



Values at T -tuples of negative integers of twisted multivariable zeta series associated to polynomials of several variables

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ABSTRACT

We consider twisted multivariable zeta series associated to polynomials of several variables. We introduce a new class of polynomials, namely *HDF*, that contains strictly non-degenerate and hypoelliptic polynomials. For polynomials belonging to the *HDF* class, we show that we can extend holomorphically our series to \mathbb{C}^T . Then, thanks to a new principle called ‘the Exchange Lemma’, we give very simple formulae for the values of our series at T -tuples of negative integers. Finally, we make the p -adic interpolation of those values. Thus, we have generalized the results of Cassou-Noguès (that she used to construct the p -adic L -functions of totally real fields) in two ways: we consider multivariable series and our series are associated to more general polynomials. In addition, our proof is completely different.

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Introduction

Let $Q, P_1, \dots, P_T \in \mathbb{R}[X_1, \dots, X_N]$ and μ_1, \dots, μ_N be complex numbers of modulus 1. To these data we can associate the following multivariable zeta series:

$$Z(Q; P_1, \dots, P_T; \mu_1, \dots, \mu_N; s_1, \dots, s_T) = \sum_{m_1 \geq 1, \dots, m_N \geq 1} \frac{(\prod_{n=1}^N \mu_n^{m_n}) Q(m_1, \dots, m_N)}{\prod_{t=1}^T P_t(m_1, \dots, m_N)^{s_t}}$$

where $(s_1, \dots, s_T) \in \mathbb{C}^T$.

In this article we will always assume that

$$\forall t \in \{1, \dots, T\}, \quad \forall \mathbf{x} \in [1, +\infty[^N, \quad P_t(\mathbf{x}) > 0 \quad \text{and} \quad \prod_{t=1}^T P_t(\mathbf{x}) \xrightarrow[\mathbf{x} \in J^N]{|\mathbf{x}| \rightarrow +\infty} +\infty.$$

Then $Z(Q; P_1, \dots, P_T; \mu_1, \dots, \mu_N; s_1, \dots, s_T)$ is an absolutely convergent series when $\Re(s_1), \dots, \Re(s_T)$ are sufficiently large. We will say that the series is twisted when all of the μ_t are different from one and non-twisted when they are all equal to one.

The complexity of these series lies in the polynomials P_t , so the authors studied these series for several classes of polynomials.

The issue of meromorphic continuation

The meromorphic continuation to \mathbb{C} of these series was proved in the non-twisted and monovaryable (i.e. $T = 1$) case by Mellin (P with positive coefficients [Mel01]), Mahler (elliptic case [Mah28]), Cassou-Noguès (positive coefficients case for polynomials with two variables [Cas83]), Sargos (non-degenerate case [Sar84]), Lichtin (hypoelliptic monovaryable case [Lic88]) and Essouabri (H_0S case [Ess97]) Definitions of some of these classes are recalled at the beginning of § 1.

In fact, these results extend easily (in their respective classes) to the multivariable case (see, for example, [Lic91] and [Ess95, p. 74]).

When the μ_n are roots of unity, the meromorphic continuation is clearly a consequence of the non-twisted case, but when they are not, we have to use the path used in [Ess97]. As a conclusion, it is a simple adaptation of the work of Essouabri to see that under H_0S the series can be meromorphically extended to \mathbb{C}^T for any μ_1, \dots, μ_N of modulus 1.

Katsurada and Matsumoto [KM96], Akiyama and Ishikawa [AI02], Matsumoto and Tanigawa [MT03], Zhao [Zha00], Ishikawa [Ish02], and Egami and Matsumoto [EM02] gave simple proofs of the existence of meromorphic continuation. However, they only considered special cases of linear forms.

The issue of values at negative integers

The monovaryable and non-twisted case when $P = P_1$ is a product of linear forms. Shintani [Shi76] showed that the negative integers are not poles and gave formulae for the values at those points. Thanks to this, he gave a new proof of a result of Klíngen and Siegel: for any totally real number field \mathbb{K} , we have $\zeta_{\mathbb{K}}(-k) \in \mathbb{Q}$ for all $k \in \mathbb{N}$.

Eie also studied this case in [Eie96].

In [Cas79], Cassou-Noguès studied the twisted case for $T = 1$ when P_1 is a product of linear forms. She gave formulae at negative integers adapted to p -adic interpolation. This allowed her to construct the p -adic L -functions associated to number fields and to solve crucial arithmetic conjectures.

In [Cas82] she generalized her work to the $T = 1$ polynomial with positive coefficients, still in the twisted case. Using similar methods, Chen and Eie (in [CE01]) gave very simple formulae for the values at negative integers, but they did not achieve the link with the formulae of Cassou-Noguès that are useful for p -adic interpolation.

The methods of Cassou-Noguès do not appear to extend easily to more general settings, that is, the case $T \geq 2$, or the case of degenerate polynomials.

The works of Akiyama, Egami and Tanigawa [AET01], Akiyama and Tanigawa [AT01], Arakawa and Kaneko [AK99], Apostol and Vu [AV84] deal with the values in the multivariable setting and non-twisted case. They deal with special cases of linear forms.

Presentation of this work

Although the H_0S class contains both non-degenerate and hypoelliptic polynomials, it is too large for our purposes. Indeed, one might hope that for any polynomial P belonging to H_0S , the continuation of any twisted zeta series $Z(Q, P, \mu, s)$ would be entire. We give an example P_{ex} that shows that this is *not* the case. This leads us to introduce a subset HDF of H_0S that still contains strictly all non-degenerate and hypoelliptic polynomials. The first main result of this paper shows that if P_1, \dots, P_T belong to HDF , then any twisted series $Z(Q; P_1, \dots, P_T; \mu_1, \dots, \mu_N; s_1, \dots, s_T)$ has a holomorphic continuation to \mathbb{C}^T (Theorem A). The second main result of this article is Theorem B. This gives a very simple expression for the value at any T -tuple of negative integers of these holomorphically continued series. This generalizes the result of Cassou-Noguès [Cas79, Cas82] to the multivariable case where the polynomials P_1, \dots, P_T , $T \geq 1$ belong to HDF . Our proof is quite different than that of Cassou-Noguès and is based on a simple ‘Exchange Lemma’. This is a new idea whose proof can only be given in a multivariable setting. The formulae we obtain also generalize those of Chen and Eie in [CE01]. Using these formulae we are then able to prove our third main result, Theorem C. This shows that the values at T -tuples of negative integers of a large class of twisted series (in T variables) can be p -adically interpolated.

Notation

Set $\mathbb{N} = \{0, 1, 2, \dots\}$, $\mathbb{N}^* = \mathbb{N} - \{0\}$, $J = [1, +\infty[$, and $\mathbb{T} = \{\alpha \in \mathbb{C} \mid |\alpha| = 1\}$. The real part of $s \in \mathbb{C}$ will be denoted by $\Re(s) = \sigma$ and its imaginary part by $\Im(s) = \tau$. If $x \in \mathbb{Q}_p$, set $v_p(x) = \text{ord}_p(x)$. Set $\mathbf{0} = (0, \dots, 0) \in \mathbb{R}^N$ and $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^N$. For $\mathbf{x} = (x_1, \dots, x_N) \in \mathbb{R}^N$ we set $|\mathbf{x}| = |x_1| + \dots + |x_N|$. For $\mathbf{z} = (z_1, \dots, z_N) \in \mathbb{C}^N$ and $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_N) \in \mathbb{R}_+^N$ we set $\mathbf{z}^\boldsymbol{\alpha} = z_1^{\alpha_1} \dots z_N^{\alpha_N}$. For $t \in \{1, \dots, T\}$ we denote $\mathbf{e}_t = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{N}^T$. Define $e: \mathbb{C} \rightarrow \mathbb{C}$ by $e(z) = \exp(2i\pi z)$. Given $P = \sum_{\boldsymbol{\alpha} \in \mathbb{N}^N} a_{\boldsymbol{\alpha}} \mathbf{X}^\boldsymbol{\alpha} \in \mathbb{R}[X_1, \dots, X_N]$, we define $P^+ \in \mathbb{R}[X_1, \dots, X_N]$ by $P^+ = \sum_{\boldsymbol{\alpha} \in \mathbb{N}^N} |a_{\boldsymbol{\alpha}}| \mathbf{X}^\boldsymbol{\alpha}$.

The notation $f(\lambda, \mathbf{y}, \mathbf{x}) \ll_{\mathbf{y}} g(\mathbf{x})$ (uniformly in $\mathbf{x} \in X$ and $\lambda \in \Lambda$) means that there exists $A = A(\mathbf{y}) > 0$, that does not depend on \mathbf{x} or λ , but could *a priori* depend on other parameters and, in particular, on \mathbf{y} , such that for all $\mathbf{x} \in X$ and all $\lambda \in \Lambda$, $|f(\lambda, \mathbf{y}, \mathbf{x})| \leq Ag(\mathbf{x})$. When there is no ambiguity, we will omit the word uniformly and the index \mathbf{y} . The notation $f \asymp g$ means that we have both $f \ll g$ and $g \ll f$.

Convention

In this work we will say that a series defined by a sum over $N \geq 1$ variables is convergent when it is absolutely convergent.

1. Statements of main results

Let us first recall a few definitions.

DEFINITION 1.1. We say that $P \in \mathbb{R}[X_1, \dots, X_N] \setminus \{0\}$ is non-degenerate if $P(\mathbf{x}) \asymp P^+(\mathbf{x})$ ($\mathbf{x} \in J^N$).

Clearly the polynomials with positive coefficients are non-degenerate.

The following proposition characterizes the non-degenerate polynomials according to their growth performance on J^N . The proof is given in [Dec03].

PROPOSITION 1.2. Let $P \in \mathbb{R}[X_1, \dots, X_N]$ satisfying $P(\mathbf{x}) > 0$ for all $\mathbf{x} \in J^N$. Then P is non-degenerate if and only if for all $\alpha \in \mathbb{N}^N$ $(\partial^\alpha P/P)(\mathbf{x}) \ll \mathbf{x}^{-\alpha}$ ($\mathbf{x} \in J^N$).

DEFINITION 1.3. We say that $P \in \mathbb{R}[X_1, \dots, X_N]$ is hypoelliptic if

$$\forall \mathbf{x} \in J^N, P(\mathbf{x}) > 0 \quad \text{and} \quad \forall \alpha \in \mathbb{N}^N \setminus \{\mathbf{0}\}, \quad \frac{\partial^\alpha P}{P}(\mathbf{x}) \xrightarrow[\substack{|\mathbf{x}| \rightarrow +\infty \\ \mathbf{x} \in J^N}]{} 0.$$

In [Ess97], Essouabri introduced a new class of polynomials as follows.

DEFINITION 1.4. We say that $P \in \mathbb{R}[X_1, \dots, X_N]$ satisfies H_0S if

$$\forall \mathbf{x} \in J^N, P(\mathbf{x}) > 0 \quad \text{and} \quad \forall \alpha \in \mathbb{N}^N, \quad \frac{\partial^\alpha P}{P}(\mathbf{x}) \ll 1 \quad (\mathbf{x} \in J^N).$$

It is clear that this class contains both non-degenerate and hypoelliptic polynomials on J^N . What is less clear is that this inclusion is strict. Essouabri gave the following example.

Example 1.5. Let $P_{ex} = (X - Y)^2 X + X \in \mathbb{R}[X, Y]$. Then P_{ex} satisfy H_0S but P is degenerate and is not hypoelliptic.

In the H_0S class, the extension of a twisted series is not always holomorphic.

PROPOSITION 1.6. We have that $Z(1, P_{ex}, -1, -1, \cdot)$ has a meromorphic extension to \mathbb{C} with a single pole at $s = 1$, which is simple. The residue at $s = 1$ is equal to $\pi/\sinh(\pi)$.

This will be proved in §3.4.

Remark 1.7. It follows from the algebraic independence of π and e^π that $\pi/\sinh(\pi)$ is transcendental.

Thus, to show that a twisted series has a holomorphic extension to \mathbb{C} , we have to restrict to a subclass of H_0S . So we introduce a new class of polynomials.

DEFINITION 1.8 (*HDF hypothesis*). Let $P \in \mathbb{R}[X_1, \dots, X_N]$. Then P is said to satisfy the weak decreasing hypothesis (denoted *HDF* in the rest of the article) if:

- for all $\mathbf{x} \in J^N$, $P(\mathbf{x}) > 0$;
- there exists $\epsilon_0 > 0$ such that for $\alpha \in \mathbb{N}^N$ and $n \in \{1, \dots, N\}$: $\alpha_n \geq 1 \Rightarrow (\partial^\alpha P/P)(\mathbf{x}) \ll x_n^{-\epsilon_0}$ ($\mathbf{x} \in J^N$).

The proof of the first point of the following remark is easy and is given in [Dec03, p. 48]. Points (2), (3) and (4) are clear.

Remark 1.9.

- (1) Let $P \in \mathbb{R}[X_1, \dots, X_N]$ satisfy *HDF*. Let us denote $I = \{n \mid P \text{ depends effectively on } X_n\}$. Then

$$P(\mathbf{x}) \xrightarrow[\substack{\sum_{n \in I} x_n \rightarrow +\infty \\ \mathbf{x} \in J^N}]{} +\infty.$$

- (2) The class *HDF* is stable under product.

As a consequence, we have the following.

(3) If P_1, \dots, P_T satisfy *HDF*, then

$$\prod_{t=1}^T P_t(\mathbf{x}) \xrightarrow[\substack{\mathbf{x} \rightarrow +\infty \\ \mathbf{x} \in J^N}]{} +\infty \iff \prod_{t=1}^T P_t(\mathbf{x}) \text{ depends effectively on all variables.}$$

(4) For $P_1, \dots, P_T \in \mathbb{R}[X_1, \dots, X_N]$ we have

$$\prod_{t=1}^T P_t(\mathbf{x}) \text{ depends effectively on all variables} \\ \iff \text{for all } n \text{ there exists } t \text{ such that } P_t \text{ depends effectively on } X_n.$$

The condition on the right-hand side is very easy to verify.

It is clear from the preceding definitions that the *HDF* class is contained in H_0S and contains both hypoelliptic and non-degenerate polynomials. We are now going to give a simple method to construct polynomials satisfying *HDF* but that are degenerate and not hypoelliptic. So the *HDF* class is strictly larger than the union of the class of non-degenerate polynomials with the class of the hypoelliptic polynomials. The result is as follows.

LEMMA 1.10. *We assume that $P \in \mathbb{R}[X_1, \dots, X_N]$ is non-degenerate and is not hypoelliptic. We assume that $Q \in \mathbb{R}[X_1, \dots, X_N]$ is hypoelliptic and degenerate. Then PQ is degenerate and is not hypoelliptic.*

Furthermore, since the class *HDF* is stable under product, PQ satisfies *HDF*, so we have obtained what was required.

