b_2 y \geq k, & y' \leq -y - 1 \end{Bmatrix} \implies \begin{cases} a_1 x + a_2 y \geq 1 + a_1 x' + a_2 y' \\ a_1 x + a_2 y \geq 0 \end{cases}$$

- Again, solvable using an extension of Farkas' Lemma, see the paper for details

A Correct and Complete Algorithm

Algorithm 2 Existence of an eventual linear ranking function

INC := the space of increasing functions for the SLC loop C
$\mathrm{DEC}(\mathbf{a}, k)$:= Farkas for the decreasing of ρ
$\mathrm{DEC}_1(\mathbf{a}), \mathrm{DEC}_2(\mathbf{a}, \lambda, \mathbf{p})$:= linearization of $\mathrm{DEC}(\mathbf{a}, k)$
$\mathrm{POS}(\mathbf{a}, k)$:= Farkas for the positivity of ρ
$\mathrm{POS}_1(\mathbf{a}), \mathrm{POS}_2(\mathbf{a}, \lambda', \mathbf{p}')$:= linearization of $\mathrm{POS}(\mathbf{a}, k)$
$\phi_{1,1} := \mathrm{DEC}_1(\mathbf{a}) \wedge \mathrm{POS}_1(\mathbf{a})$
$\phi_{1,2} := \mathrm{DEC}_1(\mathbf{a}) \wedge \mathrm{POS}_2(\mathbf{a}, \lambda', \mathbf{p}') \wedge \mathbf{p}'/\lambda' \in \mathrm{INC}$
$\phi_{2,1} := \mathrm{DEC}_2(\mathbf{a}, \lambda, \mathbf{p}) \wedge \mathbf{p}/\lambda \in \mathrm{INC} \wedge \mathrm{POS}_1(\mathbf{a})$
$\phi_{2,2} := \mathrm{DEC}_2(\mathbf{a}, \lambda, \mathbf{p}) \wedge \mathbf{p}/\lambda \in \mathrm{INC} \wedge \mathrm{POS}_2(\mathbf{a}, \lambda', \mathbf{p}') \wedge \mathbf{p}/\lambda = \mathbf{p}'/\lambda'$
if $\bigvee_{1 \leq i,j \leq 2} \phi_{i,j}$ is satisfiable **then**
  return true
else
  return false
end if

Our contributions

- Definition of eventual linear ranking function, a *strict* generalization of linear ranking function (Podelski/Rybalchenko, 2004)
- Given a rational SLC loop and a strictly increasing linear function, a correct, complete, and polynomial algorithm for the existence of an eventual ranking function
- Given a rational SLC loop, a correct and complete algorithm for the existence of an eventual ranking function
- Both algorithms implemented in Prolog+CLP(Q) within the BinTerm termination prover

Future works

- C++ implementation in the Parma Polyhedra Library (PPL)
- *Integer* SLC loops?