The preceding lemma is an obvious consequence of the following lemmas.

LEMMA 1.11. *Let P and $Q \in \mathbb{R}[X_1, \dots, X_N]$. We assume that for all $\mathbf{x} \in J^N$, $P(\mathbf{x}) > 0$ and that P is degenerate, but that Q is not. Then PQ is degenerate.*

The proof is in [Dec03].

LEMMA 1.12. *Let P and $Q \in \mathbb{R}[X_1, \dots, X_N]$. We assume that P is hypoelliptic. We assume that Q satisfies $Q(\mathbf{x}) > 0$ for all $\mathbf{x} \in J^N$ and is not hypoelliptic. Then PQ is not hypoelliptic.*

The proof is easy with the Leibniz formula.

We now come back to our series. Under the *HDF* hypothesis the twisted series Z extends holomorphically. More precisely we have the following.

THEOREM A. *Let $Q, P_1, \dots, P_T \in \mathbb{R}[X_1, \dots, X_N]$ and $\boldsymbol{\mu} \in (\mathbb{T} \setminus \{1\})^N$. We assume that P_1, \dots, P_T satisfy *HDF* and that*

$$\prod_{t=1}^T P_t(\mathbf{x}) \xrightarrow[\substack{|\mathbf{x}| \rightarrow +\infty \\ \mathbf{x} \in J^N}]{} +\infty.$$

Then $Z(Q; P_1, \dots, P_T; \boldsymbol{\mu}; \cdot)$ extends to \mathbb{C}^T as an entire function.

To study the values of $Z(Q; P_1, \dots, P_T; \boldsymbol{\mu}; \cdot)$ on $(-\mathbb{N})^T$, the key lemma is as follows.

EXCHANGE LEMMA. *Let $Q, P_1, \dots, P_T, Q_1, \dots, Q_{T'} \in \mathbb{R}[X_1, \dots, X_N]$. We assume that:*

- $P_1, \dots, P_T, Q_1, \dots, Q_{T'}$ satisfy *HDF*;
- $\prod_{t=1}^T P_t(\mathbf{x}) \xrightarrow[\substack{|\mathbf{x}| \rightarrow +\infty \\ \mathbf{x} \in J^N}]{} +\infty$ and $\prod_{t=1}^{T'} Q_t(\mathbf{x}) \xrightarrow[\substack{|\mathbf{x}| \rightarrow +\infty \\ \mathbf{x} \in J^N}]{} +\infty$.

Let $\boldsymbol{\mu} \in (\mathbb{T} \setminus \{1\})^N$ and $\mathbf{k} = (k_1, \dots, k_T) \in \mathbb{N}^T, \boldsymbol{\ell} = (\ell_1, \dots, \ell_{T'}) \in \mathbb{N}^{T'}$. Then

$$Z\left(Q \prod_{t=1}^{T'} Q_t^{\ell_t}; P_1, \dots, P_T; \boldsymbol{\mu}; -\mathbf{k}\right) = Z\left(Q \prod_{t=1}^T P_t^{k_t}; Q_1, \dots, Q_{T'}; \boldsymbol{\mu}; -\boldsymbol{\ell}\right).$$

Remark 1.13. (1) *Justification of the interest of this lemma.* Let us consider the case $T = T' = 1$ with $Q = 1$. Let us assume that P_1 is ‘complicated’ and that Q_1 is ‘simple’. The Exchange Lemma gives $Z(Q_1^{\ell_1}; P_1; \boldsymbol{\mu}; -k_1) = Z(P_1^{k_1}; Q_1; \boldsymbol{\mu}; -\ell_1)$. In principle, the left-hand side should be difficult to evaluate, whereas the right-hand side should be easier to evaluate. The equation indicates that an *a priori* hard problem (evaluation of the left-hand side) is actually easier than one might think.

(2) *Justification of the study of multivariable series.* It is true that the Exchange Lemma is meaningful for series in $T = T' = 1$ variable. However, to prove the Exchange Lemma in the monovariate setting, we need to use series in $T + T' = 2$ variables. This justifies, if required, the use of multivariable series.

Remark 1.14. In the previous works, the existence of a holomorphic continuation and the calculus of the values were simultaneously worked out. Here it is absolutely not the case: we have two independent steps.

DEFINITION 1.15. For $\mu \in \mathbb{T}$, we set $\zeta_\mu(s) = Z(1; X; \mu; s) = \sum_{m \geq 1} (\mu^m / m^s)$.

To illustrate how to use the Exchange Lemma, we easily deduce a theorem giving the values of the general series Z at points $-\mathbf{k} \in (-\mathbb{N})^T$ in terms of the values at negative integers of a much simpler series ζ_μ . This result also extends those obtained by Cassou-Noguès [Cas82] and Chen and Eie [CE01].

THEOREM B. Let $Q, P_1, \dots, P_T \in \mathbb{R}[X_1, \dots, X_N]$. We assume that: P_1, \dots, P_T satisfy HDF and that

$$\prod_{t=1}^T P_t(\mathbf{x}) \xrightarrow[|\mathbf{x}| \rightarrow +\infty]{\mathbf{x} \in J^N} +\infty.$$

Let $\mathbf{k} = (k_1, \dots, k_T) \in \mathbb{N}^T$ and write $Q \prod_{t=1}^T P_t^{k_t} = \sum_{\boldsymbol{\alpha} \in S} a_{\boldsymbol{\alpha}} \mathbf{X}^{\boldsymbol{\alpha}}$. Let $\boldsymbol{\mu} \in (\mathbb{T} \setminus \{1\})^N$. Then $Z(Q; P_1, \dots, P_T; \boldsymbol{\mu}; -\mathbf{k}) = \sum_{\boldsymbol{\alpha} \in S} a_{\boldsymbol{\alpha}} \prod_{n=1}^N \zeta_{\mu_n}(-\alpha_n)$.

An interesting corollary, arithmetic in nature, now follows as an immediate consequence of formulae for ζ_μ at negative integers (cf. Lemma 5.7) and of Theorem B.

COROLLARY 1.16. Let \mathbb{K} be a subfield of \mathbb{R} . Let $Q, P_1, \dots, P_T \in \mathbb{K}[X_1, \dots, X_N]$. We assume that P_1, \dots, P_T satisfy HDF and that

$$\prod_{t=1}^T P_t(\mathbf{x}) \xrightarrow[|\mathbf{x}| \rightarrow +\infty]{\mathbf{x} \in J^N} +\infty.$$

For any $\boldsymbol{\mu} \in (\mathbb{T} \setminus \{1\})^N$ and $\mathbf{k} \in \mathbb{N}^T$ we have $Z(Q; P_1, \dots, P_T; \boldsymbol{\mu}; -\mathbf{k}) \in \mathbb{K}(\mu_1, \dots, \mu_N)$.

The Exchange Lemma is a general principle of calculus of values at T -tuples of negative integers; we can also apply it to a class of integrals Y (see §2 and §4.3).

A suitable p -adic interpolation for the function $-\mathbf{k} \rightarrow Z(Q; P_1, \dots, P_T; \boldsymbol{\mu}; -\mathbf{k})$ is only possible provided that one restricts the series to lattice points \mathbf{m} such that $p \nmid P_t(\mathbf{m})$ for each t . The second main result of the paper is the following. Its proof is based on Theorem B.

THEOREM C. Let p be a prime number. We fix a field morphism from \mathbb{C} into \mathbb{C}_p (left implicit in the discussion and by means of which one calculates $|x|_p$ for any $x \in \mathbb{C}$). Let $Q, P_1, \dots, P_T \in \mathbb{Z}[X_1, \dots, X_N]$ and $\boldsymbol{\mu} \in (\mathbb{T} \setminus \{1\})^N$. We assume that:

(i) P_1, \dots, P_T satisfy HDF, and that

$$\prod_{t=1}^T P_t(\mathbf{x}) \xrightarrow[|\mathbf{x}| \rightarrow +\infty]{\mathbf{x} \in J^N} +\infty;$$

(ii) for all $n \in \{1, \dots, N\}$, $|1 - \mu_n|_p > p^{-1/p(p-1)}$.

We set

$$\tilde{Z}(Q; P_1, \dots, P_T; \boldsymbol{\mu}; \mathbf{s}) = \sum_{\substack{\mathbf{m} \in \mathbb{N}^{*N} \\ \forall t \in \{1, \dots, T\}, p \nmid P_t(\mathbf{m})}} \boldsymbol{\mu}^{\mathbf{m}} Q(\mathbf{m}) \prod_{t=1}^T P_t(\mathbf{m})^{-s_t}.$$

Let $\mathbf{r} \in \{0, \dots, p-2\}^T$. Then there exists $\tilde{Z}_p^{\mathbf{r}}(Q, P_1; \dots, P_T; \boldsymbol{\mu}; \cdot): \mathbb{Z}_p^T \rightarrow \mathbb{C}_p$ continuous such that for all $\mathbf{k} \in \mathbb{N}^T$ satisfying $k_t \equiv r_t \pmod{p-1}$ for all $t \in \{1, \dots, T\}$, we have

$$\tilde{Z}_p^{\mathbf{r}}(Q; P_1, \dots, P_T; \boldsymbol{\mu}; -\mathbf{k}) = \tilde{Z}(Q; P_1, \dots, P_T; \boldsymbol{\mu}; -\mathbf{k}).$$

2. Analytic properties of certain functions Y defined by means of integrals

The proof of Theorem A, given in § 3, uses an integral representation of each twisted series $Z(Q; P_1, \dots, P_T; \boldsymbol{\mu}; \mathbf{s})$ as a finite sum of integrals $Y(\mathbf{s}) = Y(Q; P_1, \dots, P_T; f_1, \dots, f_N; \mathbf{s})$, defined in § 2.1. An important ingredient in the proof is therefore a precise description of the analytic continuation of each such integral Y in \mathbf{s} . The main result of this section is proved in § 2.3. This shows that if each P_t is in the class HDF, then each function Y can be extended from some open set in which each σ_t is sufficiently large to \mathbb{C}^T as an entire function.

2.1 Precise definition of the functions Y

DEFINITION 2.1. Let $Q, P_1, \dots, P_T \in \mathbb{C}[X_1, \dots, X_N]$ and $N_1 \in \{0, \dots, N\}$. We assume that for all $t \in \{1, \dots, T\}$ and all $\mathbf{x} \in [-1, 1]^{N_1} \times J^{N-N_1}$, $P_t(\mathbf{x}) \notin \mathbb{R}_-$. Furthermore, we take $f: [-1, 1]^{N_1} \rightarrow \mathbb{C}$ continuous and $f_{N_1+1}, \dots, f_N: [1, +\infty[\rightarrow \mathbb{C}$ continuous and bounded. For $\mathbf{s} \in \mathbb{C}^T$ we define

$$\begin{aligned} Y(Q; P_1, \dots, P_T; f_{N_1+1}, \dots, f_N; f; \mathbf{s}) \\ = \int_{[-1, 1]^{N_1} \times J^{N-N_1}} Q(\mathbf{x}) \left(\prod_{t=1}^T P_t(\mathbf{x})^{-s_t} \right) f(x_1, \dots, x_{N_1}) \left(\prod_{n=N_1+1}^N f_n(x_n) \right) d\mathbf{x}. \end{aligned}$$

LEMMA 2.2. Let $Q, P_1, \dots, P_T \in \mathbb{C}[X_1, \dots, X_N]$ and $N_1 \in \{0, \dots, N\}$. We assume the following.

- (a) For all $t \in \{1, \dots, T\}$ we have:
- for all $\mathbf{x} \in [-1, 1]^{N_1} \times J^{N-N_1}$, $P_t(\mathbf{x}) \notin \mathbb{R}_-$;
 - $|P_t(\mathbf{x})| \gg 1$ ($\mathbf{x} \in [-1, 1]^{N_1} \times J^{N-N_1}$).

(b) $\prod_{t=1}^T |P_t(\mathbf{x})| \xrightarrow[|\mathbf{x}| \rightarrow +\infty]{\mathbf{x} \in [-1, 1]^{N_1} \times J^{N-N_1}} +\infty.$

Furthermore, we take $f: [-1, 1]^{N_1} \rightarrow \mathbb{C}$ continuous and $f_{N_1+1}, \dots, f_N: [1, +\infty[\rightarrow \mathbb{C}$ continuous and bounded. Then there exists $\sigma_0 > 0$ such that $\mathbf{s} \mapsto Y(Q; P_1, \dots, P_T; f_{N_1+1}, \dots, f_N; f; \mathbf{s})$ exists and is holomorphic on $\{\mathbf{s} \in \mathbb{C}^T \mid \forall t \in \{1, \dots, T\}, \sigma_t > \sigma_0\}$.

Proof. (1) *Choice of an ϵ .* Thanks to the Tarski Saidenberg theorem there exists $\epsilon > 0$ such that

$$\prod_{t=1}^T |P_t(\mathbf{x})| \gg \left(\prod_{n=N_1+1}^N x_n \right)^\epsilon \quad (\mathbf{x} \in [-1, 1]^{N_1} \times J^{N-N_1}).$$

(2) *Proof of the existence of σ_0 .* Let $\sigma_0 \in \mathbb{R}$, that will be fixed in the following. Let K be a compact of \mathbb{C}^T included in $\{\mathbf{s} \in \mathbb{C}^T \mid \forall t \in \{1, \dots, T\}, \sigma_t > \sigma_0\}$.

- Let $t \in \{1, \dots, T\}$. Then $|P_t(\mathbf{x})| \gg 1$ ($\mathbf{x} \in [-1, 1]^{N_1} \times J^{N-N_1}$) so there exists $c > 0$ such that for all $\mathbf{x} \in [-1, 1]^{N_1} \times J^{N-N_1}$ $|P_t(\mathbf{x})| \geq c$. For all, $\mathbf{x} \in [-1, 1]^{N_1} \times J^{N-N_1}$, $c^{-1}|P_t(\mathbf{x})| \geq 1$ so $\sigma_t > \sigma_0 \Rightarrow (c^{-1}|P_t(\mathbf{x})|)^{\sigma_t} \geq (c^{-1}|P_t(\mathbf{x})|)^{\sigma_0}$.

Thus, we have $|P_t(\mathbf{x})|^{\sigma_t} \gg |P_t(\mathbf{x})|^{\sigma_0}$ ($\mathbf{x} \in [-1, 1]^{N_1} \times J^{N-N_1}$, $\mathbf{s} \in K$). Then, $|P_t(\mathbf{x})^{s_t}| = |P_t(\mathbf{x})|^{\sigma_t} \exp[-\tau_t \arg P_t(\mathbf{x})]$ so

$$|P_t(\mathbf{x})^{s_t}| \gg |P_t(\mathbf{x})|^{\sigma_t} \quad (\mathbf{x} \in [-1, 1]^{N_1} \times J^{N-N_1}, \mathbf{s} \in K).$$

Thanks to what precedes, we deduce that $|P_t(\mathbf{x})^{-s_t}| \ll |P_t(\mathbf{x})|^{-\sigma_0}$.

- So we have

$$\prod_{t=1}^T P_t(\mathbf{x})^{-s_t} \ll \left(\prod_{t=1}^T |P_t(\mathbf{x})| \right)^{-\sigma_0} \quad (\mathbf{x} \in [-1, 1]^{N_1} \times J^{N-N_1}, \mathbf{s} \in K).$$

From now on we assume that $\sigma_0 > 0$. Then

$$\prod_{t=1}^T P_t(\mathbf{x})^{-s_t} \ll \left(\prod_{n=N_1+1}^N x_n \right)^{-\sigma_0 \epsilon} \quad (\mathbf{x} \in [-1, 1]^{N_1} \times J^{N-N_1}, \mathbf{s} \in K).$$

We denote $q = \max\{\deg_{X_n} Q \mid N_1 + 1 \leq n \leq N\}$ (we can obviously assume that $Q \neq 0$). We obtain

$$\begin{aligned} Q(\mathbf{x}) \left(\prod_{t=1}^T P_t(\mathbf{x})^{-s_t} \right) f(x_1, \dots, x_{N_1}) \prod_{n=N_1+1}^N f_n(x_n) \\ \ll \left(\prod_{n=N_1+1}^N x_n \right)^{q-\sigma_0 \epsilon} \quad (\mathbf{x} \in [-1, 1]^{N_1} \times J^{N-N_1}, \mathbf{s} \in K). \end{aligned}$$

We are led to make the following choice:

$$\sigma_0 = \frac{q+2}{\epsilon} > 0.$$

The theorem that guarantees the holomorphy of a function defined as an integral allows us to conclude. \square

2.2 The \mathcal{B} class

DEFINITION 2.3. For $r \in \mathbb{R}$ we define

$$\begin{aligned} \mathcal{B}(r) = \{f: [r, +\infty[\rightarrow \mathbb{C} \mid \exists (f_n)_{n \in \mathbb{N}}, f_n: [r, +\infty[\rightarrow \mathbb{C}, \\ C^\infty \text{ bounded satisfying } f_0 = f \text{ and } \forall n \in \mathbb{N}, f'_{n+1} = f_n\}. \end{aligned}$$

LEMMA 2.4. Let $r \in \mathbb{R}$ and $f \in \mathcal{B}(r)$. Then we have the following.

- (1) There is one and only one sequence $(f_n)_{n \in \mathbb{N}}$ where $f_n: [r, +\infty[\rightarrow \mathbb{C}$ such that:
 - f_n is C^∞ bounded;
 - $f_0 = f$;
 - for all $n \in \mathbb{N}$, $f'_{n+1} = f_n$.
- (2) For all $n \in \mathbb{N}$, $f_n \in \mathcal{B}(r)$.

Proof. (1) Let $(f_n)_{n \in \mathbb{N}}$ and $(g_n)_{n \in \mathbb{N}}$ both satisfy the hypotheses of the lemma. Let us prove by induction on n that for all n , $f_n = g_n$. It is clear for $n = 0$. If we have $f_n = g_n$, then $f'_{n+2} = f'_{n+1} = f_n = g_n = g'_{n+1} = g'_{n+2}$. As $f'_{n+2} = g'_{n+2}$, so $f_{n+2} - g_{n+2}$ is of the form $x \mapsto ax + b$. However, we are

on $[r, +\infty[$ and $f_{n+2} - g_{n+2}$ is bounded, so it is constant, so its derivative is null, so $f_{n+1} - g_{n+1} = 0$, whence $f_{n+1} = g_{n+1}$.

(2) This is clear. □

The following lemma will not be used in the sequel, but it answers a natural question on the $\mathcal{B}(r)$ class. The proof (given in [Dec03]) is left as an exercise.

LEMMA 2.5. *Let $r \in \mathbb{R}$ and $f: [r, +\infty[\rightarrow \mathbb{C}$. Then*

$$f \in \mathcal{B}(r) \iff \forall n \in \mathbb{N}, \exists g: [r, +\infty[\rightarrow \mathbb{C},$$

C^∞ bounded, such that $g^{(n)} = f$.

Let us give two examples of families of functions belonging to $\mathcal{B}(r)$: the first is the ‘typical’ example, the second will be used in the proof of Theorem A.

Example 2.6. Let $r \in \mathbb{R}$.

(1) Let $f: [r, +\infty[\rightarrow \mathbb{C}$, that is C^∞ and periodic with a null mean value. Then $f \in \mathcal{B}(r)$.

(2) Let $\alpha, \beta \in \mathbb{R}$ and $a \in \mathbb{C}$. We assume that $\beta \neq 0$, $\alpha/\beta \notin \mathbb{Z}$ and $|a| \neq 1$.

Then $f: [r, +\infty[\rightarrow \mathbb{C}$ defined by

$$f(x) = \frac{\exp(i\alpha x)}{1 - a \exp(i\beta x)}$$

belongs to $\mathcal{B}(r)$.

Proof. (1) The Fourier expansion of f gives the result.

(2a) If $|a| < 1$,

$$f(x) = \exp(i\alpha x) \sum_{k=0}^{+\infty} a^k \exp(ik\beta x) = \sum_{k=0}^{+\infty} a^k \exp(i(\alpha + k\beta)x).$$

So, for $n \in \mathbb{N}$, we define f_n by

$$f_n(x) = \sum_{k=0}^{+\infty} \frac{a^k}{(i(\alpha + k\beta))^n} \exp(i(\alpha + k\beta)x)$$

f_n is C^∞ and bounded, $f'_{n+1} = f_n$, $f_0 = f$; so $f \in \mathcal{B}(r)$.

(2b) If $|a| > 1$,

$$f(x) = \frac{a^{-1} \exp(-i\beta x) \exp(i\alpha x)}{a^{-1} \exp(-i\beta x) - 1} = -a^{-1} \frac{\exp(i(\alpha - \beta)x)}{1 - a^{-1} \exp(i(-\beta)x)}.$$

This reduces this case to the preceding case, so $f \in \mathcal{B}(r)$. □

2.3 Under suitable hypothesis the twisted integrals Y holomorphically extend to \mathbb{C}^T

The aim of this subsection is to prove the following theorem.

THEOREM 2.7. *Let $Q, P_1, \dots, P_T \in \mathbb{C}[X_1, \dots, X_N]$ and $N_1 \in \{0, \dots, N\}$. We assume the following.*

(a) *For all $t \in \{1, \dots, T\}$ we have:*

- *for all $\mathbf{x} \in [-1, 1]^{N_1} \times J^{N-N_1}$, $P_t(\mathbf{x}) \notin \mathbb{R}_-$;*
- *$|P_t(\mathbf{x})| \gg 1$ ($\mathbf{x} \in [-1, 1]^{N_1} \times J^{N-N_1}$).*

(b) $\prod_{t=1}^T |P_t(\mathbf{x})| \xrightarrow[\mathbf{x} \in [-1, 1]^{N_1} \times J^{N-N_1}]{|\mathbf{x}| \rightarrow +\infty} +\infty.$

(c) There exists $\epsilon_0 > 0$ such that for $\alpha \in \{0\}^{N_1} \times \mathbb{N}^{N-N_1}$ and $n \in \{N_1 + 1, \dots, N\}$ we have

$$\alpha_n \geq 1 \Rightarrow \forall t \in \{1, \dots, T\}, \frac{\partial^\alpha P_t}{P_t}(\mathbf{x}) \ll x_n^{-\epsilon_0} \quad (\mathbf{x} \in [-1, 1]^{N_1} \times J^{N-N_1}).$$

In addition, we consider $f: [-1, 1]^{N_1} \rightarrow \mathbb{C}$ continuous and $f_{N_1+1}, \dots, f_N \in \mathcal{B}(1)$. Then the following property is true for all $0 \leq N_1 \leq N$:

$\mathcal{P}(N_1, N) \stackrel{\text{def}}{=} Y(Q; P_1, \dots, P_T; f_{N_1+1}, \dots, f_N; f; \cdot)$ has an analytic extension to \mathbb{C}^T as an entire function.

Remark 2.8. When $N_1 = 0$, the hypothesis (c) is nothing but *HDF*.

Proof. (1) *Proof of the assertion $\mathcal{P}(0, N)$.* Let us agree on the following.

- We will say that a function Y is an entire combination of the functions Y_1, \dots, Y_k if there exists entire functions $\lambda, \lambda_1, \dots, \lambda_k: \mathbb{C}^T \rightarrow \mathbb{C}$ such that $Y = \lambda + \sum_{i=1}^k \lambda_i Y_i$.
- The polynomials P_1, \dots, P_T are fixed for the whole proof, so we will write $Y(Q; f_1, \dots, f_N; \cdot)$ for $Y(Q; P_1, \dots, P_T; f_1, \dots, f_N; \cdot)$.
- Here \mathcal{B} means $\mathcal{B}(1)$.

The proof is by induction on N .

Since $\mathcal{P}(0, 0)$ is obvious, it suffices to show that the implication $\mathcal{P}(0, N-1) \Rightarrow \mathcal{P}(0, N)$ is true. The proof of this assertion will then easily be seen to apply for any other value for N_1 (the details are left to the reader). The argument is made up of ten steps.

Step 1. Let $Q \in \mathbb{C}[X_1, \dots, X_N]$ and $f_1, \dots, f_N \in \mathcal{B}$. Then $Y(Q, f_1, \dots, f_N, \mathbf{s})$ is an entire combination of $Y(\partial Q / \partial x_1, f_1, \dots, f_N, \mathbf{s})$ and of functions of the type $Y(Q(\partial P_t / \partial x_1); g_1, \dots, g_N; \mathbf{s} + \mathbf{e}_t)$ where $t \in \{1, \dots, T\}$ and $g_1, \dots, g_N \in \mathcal{B}$.

Proof of Step 1. We have $f_1 \in \mathcal{B}$ so, thanks to Lemma 2.4, a sequence of functions belonging to \mathcal{B} is associated to f_1 . We denote the first term of this sequence by f_1^1 . Then

$$\begin{aligned} Y(Q; f_1, \dots; f_N; \mathbf{s}) &= \int_{J^N} Q(\mathbf{x}) \prod_{t=1}^T P_t(\mathbf{x})^{-st} \prod_{n=1}^N f_n(x_n) dx \\ &= \int_{J^{N-1}} \left\{ \int_1^{+\infty} Q(\mathbf{x}) \left(\prod_{t=1}^T P_t(\mathbf{x})^{-st} \right) f_1(x_1) dx_1 \right\} \prod_{n=2}^N f_n(x_n) \prod_{n=2}^N dx_n. \end{aligned}$$

By means of an integration by parts with respect to x_1 , the expression between braces is the difference between

$$\left[Q(\mathbf{x}) \left(\prod_{t=1}^T P_t(\mathbf{x})^{-st} \right) f_1^1(x_1) \right]_{x_1=1}^{x_1=+\infty}$$

and

$$\int_1^{+\infty} \left(\frac{\partial Q}{\partial x_1}(\mathbf{x}) \prod_{t=1}^T P_t(\mathbf{x})^{-st} + Q(\mathbf{x}) \sum_{t=1}^T (-st) \frac{\partial P_t}{\partial x_1}(\mathbf{x}) P_t(\mathbf{x})^{-(st+1)} \prod_{r \neq t} P_r(\mathbf{x})^{-sr} \right) f_1(x_1) dx_1.$$

We deduce from this that

$$\begin{aligned} Y(Q; f_1, \dots, f_N; \mathbf{s}) &= - \int_{J^{N-1}} Q(1, x_2, \dots, x_N) \left(\prod_{t=1}^T P_t(1, x_2, \dots, x_N)^{-st} \right) f_1^1(1) \prod_{n=2}^N f_n(x_n) \prod_{n=2}^N dx_n \\ &\quad - Y\left(\frac{\partial Q}{\partial x_1}; f_1, \dots, f_N; \mathbf{s}\right) + \sum_{t=1}^T s_t Y\left(Q \frac{\partial P_t}{\partial x_1}; f_1^1, f_2, \dots, f_N; \mathbf{s} + \mathbf{e}_t\right). \end{aligned}$$

The polynomials of $N - 1$ variables $P_1(1, X_2, \dots, X_N), \dots, P_T(1, X_2, \dots, X_N)$ satisfy the hypothesis in $\mathcal{P}(0, N - 1)$. Thus, the induction hypothesis implies that the term defined by an integral over J^{N-1} admits a holomorphic continuation to \mathbb{C}^T . This fact consequently implies the assertion of Step 1.

Step 2. Let $Q \in \mathbb{C}[X_1, \dots, X_N]$ and $f_1, \dots, f_N \in \mathcal{B}$. Then the following property is true for all $d \geq 1$:

$P_2(d, N) \stackrel{\text{def}}{=} Y(Q, f_1, \dots, f_N, \mathbf{s})$ is an entire combination of $Y(\partial^d Q / \partial x_1^d; f_1, \dots, f_N; \mathbf{s})$ and of functions of the type

$$Y((\partial^i Q / \partial x_1^i)(\partial P_t / \partial x_1); g_1, \dots, g_N; \mathbf{s} + \mathbf{e}_t)$$

where $i \in \mathbb{N}, t \in \{1, \dots, T\}$, and $g_1, \dots, g_N \in \mathcal{B}$.

Proof of Step 2. The proof is by induction on d . The assertion $P_2(1, N)$ is implied by Step 1. The implication $P_2(d, N) \Rightarrow P_2(d + 1, N)$ is proved by combining Step 1, applied to the polynomial $\partial^d Q / \partial x_1^d$, with the result that is assumed to be true in the property $P_2(d, N)$.

Step 3. Let $Q \in \mathbb{C}[X_1, \dots, X_N]$ and $f_1, \dots, f_N \in \mathcal{B}$. Then for all $n \in \{1, \dots, N\}$, $Y(Q; f_1, \dots, f_N; \mathbf{s})$ is an entire combination of functions of the type $Y((\partial^i Q / \partial x_n^i)(\partial P_t / \partial x_n); g_1, \dots, g_N; \mathbf{s} + \mathbf{e}_t)$ where $i \in \mathbb{N}, t \in \{1, \dots, T\}$ and $g_1, \dots, g_N \in \mathcal{B}$.

Proof of Step 3. Of course, it is enough to deal with the case $n = 1$. In order to deduce the result for $n = 1$, it is sufficient to apply Step 2 with $d = \deg_{X_1} Q + 1$.

Step 4. For $n \in \{1, \dots, N\}, \mathbf{u} = (u_1, \dots, u_T) \in \mathbb{N}^T$ and $Q \in \mathbb{C}[X_1, \dots, X_N]$, we define $\mathcal{E}_{\mathbf{u}}^n(Q)$ to be the subspace of $\mathbb{C}[X_1, \dots, X_N]$ generated by the polynomials of the form

$$\partial^\beta Q \prod_{k=1}^n \frac{\partial^{|\alpha_k|+1} P_{t_k}}{\partial \mathbf{x}^{\alpha_k} \partial x_k}$$

where $\beta \in \mathbb{N}^N, \alpha_1, \dots, \alpha_n \in \mathbb{N}^N$ and $t_1, \dots, t_n \in \{1, \dots, T\}$ verify that for all $t \in \{1, \dots, T\}$

$$u_t = \text{card}\{k \in \{1, \dots, n\} \mid t_k = t\}.$$

It is clear that $n \neq |\mathbf{u}| \Rightarrow \mathcal{E}_{\mathbf{u}}^n(Q) = \{0\}$.

The following two observations are satisfied:

- $\mathcal{E}_{\mathbf{u}}^n(Q)$ is stable under derivation;
- for any $n \in \{1, \dots, N - 1\}, t \in \{1, \dots, T\}$ and $Q \in \mathbb{C}[X_1, \dots, X_N]$, we have

$$\frac{\partial P_t}{\partial x_{n+1}} \mathcal{E}_{\mathbf{u}}^n(Q) \subset \mathcal{E}_{\mathbf{u} + \mathbf{e}_t}^{n+1}(Q).$$

Step 5. Let $n \in \{1, \dots, N\}, Q \in \mathbb{C}[X_1, \dots, X_N]$ and $f_1, \dots, f_N \in \mathcal{B}$. Define the property $P_5(n, N)$: $Y(Q; f_1, \dots, f_N; \mathbf{s})$ is an entire combination of functions of the type $Y(R; g_1, \dots, g_N; \mathbf{s} + \mathbf{u})$, where $\mathbf{u} \in \mathbb{N}^T, R \in \mathcal{E}_{\mathbf{u}}^n(Q)$ and $g_1, \dots, g_N \in \mathcal{B}$.

Then $P_5(n, N)$ is true for all $n \in \{1, \dots, N\}$.

Proof of Step 5. The proof is by induction on n . Step 3 shows that the property $P_5(1, N)$ is true. Let us assume that $P_5(n, N)$ is true for any $n \in \{1, \dots, N - 1\}$. Thus, $Y(Q; f_1, \dots, f_N; \mathbf{s})$ is an entire combination of functions of the type: $Y(R; g_1, \dots, g_N; \mathbf{s} + \mathbf{u})$ where $\mathbf{u} \in \mathbb{N}^T, R \in \mathcal{E}_{\mathbf{u}}^n(Q)$ and $g_1, \dots, g_N \in \mathcal{B}$. By Step 3, $Y(R, g_1, \dots, g_N, \mathbf{s} + \mathbf{u})$ is an entire combination of functions of the type

$$Y\left(\frac{\partial^i R}{\partial x_{n+1}^i} \frac{\partial P_t}{\partial x_{n+1}}; h_1, \dots, h_N; \mathbf{s} + \mathbf{u} + \mathbf{e}_t\right)$$

where $i \in \mathbb{N}, t \in \{1, \dots, T\}$ and $h_1, \dots, h_N \in \mathcal{B}$.

Thanks to the two observations made in Step 4,

$$\frac{\partial^i R}{\partial x_{n+1}^i} \frac{\partial P_t}{\partial x_{n+1}} \in \mathcal{E}_{\mathbf{u}+\mathbf{e}_t}^{n+1}(Q).$$

This now shows that $P_5(n+1, N)$ is also true.

Step 6. For $\mathbf{u} \in \mathbb{N}^T$ and $Q \in \mathbb{C}[X_1, \dots, X_N]$, we define $\mathcal{E}_{\mathbf{u}}(Q)$ to denote the subspace $\mathbb{C}[X_1, \dots, X_N]$ generated by all polynomials of the form $\partial^\beta Q \prod_{t=1}^T \prod_{k \in F_t} \partial^{f_t(k)} P_t$, where:

- $\beta \in \mathbb{N}^N$;
- the F_t are finite and pairwise disjoint subsets of \mathbb{N} , satisfying $|F_t| = u_t$;
- for all $t \in \{1, \dots, T\}$, f_t is a function from F_t to \mathbb{N}^N ;
- we can associate to the f_t finite and pairwise disjoint subsets of \mathbb{N} , D_1, \dots, D_N such that
 - $|D_1| = \dots = |D_N|$,
 - $\bigsqcup_{n=1}^N D_n = \bigsqcup_{t=1}^T F_t$,
 - $t \in \{1, \dots, T\}$, $n \in \{1, \dots, N\}$ and $k \in D_n \cap F_t \Rightarrow f_t(k) \in \mathbb{N}^{n-1} \times \mathbb{N}^* \times \mathbb{N}^{N-n}$.

We note that $\mathcal{E}_{\mathbf{u}}(Q)$ is stable under derivation.

Example 2.9 (The case $T = 4$ and $N = 3$). Let us take $\mathbf{u} = (1, 3, 2, 3)$ and $F_1 = \{1\}$, $F_2 = \{2, 3, 4\}$, $F_3 = \{5, 6\}$ and $F_4 = \{7, 8, 9\}$. Let us take f_1, f_2, f_3 and f_4 defined as follows:

- $f_1: F_1 \rightarrow \mathbb{N}^3$ is defined by $f_1(1) = (1, 0, 0)$;
- $f_2: F_2 \rightarrow \mathbb{N}^3$ is defined by $f_2(2) = (1, 2, 3)$, $f_2(3) = (0, 2, 1)$, $f_2(4) = (2, 0, 1)$;
- $f_3: F_3 \rightarrow \mathbb{N}^3$ is defined by $f_3(5) = (2, 1, 0)$, $f_3(6) = (0, 3, 0)$;
- $f_4: F_4 \rightarrow \mathbb{N}^3$ is defined by: $f_4(7) = (0, 1, 2)$, $f_4(8) = (4, 1, 2)$, $f_4(9) = (1, 0, 2)$.

Then it is easy to check that $D_1 = \{1, 2, 5\}$, $D_2 = \{3, 6, 7\}$, $D_3 = \{4, 8, 9\}$ satisfy the conditions we require.

Step 7. Let $\mathbf{u}, \mathbf{v} \in \mathbb{N}^T$ and $R, S \in \mathbb{C}[X_1, \dots, X_N]$. Then $R \in \mathcal{E}_{\mathbf{u}}(Q)$ and $S \in \mathcal{E}_{\mathbf{v}}(R)$ implies $S \in \mathcal{E}_{\mathbf{u}+\mathbf{v}}(Q)$.

Proof of Step 7. Let S be an entire combination of terms of the form $\partial^\beta R \prod_{t=1}^T \prod_{k \in F'_t} \partial^{f'_t(k)} P_t$, where:

- $\beta \in \mathbb{N}^N$, $|F'_t| = v_t$, $f'_t: F'_t \rightarrow \mathbb{N}^N$ and D'_1, \dots, D'_N are as in Step 6;
- $R \in \mathcal{E}_{\mathbf{u}}(Q)$, so $\partial^\beta R \in \mathcal{E}_{\mathbf{u}}(Q)$, so $\partial^\beta R$ is a linear combination of terms of the form

$$\partial^\gamma Q \prod_{t=1}^T \prod_{k \in F_t} \partial^{f_t(k)} P_t,$$

where $\gamma \in \mathbb{N}^N$, $|F_t| = u_t$, $f_t: F_t \rightarrow \mathbb{N}^N$ and D_1, \dots, D_N are as in Step 6.

We can assume that for all t_1, t_2 , $F_{t_1} \cap F'_{t_2} = \emptyset$. This implies that for all n, t $D_n \cap F'_t = F_t \cap D'_n = \emptyset$ and for all n, n' , $D_n \cap D'_{n'} = \emptyset$. So, as to conclude, it is enough for us to prove that

$$U \stackrel{\text{def}}{=} \partial^\gamma Q \left(\prod_{t=1}^T \prod_{k \in F_t} \partial^{f_t(k)} P_t \right) \left(\prod_{t=1}^T \prod_{k \in F'_t} \partial^{f'_t(k)} P_t \right)$$

is in $\mathcal{E}_{\mathbf{u}+\mathbf{v}}(Q)$.

For $t \in \{1, \dots, T\}$, we define $g_t: F_t \sqcup F'_t \rightarrow \mathbb{N}^N$ by

$$g_t(k) = \begin{cases} f_t(k) & \text{if } k \in F_t, \\ f'_t(k) & \text{if } k \in F'_t. \end{cases}$$

Then

$$U = \partial^\gamma Q \prod_{t=1}^T \prod_{k \in F_t \sqcup F'_t} \partial^{g_t(k)} P_t.$$

Thanks to this expression we now show that $U \in \mathcal{E}_{\mathbf{u}+\mathbf{v}}(Q)$. The following points justify this assertion:

- the $F_t \sqcup F'_t$ are pairwise disjoint and for all $t \in \{1, \dots, T\}$, $|F_t \sqcup F'_t| = u_t + v_t$;
- the $D_n \sqcup D'_n$ are pairwise disjoint, $\bigsqcup_{t=1}^T (F_t \sqcup F'_t) = \bigsqcup_{n=1}^N (D_n \sqcup D'_n)$, and $|D_1 \sqcup D'_1| = \dots = |D_N \sqcup D'_N|$;
- if $k \in (D_n \sqcup D'_n) \cap (F_t \sqcup F'_t) = (D_n \cap F_t) \sqcup (D'_n \cap F'_t)$, then either,
 - $k \in D_n \cap F_t$ and then $g_t(k) = f_t(k) \in \mathbb{N}^{n-1} \times \mathbb{N}^* \times \mathbb{N}^{N-n}$, or
 - $k \in D'_n \cap F'_t$ and then $g_t(k) = f'_t(k) \in \mathbb{N}^{n-1} \times \mathbb{N}^* \times \mathbb{N}^{N-n}$.

So, we actually obtain the conclusion that $U \in \mathcal{E}_{\mathbf{u}+\mathbf{v}}(Q)$.

Step 8. We have $Q \in \mathbb{C}[X_1, \dots, X_N]$ and $\mathbf{u} \in \mathbb{N}^T \Rightarrow \mathcal{E}_{\mathbf{u}}^N(Q) \subset \mathcal{E}_{\mathbf{u}}(Q)$.

Proof of Step 8. We set

$$S = \partial^\beta Q \prod_{k=1}^N \frac{\partial^{|\alpha_k|+1} P_{t_k}}{\partial \mathbf{x}^{\alpha_k} \partial x_k},$$

where $\beta \in \mathbb{N}^N$, $\alpha_1, \dots, \alpha_N \in \mathbb{N}^N$ and $t_1, \dots, t_N \in \{1, \dots, T\}$ satisfy: $u_t = \text{card}\{k \in \{1, \dots, N\} \mid t_k = t\}$ for all $t \in \{1, \dots, T\}$. So, as to conclude, it is enough to show that $S \in \mathcal{E}_{\mathbf{u}}(Q)$.

For $t \in \{1, \dots, T\}$, we set $F_t = \{k \in \{1, \dots, N\} \mid t_k = t\}$, so that $|F_t| = u_t$. We see that the F_t are pairwise disjoint, and that $\bigsqcup_{t=1}^T F_t = \{1, \dots, N\}$. We define $f_t: F_t \rightarrow \mathbb{N}^N$ by $f_t(k) = \alpha_k + \mathbf{e}_k$. We set $D_n = \{n\}$. Then we see that:

- $|D_1| = \dots = |D_N|$;
- $\bigsqcup_{n=1}^N D_n = \{1, \dots, N\}$;
- if $k \in D_n \cap F_t$, then $k = n$, which implies $f_t(k) = \alpha_n + \mathbf{e}_n \in \mathbb{N}^{n-1} \times \mathbb{N}^* \times \mathbb{N}^{N-n}$.

Thus,

$$S = \partial^\beta Q \prod_{t=1}^T \prod_{k \in F_t} \frac{\partial^{|\alpha_k|+1} P_{t_k}}{\partial \mathbf{x}^{\alpha_k} \partial x_k} = \partial^\beta Q \prod_{t=1}^T \prod_{k \in F_t} \partial^{f_t(k)} P_t.$$

This implies $S \in \mathcal{E}_{\mathbf{u}}(Q)$.

Step 9. Let $Q \in \mathbb{C}[X_1, \dots, X_N]$ and $f_1, \dots, f_N \in \mathcal{B}$. Then the following property is true for all $m \geq 1$:

$P_9(m, N) \stackrel{\text{def}}{=} Y(Q; f_1, \dots, f_N; \mathbf{s})$ is an entire combination of functions of the type $Y(R; g_1, \dots, g_N; \mathbf{s} + \mathbf{u})$, where $\mathbf{u} \in \mathbb{N}^T$, $|\mathbf{u}| = mN$, $R \in \mathcal{E}_{\mathbf{u}}(Q)$ and $g_1, \dots, g_N \in \mathcal{B}$.

Proof of Step 9. The proof is by induction on $m \geq 1$.

- $P_9(1, N)$ is true. Thanks to Step 5, $Y(Q; f_1, \dots, f_N; \mathbf{s})$ is an entire combination of functions of the type $Y(R, g_1, \dots, g_N, \mathbf{s} + \mathbf{u})$, where $\mathbf{u} \in \mathbb{N}^T$, $R \in \mathcal{E}_{\mathbf{u}}^N(Q)$ and $g_1, \dots, g_N \in \mathcal{B}$. We can assume $|\mathbf{u}| = N$ since $|\mathbf{u}| \neq N$ would imply $\mathcal{E}_{\mathbf{u}}^N(Q) = \{0\}$. Thus, Step 8 gives the result.

- $P_9(m, N) \Rightarrow P_9(m+1, N)$ is true. Let us assume that $Y(Q; f_1, \dots, f_N; \mathbf{s})$ is an entire combination of functions of the type $Y(R; g_1, \dots, g_N; \mathbf{s} + \mathbf{u})$ where $\mathbf{u} \in \mathbb{N}^T$, $|\mathbf{u}| = mN$, $R \in \mathcal{E}_{\mathbf{u}}(Q)$ and $g_1, \dots, g_N \in \mathcal{B}$. The application of the argument with $m = 1$ shows that $Y(R; g_1, \dots, g_N; \mathbf{s} + \mathbf{u})$ is an entire combination of functions of the type $Y(S; h_1, \dots, h_N; \mathbf{s} + \mathbf{u} + \mathbf{v})$ where $\mathbf{v} \in \mathbb{N}^T$, $|\mathbf{v}| = N$, $S \in \mathcal{E}_{\mathbf{v}}(R)$ and $h_1, \dots, h_N \in \mathcal{B}$. Step 7 then implies that $S \in \mathcal{E}_{\mathbf{u}+\mathbf{v}}(Q)$. Since $|\mathbf{u} + \mathbf{v}| = (m+1)N$, this shows that $P_9(m+1, N)$ is true.

Step 10 (Conclusion of the proof). We fix $Q \in \mathbb{C}[X_1, \dots, X_N] \setminus \{0\}$ and $f_1, \dots, f_N \in \mathcal{B}$ until the end. Let $m \geq 1$. Thanks to Step 9, $Y(Q; f_1, \dots, f_N; \mathbf{s})$ is an entire combination of functions of the type $Y(R; g_1, \dots, g_N; \mathbf{s} + \mathbf{u})$ where $\mathbf{u} \in \mathbb{N}^T$, $|\mathbf{u}| = mN$, $R \in \mathcal{E}_{\mathbf{u}}(Q)$ and $g_1, \dots, g_N \in \mathcal{B}$.

Since $R \in \mathcal{E}_{\mathbf{u}}(Q)$, it follows that R is equal to a linear combination of polynomials of the type $\partial^{\beta} Q \prod_{t=1}^T \prod_{k \in F_t} \partial^{f_t(k)} P_t$ with $\beta \in \mathbb{N}^N$, $|F_t| = u_t$, $f_t: F_t \rightarrow \mathbb{N}^N$ and D_1, \dots, D_N as in Step 6.

We have $|D_n| = m$ for all $n \in \{1, \dots, N\}$. It then follows that

$$\begin{aligned} \prod_{t=1}^T \prod_{k \in F_t} \partial^{f_t(k)} P_t(\mathbf{x}) &= \prod_{t=1}^T \prod_{n=1}^N \prod_{k \in F_t \cap D_n} \partial^{f_t(k)} P_t(\mathbf{x}) \\ &\ll \prod_{t=1}^T \prod_{n=1}^N \prod_{k \in F_t \cap D_n} x_n^{-\epsilon_0} P_t(\mathbf{x}) \quad (\mathbf{x} \in J^N) \\ &\ll \prod_{t=1}^T \prod_{n=1}^N (x_n^{-\epsilon_0} P_t(\mathbf{x}))^{|F_t \cap D_n|} \quad (\mathbf{x} \in J^N) \\ &\ll \prod_{n=1}^N \prod_{t=1}^T x_n^{-\epsilon_0 |F_t \cap D_n|} \prod_{t=1}^T \prod_{n=1}^N P_t(\mathbf{x})^{|F_t \cap D_n|} \quad (\mathbf{x} \in J^N) \\ &\ll \prod_{n=1}^N x_n^{-\epsilon_0 |D_n|} \prod_{t=1}^T P_t(\mathbf{x})^{|F_t|} \quad (\mathbf{x} \in J^N) \\ &\ll \prod_{n=1}^N x_n^{-\epsilon_0 m} \prod_{t=1}^T P_t(\mathbf{x})^{u_t} \quad (\mathbf{x} \in J^N). \end{aligned}$$

We set $q = \max\{\deg_{X_n} Q \mid 1 \leq n \leq N\}$. We also set $p = \max\{\deg_{X_n} P_t \mid 1 \leq n \leq N, 1 \leq t \leq T\}$. We introduce a parameter $a > 0$ whose value will be determined in the following.

Let K be a compact subset of \mathbb{C}^T included in $\{\mathbf{s} \in \mathbb{C}^T \mid \forall t \in \{1, \dots, T\}, \sigma_t > -a\}$.

- Let $t \in \{1, \dots, T\}$. As in the proof of the existence of σ_0 in the proof of the Lemma 2.2,

$$|P_t(\mathbf{x})^{-st}| \ll |P_t(\mathbf{x})|^a \quad (\mathbf{x} \in J^N, \mathbf{s} \in K).$$

Since $a > 0$, it follows that

$$P_t(\mathbf{x})^a \ll \left(\prod_{n=1}^N x_n \right)^{pa} \quad (\mathbf{x} \in J^N).$$

- From the previous inequalities we deduce that

$$P_t(\mathbf{x})^{-st} \ll \left(\prod_{n=1}^N x_n \right)^{pa} \quad (\mathbf{x} \in J^N, \mathbf{s} \in K).$$

We set

$$S = \partial^{\beta} Q \prod_{t=1}^T \prod_{k \in F_t} \partial^{f_t(k)} P_t.$$

By combining the preceding estimates, we obtain

$$\begin{aligned} S(\mathbf{x}) \prod_{t=1}^T P_t(\mathbf{x})^{-(s_t+u_t)} &\ll \partial^\beta Q(\mathbf{x}) \prod_{n=1}^N x_n^{-\epsilon_0 m} \prod_{t=1}^T P_t(\mathbf{x})^{u_t} \prod_{t=1}^T P_t(\mathbf{x})^{-(s_t+u_t)} \quad (\mathbf{x} \in J^N, \mathbf{s} \in K) \\ &\ll \left(\prod_{n=1}^N x_n \right)^q \left(\prod_{n=1}^N x_n \right)^{-\epsilon_0 m} \left(\prod_{n=1}^N x_n \right)^{Tpa} \quad (\mathbf{x} \in J^N, \mathbf{s} \in K) \\ &\ll \left(\prod_{n=1}^N x_n \right)^{q+Tpa-\epsilon_0 m} \quad (\mathbf{x} \in J^N, \mathbf{s} \in K). \end{aligned}$$

From now on we choose m so that $m > (q+2)/\epsilon_0$. We then choose a so that $a = (\epsilon_0 m - (q+2))/Tp$ (clearly $a > 0$). The above estimates then show that $Y(S; g_1, \dots, g_N; \mathbf{s} + \mathbf{u})$ is holomorphic on

$$\{\mathbf{s} \in \mathbb{C}^T \mid \forall t \in \{1, \dots, T\}, \sigma_t > -a\}.$$

Since R is a linear combination of S as above, it then follows that $Y(Q, f_1, \dots, f_N, \cdot)$ can be extended analytically to

$$\left\{ \mathbf{s} \in \mathbb{C}^T \mid \forall t \in \{1, \dots, T\}, \sigma_t > \frac{q+2-\epsilon_0 m}{Tp} \right\}.$$

Since this is true for all $m > (q+2)/\epsilon_0$, one concludes that $Y(Q; f_1, \dots, f_N; \cdot)$ can be extended to \mathbb{C}^T as an analytic function.

This completes the proof that $\mathcal{P}(0, N-1) \Rightarrow \mathcal{P}(0, N)$ is true. Thus, $\mathcal{P}(0, N)$ is true for all N . The details needed to verify the similar argument when $N_1 \geq 1$ are left to the reader. \square

2.4 Proof that $Y(1, P_{e\mathbf{x}}, \mathbf{x} \mapsto e^{i\mathbf{x}}, \mathbf{y} \mapsto e^{-i\mathbf{y}}, \cdot)$ has a pole

As the following example shows, the H_0S hypothesis is not enough to guarantee the holomorphy of the continuation of the twisted Y .

Example 2.10. We define $f_1: J \rightarrow \mathbb{C}$ by $f_1(x) = e^{ix}$ and $f_2: J \rightarrow \mathbb{C}$ by $f_2(y) = e^{-iy}$; f_1 and f_2 belong to $\mathcal{B}(1)$. Then $Y(1; P; f_1, f_2; \cdot)$ has a meromorphic extension to \mathbb{C} with a single pole at $s = 1$ which is simple. The residue at $s = 1$ is equal to π/e .

Proof. By definition,

$$Y(1; P; f_1, f_2; s) = \int_{J^2} P(x, y)^{-s} e^{i(x-y)} dx dy.$$

We set

$$Y_1(s) = \int_{\{(x,y) \mid 1 < x < y\}} P(x, y)^{-s} e^{i(x-y)} dx dy.$$

Let $g_1:]1, +\infty[\times \mathbb{R}_+^* \rightarrow \{(x, y) \mid 1 < x < y\}$ be defined by $g_1(u, v) = (u, u+v)$; g_1 is a diffeomorphism with Jacobian equal to 1. Thanks to g_1 , we see that

$$Y_1(s) = \int_{]1, +\infty[\times \mathbb{R}_+^*} [(u - (u+v))^2 u + u]^{-s} e^{i(u-(u+v))} du dv.$$

So

$$Y_1(s) = \int_1^{+\infty} u^{-s} du \int_0^{+\infty} (v^2 + 1)^{-s} e^{-iv} dv = \frac{1}{s-1} \int_0^{+\infty} (v^2 + 1)^{-s} e^{-iv} dv.$$

We now set

$$Y_2(s) = \int_{\{(x,y) \mid 1 < y < x\}} P(x, y)^{-s} e^{i(x-y)} dx dy.$$

Let $g_2:]1, +\infty[\times \mathbb{R}_+^* \rightarrow \{(x, y) \mid 1 < y < x\}$ be defined by $g_2(u, v) = (u + v, u)$; g_2 is also a diffeomorphism with Jacobian equal to -1 . Thanks to g_2 , we see that

$$Y_2(s) = \int_{]1, +\infty[\times \mathbb{R}_+^*} [(u + v - u)^2(u + v) + (u + v)]^{-s} e^{i(u+v-u)} du dv.$$

Thus,

$$\begin{aligned} Y_2(s) &= \int_{]1, +\infty[\times \mathbb{R}_+^*} (v^2 + 1)^{-s} (u + v)^{-s} e^{iv} du dv = \int_0^{+\infty} (v^2 + 1)^{-s} e^{iv} \left\{ \int_1^{+\infty} (u + v)^{-s} du \right\} dv \\ &= \int_0^{+\infty} (v^2 + 1)^{-s} e^{iv} \frac{(1 + v)^{-s+1}}{s - 1} dv = \frac{1}{s - 1} \int_0^{+\infty} (v^2 + 1)^{-s} (v + 1)^{-s+1} e^{iv} dv. \end{aligned}$$

Let us now set

$$Y(s) = \int_0^{+\infty} (v^2 + 1)^{-s} e^{-iv} dv + \int_0^{+\infty} (v^2 + 1)^{-s} (v + 1)^{-s+1} e^{iv} dv.$$

Thanks to Theorem 2.7, Y has a holomorphic continuation to \mathbb{C} . Since $Y(1; P, f_1, f_2; s) = (s - 1)^{-1} Y(s)$, we now evaluate $Y(1)$ as follows:

$$Y(1) = \int_0^{+\infty} (v^2 + 1)^{-1} e^{-iv} dv + \int_0^{+\infty} (v^2 + 1)^{-1} e^{iv} dv = \int_{-\infty}^{+\infty} (v^2 + 1)^{-1} e^{iv} dv.$$

Showing that this integral is equal to π/e is a classical application of the residue theorem, and so we are through. \square

3. Analytic properties of the series Z

The main result of this section is Theorem A, proved in § 3.3. The proof is based on a simple integral representation for the sum of values of any holomorphic function at integral points, proved in § 3.2, and on the main result of § 2.

3.1 Holomorphy of Z on this set of convergence

In this subsection we establish some easy properties of the set of convergence of the series defining Z . The proofs are easy and can be found in [Dec03].

DEFINITION 3.1. Let $Q, P_1, \dots, P_T \in \mathbb{R}[X_1, \dots, X_N]$ satisfy $P_t(\mathbf{x}) > 0$ for all $t \in \{1, \dots, T\}$ and all $\mathbf{x} \in J^N$. We set

$$\mathcal{C}(Q, P_1, \dots, P_T) = \{(\sigma_1, \dots, \sigma_T) \in \mathbb{R}^T \mid Z(Q, P_1, \dots, P_T, \mathbf{1}, \sigma_1, \dots, \sigma_T) \text{ converges}\}.$$

The set of convergence of Z does not depend on $\boldsymbol{\mu}$.

Remark 3.2. If, moreover, $\boldsymbol{\mu}$ belongs to \mathbb{T}^N , then we have

$$Z(Q, P_1, \dots, P_T, \boldsymbol{\mu}, s_1, \dots, s_T) \text{ converges} \iff (\sigma_1, \dots, \sigma_T) \in \mathcal{C}(Q, P_1, \dots, P_T).$$

PROPOSITION 3.3. Let $Q, P_1, \dots, P_T \in \mathbb{R}[X_1, \dots, X_N]$ such that $P_t(\mathbf{x}) \gg 1$ ($\mathbf{x} \in J^N$) for all $t \in \{1, \dots, T\}$. Let $1 \leq T_0 \leq T$. We assume that

$$\prod_{t=1}^{T_0} P_t(\mathbf{x}) \xrightarrow[\substack{|\mathbf{x}| \rightarrow +\infty \\ \mathbf{x} \in J^N}]{} +\infty.$$

Let $\sigma_{T_0+1}, \dots, \sigma_T \in \mathbb{R}$. Then there exists $\sigma_0 \in \mathbb{R}$ such that: $\sigma_1, \dots, \sigma_{T_0} \geq \sigma_0 \Rightarrow (\sigma_0, \dots, \sigma_{T_0}, \sigma_{T_0+1}, \dots, \sigma_T) \in \text{int}(\mathcal{C}(Q, P_1, \dots, P_T))$.

PROPOSITION 3.4. Let $Q, P_1, \dots, P_T \in \mathbb{R}[X_1, \dots, X_N]$ satisfy $P_t(\mathbf{x}) \gg 1$ ($\mathbf{x} \in J^N$) for all $t \in \{1, \dots, T\}$. Let $\boldsymbol{\mu} \in \mathbb{T}^N$. Then $Z(Q, P_1, \dots, P_T, \boldsymbol{\mu}, \cdot)$ is holomorphic on $\text{int}(\mathcal{C}(Q, P_1, \dots, P_T)) + i\mathbb{R}^T$.

Remark 3.5. Since $\text{int}(\mathcal{C}(Q, P_1, \dots, P_T)) + i\mathbb{R}^T$ is convex and, therefore, connex, we can speak without ambiguity of the meromorphic continuation of $Z(Q, P_1, \dots, P_T, \boldsymbol{\mu}, \cdot)$ (if it exists).

3.2 An integral representation for a sum

NOTATION/DEFINITION 3.6. For $\epsilon > 0$, we define

$$\lambda_\epsilon: [\frac{3}{2}, +\infty[\rightarrow \mathbb{C} \text{ by } \lambda_\epsilon(x) = x + i\epsilon \text{ and } \overline{\lambda}_\epsilon: [\frac{3}{2}, +\infty[\rightarrow \mathbb{C} \text{ by } \overline{\lambda}_\epsilon = x - i\epsilon.$$

Let k denote an integer belonging to $[2, +\infty[$ and set

$$\lambda_{\epsilon,k} = \lambda_{\epsilon, [3/2, k+1/2]} \text{ and } \overline{\lambda}_{\epsilon,k} = \overline{\lambda}_{\epsilon, [3/2, k+1/2]}.$$

We define $\gamma_{\epsilon,k}: [-1, 1] \rightarrow \mathbb{C}$ by $\gamma_{\epsilon,k}(x) = k + \frac{1}{2} + i\epsilon x$ (even for $k = 1$).

The following is a straightforward application of residue calculus and induction.

LEMMA 3.7. Let U be an open set of \mathbb{C} containing $[\frac{3}{2}, k + \frac{1}{2}] + i[-\epsilon, \epsilon]$.

- Let $f: U \rightarrow \mathbb{C}$ be holomorphic. Then

$$\sum_{m=2}^k f(m) = - \int_{\gamma_{\epsilon,1}} \frac{f(z)}{e(z)-1} dz + \int_{\overline{\lambda}_{\epsilon,k}} \frac{f(z)}{e(z)-1} dz + \int_{\gamma_{\epsilon,k}} \frac{f(z)}{e(z)-1} dz - \int_{\lambda_{\epsilon,k}} \frac{f(z)}{e(z)-1} dz.$$

- Let $f: U^N \rightarrow \mathbb{C}$ be holomorphic. For $\tau \in \mathcal{S}_N$, we define $f_\tau: U^N \rightarrow \mathbb{C}$ by $f_\tau(z_1, \dots, z_N) = f(z_{\tau(1)}, \dots, z_{\tau(N)})$. Then $\sum_{\mathbf{m} \in \{2, \dots, k\}^N} f(\mathbf{m})$ is a sum of a finite numbers of terms, each of the form

$$\pm \int_{(\gamma_{\epsilon,1})^{N_1} \times (\lambda_{\epsilon,k})^{N_2} \times (\overline{\lambda}_{\epsilon,k})^{N_3} \times (\gamma_{\epsilon,k})^{N_4}} f_\tau(z_1, \dots, z_N) \prod_{n=1}^N \frac{1}{e(z_n)-1} dz$$

where $N_1, N_2, N_3, N_4 \in \mathbb{N}$ satisfy $N_1 + N_2 + N_3 + N_4 = N$, and $\tau \in \mathcal{S}_N$.

3.3 Proof of Theorem A

Before applying Lemma 3.7 to the proof of the theorem, two preliminaries are needed.

The next result follows from [Ess95]. The complete proof is given in [Dec03].

LEMMA 3.8. Let $P \in \mathbb{R}[X_1, \dots, X_N]$ satisfying:

- (i) for all $\mathbf{x} \in J^N$, $P(\mathbf{x}) > 0$;
- (ii) there exists $\epsilon_0 > 0$ such that for all $\boldsymbol{\alpha} \in \mathbb{N}^N, \alpha_n \geq 1 \Rightarrow \partial^\alpha P(\mathbf{x}) \ll x_n^{-\epsilon_0} P(\mathbf{x})$ ($\mathbf{x} \in J^N$).

Then there exists $\epsilon > 0$ such that:

- (i') $\mathbf{x} \in J^N$ and $\mathbf{y} \in [-2\epsilon, 2\epsilon]^N \Rightarrow \Re(P(\mathbf{x} + i\mathbf{y})) \geq \frac{1}{2}P(\mathbf{x})$;
- (ii') for all $\boldsymbol{\alpha} \in \mathbb{N}^N, \alpha_n \geq 1 \Rightarrow \partial^\alpha P(\mathbf{x} + i\mathbf{y}) \ll x_n^{-\epsilon_0} P(\mathbf{x})$ ($\mathbf{x} \in J^N, \mathbf{y} \in [-2\epsilon, 2\epsilon]^N$).

The second preliminary result is evident.

LEMMA 3.9. We can partition \mathbb{N}^{*N} in the following way:

$$\mathbb{N}^{*N} = \bigsqcup_{c=1}^C A_c, \text{ where for all } c, A_c \text{ is of the form } \prod_{n=1}^N B_n \text{ with } B_n = \{1\} \text{ or } B_n = [2, +\infty[\cap \mathbb{N}.$$

Proof of Theorem A. The proof is divided into two steps.

Step 1. We have that

$$\mathbf{s} \mapsto Z^*(\mathbf{s}) \stackrel{\text{def}}{=} \sum_{\mathbf{m} \geq \mathbf{2}} \boldsymbol{\mu}^{\mathbf{m}} Q(\mathbf{m}) \prod_{t=1}^T P_t(\mathbf{m})^{-s_t}$$

can be holomorphically extended to \mathbb{C}^T .

Proof of Step 1. Since each P_t belongs to *HDF* it follows that we can choose $\epsilon_0 > 0$ such that:

- $\prod_{t=1}^T P_t(\mathbf{x}) \gg \prod_{n=1}^N x_n^{\epsilon_0}$ ($\mathbf{x} \in J^N$);
- $\boldsymbol{\alpha} \in \mathbb{N}^N$, $\alpha_n \geq 1 \Rightarrow (\partial^{\boldsymbol{\alpha}} P_t / P_t)(\mathbf{x}) \ll x_n^{-\epsilon_0}$ ($\mathbf{x} \in J^N$).

There exists $\sigma_0 > 0$ such that if $\sigma_1, \dots, \sigma_T > \sigma_0$, then $Z^*(\mathbf{s})$ converges. Starting with any \mathbf{s} whose real part belongs to this set, one then proceeds as follows.

Applying Lemma 3.8, we obtain for each $t \in \{1, \dots, T\}$ an $\epsilon_t > 0$, and then set $\epsilon = \min_t \{\epsilon_t\}$. For $\mathbf{s} \in \mathbb{C}^N$, we define $f_{\mathbf{s}}: (]1, +\infty[+i] - 2\epsilon, 2\epsilon[)^N \rightarrow \mathbb{C}$ by $f_{\mathbf{s}}(\mathbf{z}) = Q(\mathbf{z}) \prod_{t=1}^T P_t(\mathbf{z})^{-s_t} \prod_{n=1}^N e^{i\theta_n z_n}$, where we have chosen $\theta_n \in \mathbb{R} \setminus 2\pi\mathbb{Z}$ so that $\mu_n = e^{i\theta_n}$ for each n . Thus, for each integer $k \geq 2$,

$$\sum_{\mathbf{m} \in \{2, \dots, k\}^N} Q(\mathbf{m}) \prod_{n=1}^N e^{i\theta_n m_n} \prod_{t=1}^T P_t(\mathbf{m})^{-s_t} = \sum_{\mathbf{m} \in \{2, \dots, k\}^N} f_{\mathbf{s}}(\mathbf{m}).$$

By applying Lemma 3.7 to $f_{\mathbf{s}}$ we conclude that $\sum_{\mathbf{m} \in \{2, \dots, k\}^N} f_{\mathbf{s}}(\mathbf{m})$ can be written as a sum/difference of finitely many integrals, each of which is indexed by a permutation τ on $\{1, \dots, N\}$ and a choice of $N_1, N_2, N_3, N_4 \in \mathbb{N}$ whose sum equals N . We will assume that τ is the identity since the argument is the same for any other permutation. Each integral is therefore an expression of the form

$$\int_{(\gamma_{\epsilon,1})^{N_1} \times (\lambda_{\epsilon,k})^{N_2} \times (\overline{\lambda_{\epsilon,k}})^{N_3} \times (\gamma_{\epsilon,k})^{N_4}} Q(\mathbf{z}) \prod_{t=1}^T P_t(\mathbf{z})^{-s_t} \prod_{n=1}^N \frac{\exp(i\theta_n z_n)}{e(z_n) - 1} dz.$$

We now conclude by dominated convergence that there exists $r > 0$ such that any integral with $N_4 \geq 1$ tends to zero (as $k \rightarrow \infty$) on $\{\mathbf{s} \in \mathbb{C}^T : \sigma_1, \dots, \sigma_T > r\}$. Thus, in this open set, Z^* is a linear combination of integrals of the form $Y^{N_1, N_2, N_3}(\mathbf{s})$ where

$$Y^{N_1, N_2, N_3}(\mathbf{s}) \stackrel{\text{def}}{=} \int_{(\gamma_{\epsilon,1})^{N_1} \times (\lambda_{\epsilon})^{N_2} \times (\overline{\lambda_{\epsilon,k}})^{N_3}} Q(\mathbf{z}) \prod_{t=1}^T P_t(\mathbf{z})^{-s_t} \prod_{n=1}^N \frac{\exp(i\theta_n z_n)}{e(z_n) - 1} dz.$$

To finish the proof of Theorem A, it suffices to show that any $Y^{N_1, N_2, N_3}(\mathbf{s})$ satisfies the hypotheses of Theorem 2.7.

- For $1 \leq n \leq N_1$, define $f_n: [-1, 1] \rightarrow \mathbb{C}$ by

$$f_n(x) = \frac{\exp(i\theta_n \gamma_{\epsilon,1}(x))}{e(\gamma_{\epsilon,1}(x)) - 1} = \frac{\exp(i\theta_n(3/2 + i\epsilon x))}{\exp(2i\pi(3/2 + i\epsilon x)) - 1} = -\exp\left(\frac{3}{2}i\theta_n\right) \frac{\exp(-\epsilon\theta_n x)}{\exp(-2\pi\epsilon x) + 1}.$$

The function $f: [-1, 1]^{N_1} \rightarrow \mathbb{C}$ defined by $f(x_1, \dots, x_{N_1}) = \prod_{n=1}^{N_1} f_n(x_n)$ is evidently continuous.

- For $N_1 + 1 \leq n \leq N_1 + N_2$, define $f_n: [\frac{3}{2}, +\infty[\rightarrow \mathbb{C}$ by

$$f_n(x) = \frac{\exp(i\theta_n \lambda_{\epsilon}(x))}{e(\lambda_{\epsilon}(x)) - 1} = \frac{\exp(i\theta_n(x + i\epsilon))}{\exp(2i\pi(x + i\epsilon)) - 1} = -\exp(-\epsilon\theta_n) \frac{\exp(i\theta_n x)}{1 - \exp(-2\pi\epsilon) \exp(i2\pi x)}.$$

- For $N_1 + N_2 + 1 \leq n \leq N$, define $f_n: [\frac{3}{2}, +\infty[\rightarrow \mathbb{C}$ by

$$f_n(x) = \frac{\exp(i\theta_n \overline{\lambda_{\epsilon}}(x))}{e(\overline{\lambda_{\epsilon}}(x)) - 1} = \frac{\exp(i\theta_n(x - i\epsilon))}{\exp(2i\pi(x - i\epsilon)) - 1} = -\exp(\epsilon\theta_n) \frac{\exp(i\theta_n x)}{1 - \exp(2\pi\epsilon) \exp(i2\pi x)}.$$

Since $\theta_n/2\pi \notin \mathbb{Z}$, it follows that $f_n \in \mathcal{B}(\frac{3}{2})$ for any $n \geq N_1 + 1$.

For any $P \in \mathbb{C}[X_1, \dots, X_N]$ and N_1, N_2, N_3 of sum N , we define $P^{N_1, N_2, N_3} \in \mathbb{C}[X_1, \dots, X_N]$ by

$$\begin{aligned} &P^{N_1, N_2, N_3}(\mathbf{x}) \\ &= P(\gamma_{\epsilon, 1}(x_1), \dots, \gamma_{\epsilon, 1}(x_{N_1}), \lambda_{\epsilon}(x_{N_1+1}), \dots, \lambda_{\epsilon}(x_{N_1+N_2}), \overline{\lambda_{\epsilon}}(x_{N_1+N_2+1}), \dots, \overline{\lambda_{\epsilon}}(x_N)) \\ &= P\left(\frac{3}{2} + i\epsilon x_1, \dots, \frac{3}{2} + i\epsilon x_{N_1}, x_{N_1+1} + i\epsilon, \dots, x_{N_1+N_2} + i\epsilon, x_{N_1+N_2+1} - i\epsilon, \dots, x_N - i\epsilon\right). \end{aligned}$$

Applying this to each P_t and using the defining property of ϵ from Lemma 3.8, it follows that

$$P_t^{N_1, N_2, N_3}(\mathbf{x}) = P_t\left(\frac{3}{2}, \dots, \frac{3}{2}, x_{N_1+1}, \dots, x_N\right) + i(\epsilon x_1, \dots, \epsilon x_{N_1}, \epsilon, \dots, \epsilon, -\epsilon, \dots, -\epsilon)$$

satisfies

$$\Re(P_t^{N_1, N_2, N_3}(x_1, \dots, x_N)) \geq \frac{1}{2} P_t\left(\frac{3}{2}, \dots, \frac{3}{2}, x_{N_1+1}, \dots, x_N\right) \quad \forall \mathbf{x} \in [-1, 1]^{N_1} \times J^{N-N_1}.$$

Thus, for all $\mathbf{x} \in [-1, 1]^{N_1} \times [\frac{3}{2}, +\infty]^{N-N_1}$, we have:

- $\Re(P_t^{N_1, N_2, N_3}(\mathbf{x})) > 0$;
- $|P_t^{N_1, N_2, N_3}(\mathbf{x})| \geq \frac{1}{2} P_t\left(\frac{3}{2}, \dots, \frac{3}{2}, x_{N_1+1}, \dots, x_N\right)$; and
- $\prod_{t=1}^T |P_t^{N_1, N_2, N_3}(\mathbf{x})| \gg \prod_{n=N_1+1}^N x_n^{\epsilon_0} (\mathbf{x} \in [-1, 1]^{N_1} \times [\frac{3}{2}, +\infty]^{N-N_1})$.

Finally, if $\alpha \in \{0\}^{N_1} \times \mathbb{N}^{N-N_1}$ and $N_1 + 1 \leq n \leq N$ is such that $\alpha_n \geq 1$, then it also follows from Lemma 3.8 that

$$\begin{aligned} \partial^\alpha P_t^{N_1, N_2, N_3}(\mathbf{x}) &\ll x_n^{-\epsilon_0} P_t\left(\frac{3}{2}, \dots, \frac{3}{2}, x_{N_1+1}, \dots, x_N\right) \quad (\mathbf{x} \in [-1, 1]^{N_1} \times [\frac{3}{2}, +\infty]^{N-N_1}) \\ &\ll x_n^{-\epsilon_0} |P_t^{N_1, N_2, N_3}(\mathbf{x})| \quad (\mathbf{x} \in [-1, 1]^{N_1} \times [\frac{3}{2}, +\infty]^{N-N_1}). \end{aligned}$$

Since

$$Y^{N_1, N_2, N_3}(\mathbf{s}) = (i\epsilon)^{N_1} \int_{[-1, 1]^{N_1} \times [3/2, +\infty]^{N-N_1}} Q^{N_1, N_2, N_3}(\mathbf{x}) \prod_{t=1}^T P_t^{N_1, N_2, N_3}(\mathbf{x})^{-s_t} \prod_{n=1}^N f_n(x_n) d\mathbf{x}$$

the hypotheses of Theorem 2.7 guarantee the existence of an holomorphic continuation for each $Y^{N_1, N_2, N_3}(\mathbf{s})$ to \mathbb{C}^T . This completes the proof of Step 1.

Step 2 (Conclusion). A simple induction argument (on N) completes the proof of Theorem A.

- For $N = 1$, we only need to write

$$Z(Q; P_1, \dots, P_T; \mu; \mathbf{s}) = \mu Q(1) \prod_{t=1}^T P_t(1)^{-s_t} + \sum_{m \geq 2} \mu^m Q(m) \prod_{t=1}^T P_t(m)^{-s_t}$$

and then we apply Step 1.

- If the result is true for each any number of variables between 1 and $N - 1$, then, thanks to Lemma 3.9 and Step 1, we see that it is true for N variables. □

3.4 Proof that $Z(1, P_{\mathbf{e}\mathbf{x}}, -1, -1, \cdot)$ has a pole

Proof. For this proof we set $Z(s) = Z(1; P; -1, -1; s)$. Thus,

$$\begin{aligned} Z(s) &= \sum_{m, n \geq 1} (-1)^m (-1)^n [(m-n)^2 m + m]^{-s} \\ &= \sum_{m, n \geq 1} (-1)^{m-n} m^{-s} [(m-n)^2 + 1]^{-s} \\ &= \sum_{1 \leq m \leq n} (-1)^{m-n} m^{-s} [(m-n)^2 + 1]^{-s} + \sum_{1 \leq n < m} (-1)^{m-n} m^{-s} [(m-n)^2 + 1]^{-s}. \end{aligned}$$

By setting $n = m + u$ in the first sum and $m = n + u$ in the second sum, we obtain

$$\begin{aligned} Z(s) &= \sum_{\substack{m \geq 1 \\ u \geq 0}} (-1)^u m^{-s} (u^2 + 1)^{-s} + \sum_{n, u \geq 1} (-1)^u (n + u)^{-s} (u^2 + 1)^{-s} \\ &= \zeta(s) \sum_{u \geq 0} (-1)^u (u^2 + 1)^{-s} + \sum_{u \geq 1} (-1)^u (u^2 + 1)^{-s} \sum_{n \geq 1} (n + u)^{-s} \\ &= \zeta(s) \sum_{u \geq 0} (-1)^u (u^2 + 1)^{-s} + \sum_{u \geq 1} (-1)^u (u^2 + 1)^{-s} \left[\zeta(s) - \sum_{1 \leq k \leq u} k^{-s} \right] \\ &= \zeta(s) \sum_{u \in \mathbb{Z}} (-1)^u (u^2 + 1)^{-s} - \sum_{1 \leq k \leq u} (-1)^u (u^2 + 1)^{-s} k^{-s} \\ &= \zeta(s) \sum_{u \in \mathbb{Z}} (-1)^u (u^2 + 1)^{-s} - \sum_{\substack{k \geq 1 \\ \ell \geq 0}} (-1)^{k+\ell} [(k + \ell)^2 + 1]^{-s} k^{-s}. \end{aligned}$$

The following facts suffice to show that Z has a simple pole at $s = 1$:

- a classical application of the residue theorem is $\sum_{u \in \mathbb{Z}} (-1)^u (u^2 + 1)^{-1} = \pi / \sinh(\pi)$;
- Theorem A implies that

$$s \mapsto \sum_{u \in \mathbb{Z}} (-1)^u (u^2 + 1)^{-s} \quad \text{and} \quad s \mapsto \sum_{\substack{k \geq 1 \\ \ell \geq 0}} (-1)^{k+\ell} [(k + \ell)^2 + 1]^{-s} k^{-s}$$

can be holomorphically extended to \mathbb{C} . □

4. Values at T -tuples of negative integers

4.1 Proof of the Exchange Lemma

The proof is a simple consequence of the following (in which the notation $Z(Q; P_1, \dots, P_T; \boldsymbol{\mu}; \cdot)$ is understood to denote the analytically continued series to \mathbb{C}^T).

PROPOSITION 4.1. *Let $Q, P_1, \dots, P_T \in \mathbb{R}[X_1, \dots, X_N]$ and $T_0 \in \{1, \dots, T - 1\}$ for a given $T \geq 2$. We assume that P_1, \dots, P_T satisfy HDF and that*

$$\prod_{t=1}^{T_0} P_t(\mathbf{x}) \xrightarrow[\substack{|\mathbf{x}| \rightarrow +\infty \\ \mathbf{x} \in J^N}]{} +\infty.$$

Let $\boldsymbol{\mu} \in (\mathbb{T} \setminus \{1\})^N$ and $k_1, \dots, k_T \in \mathbb{N}$. Then

$$Z(Q; P_1, \dots, P_T; \boldsymbol{\mu}; -k_1, \dots, -k_T) = Z\left(Q \prod_{t=T_0+1}^T P_t^{k_t}; P_1, \dots, P_{T_0}; \boldsymbol{\mu}; -k_1, \dots, -k_{T_0}\right).$$

Proof. We define the holomorphic function $f: \mathbb{C}^{T_0} \rightarrow \mathbb{C}$ by

$$f(s_1, \dots, s_{T_0}) = Z(Q; P_1, \dots, P_T; \boldsymbol{\mu}; s_1, \dots, s_{T_0}; -k_{T_0+1}, \dots, -k_T).$$

Thanks to Proposition 3.3, there exists $\sigma_0 \in \mathbb{R}$ (depending on (k_{T_0+1}, \dots, k_T)) such that for any (s_1, \dots, s_{T_0}) with $\sigma_1, \dots, \sigma_{T_0} \geq \sigma_0$ we have

$$f(s_1, \dots, s_{T_0}) = \sum_{\mathbf{m} \in \mathbb{N}^{*N}} \boldsymbol{\mu}^{\mathbf{m}} Q(\mathbf{m}) \prod_{t=T_0+1}^T P_t(\mathbf{m})^{k_t} \prod_{t=1}^{T_0} P_t(\mathbf{m})^{-s_t}.$$

Next, define the function $g: \mathbb{C}^{T_0} \rightarrow \mathbb{C}$ by

$$g(s_1, \dots, s_{T_0}) = Z\left(Q \prod_{t=T_0+1}^T P_t^{k_t}; P_1, \dots, P_{T_0}; \boldsymbol{\mu}; s_1, \dots, s_{T_0}\right).$$

That is, g is the analytic continuation of the twisted series in (s_1, \dots, s_{T_0}) , with the role of Q now played by $Q \prod_{t=T_0+1}^T P_t^{k_t}$. Theorem A also applies to this series. Thus, g is an entire function on \mathbb{C}^{T_0} . Proposition 3.3 therefore applies to g . As a result, there exists $\sigma'_0 \in \mathbb{R}$ such that for any (s_1, \dots, s_{T_0}) with $\sigma_1, \dots, \sigma_{T_0} \geq \sigma'_0$ we have

$$g(s_1, \dots, s_{T_0}) = \sum_{\mathbf{m} \in \mathbb{N}^{*N}} \boldsymbol{\mu}^{\mathbf{m}} Q(\mathbf{m}) \prod_{t=T_0+1}^T P_t^{k_t} \prod_{t=1}^{T_0} P_t(\mathbf{m})^{-s_t}.$$

Thus, $f(s_1, \dots, s_{T_0}) = g(s_1, \dots, s_{T_0})$ in the open set consisting of all (s_1, \dots, s_{T_0}) such that each $\sigma_t > \max(\sigma_0, \sigma'_0)$. The uniqueness of the analytic continuation then ensures that $f = g$ on \mathbb{C}^{T_0} . In particular, $f(-k_1, \dots, -k_{T_0}) = g(-k_1, \dots, -k_{T_0})$, as claimed. \square

Proof of the Exchange Lemma. Proposition 4.1 tells us that both quantities are equal to

$$Z(Q; P_1, \dots, P_T, Q_1, \dots, Q_{T'}; \boldsymbol{\mu}; -k_1, \dots, -k_T, -\ell_1, \dots, -\ell_{T'}). \quad \square$$

4.2 An application of the Exchange Lemma: the proof of Theorem B

Theorem B illustrates how one can use the Exchange Lemma. Its proof is a simple consequence of the following.

LEMMA 4.2. *Let $Q = \sum_{\alpha \in S} a_\alpha \mathbf{X}^\alpha \in \mathbb{R}[X_1, \dots, X_N]$ and $\boldsymbol{\mu} \in (\mathbb{T} \setminus \{1\})^N$. Then*

$$Z(Q; X_1, \dots, X_N; \boldsymbol{\mu}; 0, \dots, 0) = \sum_{\alpha \in S} a_\alpha \prod_{n=1}^N \zeta_{\mu_n}(-\alpha_n).$$

Proof. Set $T = N$ and $P_t = X_t$ for each $t = 1, \dots, T$. These polynomials evidently belong to *HDF*. Thus, if $\sigma_1, \dots, \sigma_N$ are large enough, we have

$$\begin{aligned} Z(Q; X_1, \dots, X_N; \boldsymbol{\mu}; \mathbf{s}) &= Z\left(\sum_{\alpha \in S} a_\alpha \mathbf{X}^\alpha; X_1, \dots, X_N; \boldsymbol{\mu}; \mathbf{s}\right) = \sum_{\alpha \in S} a_\alpha Z(\mathbf{X}^\alpha; X_1, \dots, X_N; \boldsymbol{\mu}; \mathbf{s}) \\ &= \sum_{\alpha \in S} a_\alpha \sum_{\mathbf{m} \in \mathbb{N}^{*N}} \boldsymbol{\mu}^{\mathbf{m}} \mathbf{m}^\alpha \prod_{n=1}^N m_n^{-s_n} = \sum_{\alpha \in S} a_\alpha \sum_{m_1, \dots, m_N \geq 1} \prod_{n=1}^N \mu_n^{m_n} m_n^{\alpha_n - s_n} \\ &= \sum_{\alpha \in S} a_\alpha \prod_{n=1}^N \sum_{m_n \geq 1} \mu_n^{m_n} m_n^{\alpha_n - s_n} = \sum_{\alpha \in S} a_\alpha \prod_{n=1}^N \zeta_{\mu_n}(s_n - \alpha_n). \end{aligned}$$

The uniqueness of analytic continuation then implies

$$Z(Q; X_1, \dots, X_N; \boldsymbol{\mu}; \mathbf{s}) = \sum_{\alpha \in S} a_\alpha \prod_{n=1}^N \zeta_{\mu_n}(s_n - \alpha_n) \quad \forall \mathbf{s} \in \mathbb{C}^N.$$

Setting $\mathbf{s} = \mathbf{0}$ in this equality completes the proof. \square

Proof of Theorem B. The argument is now very simple and goes as follows:

$$\begin{aligned} Z(Q; P_1, \dots, P_T; \boldsymbol{\mu}; -k_1, \dots, -k_T) &= Z\left(Q \prod_{n=1}^N X_n^0; P_1, \dots, P_T; \boldsymbol{\mu}; -k_1, \dots, -k_T\right) \\ &= Z\left(Q \prod_{t=1}^T P_t^{k_t}; X_1, \dots, X_N; \boldsymbol{\mu}; 0, \dots, 0\right) \\ &= \sum_{\boldsymbol{\alpha} \in S} a_{\boldsymbol{\alpha}} \prod_{n=1}^N \zeta_{\mu_n}(-\alpha_n). \end{aligned}$$

The Exchange Lemma implies the second equality, and Lemma 4.2 implies the third equality. \square

4.3 Values at T -tuples of integers for Y

We gave the values at T -tuples of negative integers for general Y in terms of values at negative integers of the simplest Y . The proof of the following theorem follows exactly the same process as that of Theorem B.

THEOREM 4.3. *Let $Q, P_1, \dots, P_T \in \mathbb{R}[X_1, \dots, X_N]$ and $f_1, \dots, f_N \in \mathcal{B}(1)$. We assume that P_1, \dots, P_T satisfy HDF and that*

$$\prod_{t=1}^T P_t(\mathbf{x}) \xrightarrow[|\mathbf{x}| \rightarrow +\infty]{\mathbf{x} \in J^N} +\infty.$$

Let $k_1, \dots, k_T \in \mathbb{N}$. We denote $Q \prod_{t=1}^T P_t^{k_t} = \sum_{\boldsymbol{\alpha} \in S} a_{\boldsymbol{\alpha}} \mathbf{X}^{\boldsymbol{\alpha}}$. Then

$$Y(Q; P_1, \dots, P_T; f_1, \dots, f_N; -k_1, \dots, -k_T) = \sum_{\boldsymbol{\alpha} \in S} a_{\boldsymbol{\alpha}} \prod_{n=1}^N Y(1; X; f_n; -\alpha_n).$$

Remark 4.4. For example, if f is given by $f(x) = e^{i\theta x}$, where $\theta \in \mathbb{R}^*$, then the values at negative integers of $Y(1; X; f; \cdot)$ are very easy to calculate by induction thanks to an integration by parts.

5. p -adic interpolation

The main result of this section is Theorem C. The proof is based on Theorem B and a precise description of each $\zeta_{\mu}(-k)$, proved in § 5.1.

5.1 A formula for the values of $\zeta_{\mu}(-k)$

The first ingredient is a classical lemma [Zag77].

LEMMA 5.1. *Let $(a_m)_{m \in \mathbb{N}^*}$ be a sequence of complex numbers and define*

$$Z(s) = \sum_{m=1}^{+\infty} \frac{a_m}{m^s}.$$

Let us assume that there exists $s \in \mathbb{C}$ such that the series converges, from which it follows that the series $f(x) = \sum_{m=1}^{+\infty} a_m e^{-mx}$ converges if $x > 0$.

We assume that there is a sequence $(c_k)_{k \in \mathbb{N}}$ of complex numbers such that, for all $K \in \mathbb{N}^*$, we have in a neighborhood of zero

$$f(x) = \sum_{k=0}^{K-1} c_k x^k + O(x^K).$$

Then Z can be holomorphically extended to \mathbb{C} and $Z(-k) = (-1)^k k! c_k$ for all $k \in \mathbb{N}$.

We will also need the Stirling numbers of the second kind, as well as some of their elementary properties. Let us recall the following definition.

DEFINITION 5.2. Let $k, \ell \in \mathbb{N}$. The *Stirling number of the second kind* (associated to (k, ℓ)) is the number of partitions in ℓ parts of a set with k elements. This integer is denoted by $S(k, \ell)$.

Example 5.3. We have $S(0, 0) = 1$; for $k \in \mathbb{N}$, $S(k, k) = 1$; if $0 \leq k < \ell$, then $S(k, \ell) = 0$.

The proofs of the next two results can be found in [Com70].

LEMMA 5.4. For all $k \in \mathbb{N}$ and all $\ell \in \mathbb{N}^*$, $S(k + 1, \ell) = \ell S(k, \ell) + S(k, \ell - 1)$.

LEMMA 5.5. For all $k, \ell \in \mathbb{N}$, we have

$$S(k, \ell) = \frac{1}{\ell!} \sum_{j=0}^{\ell} (-1)^{\ell-j} \binom{\ell}{j} j^k.$$

Finally, we need a general expression for each derivative of the composition of a smooth function with the exponential function.

LEMMA 5.6. Let $g: \mathbb{R}_+^* \rightarrow \mathbb{C}$ be smooth, and define $f = g \circ \exp$. Then, for all $k \in \mathbb{N}$, we have: $f^{(k)}(x) = \sum_{\ell=0}^k S(k, \ell) e^{\ell x} g^{(\ell)}(e^x)$ for all $x \in \mathbb{R}$.

Proof. The proof is by induction on $k \in \mathbb{N}$.

- For $k = 0$, the formula is true because $S(0, 0) = 1$.
- Assuming that the formula holds for a given k , and differentiating one more time, it follows that for all $x \in \mathbb{R}$,

$$\begin{aligned} f^{(k+1)}(x) &= \sum_{\ell=0}^k S(k, \ell) (\ell e^{\ell x} g^{(\ell)}(e^x) + e^{\ell x} e^x g^{(\ell+1)}(e^x)) \\ &= \sum_{\ell=0}^k S(k, \ell) \ell e^{\ell x} g^{(\ell)}(e^x) + \sum_{\ell=1}^{k+1} S(k, \ell - 1) e^{\ell x} g^{(\ell)}(e^x). \end{aligned}$$

Since $S(k, k + 1) = 0$, one concludes that

$$\begin{aligned} f^{(k+1)}(x) &= \sum_{\ell=1}^{k+1} [\ell S(k, \ell) + S(k, \ell - 1)] e^{\ell x} g^{(\ell)}(e^x) \\ &= \sum_{\ell=1}^{k+1} S(k + 1, \ell) e^{\ell x} g^{(\ell)}(e^x) = \sum_{\ell=0}^{k+1} S(k + 1, \ell) e^{\ell x} g^{(\ell)}(e^x). \end{aligned}$$

This proves the formula for $k + 1$. □

We can now express each $\zeta_{\mu}(-k)$ in terms of the $S(k, \ell)$ as follows.

LEMMA 5.7. Let $\mu \in \mathbb{T} \setminus \{1\}$. Then, for all $k \in \mathbb{N}$, we have

$$\zeta_{\mu}(-k) = \frac{(-1)^k \mu}{1 - \mu} \sum_{\ell=0}^k \frac{\ell! S(k, \ell)}{(\mu - 1)^{\ell}}.$$

Proof. For all

$$x > 0, \quad \sum_{m=1}^{+\infty} \mu^m e^{-mx} = \sum_{m=1}^{+\infty} (\mu e^{-x})^m = \mu e^{-x} \frac{1}{1 - \mu e^{-x}} = \frac{\mu}{e^x - \mu}.$$

We define $f: \mathbb{R} \rightarrow \mathbb{C}$ by $f(x) = \mu/(e^x - \mu)$ and $g: \mathbb{R}_+^* \rightarrow \mathbb{C}$ by $g(y) = \mu/(y - \mu)$. Then g is smooth and $f = g \circ \exp$, so (5.6) gives: $f^{(k)}(x) = \sum_{\ell=0}^k S(k, \ell)e^{\ell x}g^{(\ell)}(e^x)$ for all $x \in \mathbb{R}$. Writing $g(y) = -\mu(1/(\mu - y))$, it is clear that for each ℓ , $g^{(\ell)}(y) = -\mu(\ell!/(\mu - y)^{\ell+1})$. Thus,

$$f^{(k)}(0) = \sum_{\ell=0}^k S(k, \ell) \left(-\mu \frac{\ell!}{(\mu - 1)^{\ell+1}} \right),$$

we then apply (5.1) to finish the proof. □

5.2 Proof of Theorem C

To prove Theorem C, we need to have a formula adapted to p -adic interpolation: we want to obtain a formula similar to that appearing in the proof of Theorem 20 in [Cas82]. In the present work, such a formula is obtained during the proof of Lemma 5.9: this is the formula (7) for $Z_\ell(-\mathbf{k})$.

However, for the p -adic control of $Z_\ell(-\mathbf{k})$ we do not use the formula (7) but the formula (1) (cf. the proof of the Lemma 5.9), which come from Theorem B and that contain the Stirling numbers. This explains why we obtain the bound $p^{-1/p(p-1)}$, which is better than the bound 1 obtained in the work of Cassou-Noguès.

We first rewrite \tilde{Z} as follows:

$$\begin{aligned} \tilde{Z}(Q; P_1, \dots, P_T; \mu; \mathbf{s}) &= \sum_{\substack{\mathbf{m} \in \mathbb{N}^{*N} \\ \forall t \in \{1, \dots, T\}, p \nmid P_t(\mathbf{m})}} \mu^{\mathbf{m}} Q(\mathbf{m}) \prod_{t=1}^T P_t(\mathbf{m})^{-s_t} \\ &= \sum_{\mathbf{u} \in \{1, \dots, p\}^N} \sum_{\substack{\mathbf{m} \in \mathbb{N}^{*N} \\ \forall t \in \{1, \dots, T\}, p \nmid P_t(\mathbf{m}) \\ \forall n, m_n \equiv u_n \pmod{p}}} \mu^{\mathbf{m}} Q(\mathbf{m}) \prod_{t=1}^T P_t(\mathbf{m})^{-s_t} \\ &= \sum_{\mathbf{u} \in \{1, \dots, p\}^N} \sum_{\substack{\mathbf{m} \in \mathbb{N}^N \\ \forall t \in \{1, \dots, T\}, p \nmid P_t(\mathbf{u} + p\mathbf{m})}} \mu^{\mathbf{u} + p\mathbf{m}} Q(\mathbf{u} + p\mathbf{m}) \prod_{t=1}^T P_t(\mathbf{u} + p\mathbf{m})^{-s_t} \\ &= \sum_{\mathbf{u} \in \{1, \dots, p\}^N} \sum_{\substack{\mathbf{m} \in \mathbb{N}^N \\ \forall t \in \{1, \dots, T\}, p \nmid P_t(\mathbf{u})}} \mu^{\mathbf{u} + p\mathbf{m}} Q(\mathbf{u} + p\mathbf{m}) \prod_{t=1}^T P_t(\mathbf{u} + p\mathbf{m})^{-s_t} \\ &= \sum_{\substack{\mathbf{u} \in \{1, \dots, p\}^N \\ \forall t \in \{1, \dots, T\}, p \nmid P_t(\mathbf{u})}} \mu^{\mathbf{u}} \tilde{Z}(Q_{\mathbf{u}}; P_{1,\mathbf{u}}, \dots, P_{T,\mathbf{u}}; \mu^p; \mathbf{s}), \end{aligned}$$

where $Q_{\mathbf{u}} = Q(\mathbf{u} + p\mathbf{X})$ and $P_{t,\mathbf{u}} = P_t(\mathbf{u} + p\mathbf{X})$. Note that each $P_{t,\mathbf{u}}$ satisfies the property that $p \nmid P_{t,\mathbf{u}}(\mathbf{m})$ for all integral vectors \mathbf{m} , and that the twist is now determined by the vector μ^p rather than μ .

Two lemmas are now needed to complete the proof of Theorem C.

LEMMA 5.8. *Let $x \in \mathbb{C}_p$. Then $|x - 1|_p > p^{-1/(p-1)} \Rightarrow |x^p - 1|_p = (|x - 1|_p)^p$.*

Proof. Set $z = x - 1$. We have

$$x^p - 1 = (z + 1)^p - 1 = \sum_{k=1}^p \binom{p}{k} z^k = z \left(\sum_{k=1}^{p-1} \binom{p}{k} z^{k-1} + z^{p-1} \right).$$

Let $k \in \{1, \dots, p-1\}$. We want to show that

$$\left| \binom{p}{k} z^{k-1} \right|_p < |z^{p-1}|_p.$$

Since p is prime,

$$\left| \binom{p}{k} \right|_p \leq p^{-1}.$$

In addition,

$$\left| \binom{p}{k} z^{k-1} \right|_p = \left| \binom{p}{k} \right|_p |z|_p^{k-1},$$

so it is enough to show that $p^{-1}|z|_p^{k-1} < |z|_p^{p-1}$. To show this, we are going to study two cases.

- *Case $|z|_p > 1$:* $p^{-1}|z|_p^{k-1} < |z|_p^{k-1} \leq |z|_p^{p-2} < |z|_p^{p-1}$.
- *Case $0 < |z|_p \leq 1$:* $p^{-1}|z|_p^{k-1} \leq p^{-1}$. Since $|z|_p > p^{-1/(p-1)}$, $|z|_p^{p-1} > p^{-1}$ and so we see that $p^{-1}|z|_p^{k-1} < |z|_p^{p-1}$.

From

$$\left| \binom{p}{k} z^{k-1} \right|_p < |z^{p-1}|_p \quad \text{for all } k \in \{1, \dots, p-1\},$$

we deduce that

$$\left| \sum_{k=1}^{p-1} \binom{p}{k} z^{k-1} + z^{p-1} \right|_p = |z^{p-1}|_p.$$

The conclusion follows. □

LEMMA 5.9. *We make the same hypothesis as that in Theorem C, except that part (ii) is replaced by part (ii'): $|1 - \mu_n|_p > p^{-1/(p-1)}$. However, impose the additional property that $p \nmid P_t(\mathbf{m})$ for all $\mathbf{m} \in \mathbb{N}^N$. Then for each $\mathbf{r} \in \{0, \dots, p-2\}^T$ there exists $Z_p^{(\mathbf{r})}(Q, P_1, \dots, P_T, \boldsymbol{\mu}, \cdot): \mathbb{Z}_p^T \rightarrow \mathbb{C}_p$ continuous such that for all $\mathbf{k} \in \mathbb{N}^T$ satisfying $k_t \equiv r_t \pmod{p-1}$ for all $t \in \{1, \dots, T\}$, we have*

$$Z_p^{(\mathbf{r})}(Q; P_1, \dots, P_T; \boldsymbol{\mu}; -\mathbf{k}) = Z(Q; P_1, \dots, P_T; \boldsymbol{\mu}; -\mathbf{k}).$$

Proof. Let $\mathbf{k} \in \mathbb{N}^T$ and write $Q \prod_{t=1}^T P_t^{k_t} = \sum_{\boldsymbol{\alpha}} a_{\boldsymbol{\alpha}} \mathbf{X}^{\boldsymbol{\alpha}}$. Set $\mathcal{S}_{\mathbf{k}} = \{\boldsymbol{\alpha} : a_{\boldsymbol{\alpha}} \neq 0\}$. Thanks to Theorem B and Lemma 5.7, we know the following:

$$\begin{aligned} Z(Q, P_1, \dots, P_T, \boldsymbol{\mu}, -\mathbf{k}) &= \sum_{\boldsymbol{\alpha} \in \mathcal{S}_{\mathbf{k}}} \left[a_{\boldsymbol{\alpha}} \prod_{n=1}^N \left(\frac{(-1)^{\alpha_n} \mu_n}{1 - \mu_n} \sum_{\ell_n=0}^{\alpha_n} \frac{\ell_n! S(\alpha_n, \ell_n)}{(\mu_n - 1)^{\ell_n}} \right) \right] \\ &= \sum_{\boldsymbol{\alpha} \in \mathcal{S}_{\mathbf{k}}} \left[a_{\boldsymbol{\alpha}} (-1)^{|\boldsymbol{\alpha}|} \frac{\boldsymbol{\mu}^{\mathbf{1}}}{(\mathbf{1} - \boldsymbol{\mu})^{\mathbf{1}}} \prod_{n=1}^N \left(\sum_{\ell_n=0}^{\alpha_n} \frac{\ell_n! S(\alpha_n, \ell_n)}{(\mu_n - 1)^{\ell_n}} \right) \right] \\ &= \frac{\boldsymbol{\mu}^{\mathbf{1}}}{(\mathbf{1} - \boldsymbol{\mu})^{\mathbf{1}}} \sum_{\boldsymbol{\alpha} \in \mathcal{S}_{\mathbf{k}}} \left[(-1)^{|\boldsymbol{\alpha}|} a_{\boldsymbol{\alpha}} \prod_{n=1}^N \sum_{\ell_n \leq \alpha_n} \frac{\ell_n! S(\alpha_n, \ell_n)}{(\mu_n - 1)^{\ell_n}} \right] \\ &= \frac{\boldsymbol{\mu}^{\mathbf{1}}}{(\mathbf{1} - \boldsymbol{\mu})^{\mathbf{1}}} \sum_{\boldsymbol{\alpha} \in \mathcal{S}_{\mathbf{k}}} \left[(-1)^{|\boldsymbol{\alpha}|} a_{\boldsymbol{\alpha}} \sum_{\boldsymbol{\ell} \leq \boldsymbol{\alpha}} \prod_{n=1}^N \frac{\ell_n! S(\alpha_n, \ell_n)}{(\mu_n - 1)^{\ell_n}} \right] \end{aligned}$$

$$\begin{aligned} &= \frac{\boldsymbol{\mu}^{\mathbf{1}}}{(\mathbf{1} - \boldsymbol{\mu})^{\mathbf{1}}} \sum_{\boldsymbol{\alpha} \in S_{\mathbf{k}}} \left[\sum_{\boldsymbol{\ell} \leq \boldsymbol{\alpha}} \left((-1)^{|\boldsymbol{\alpha}|} a_{\boldsymbol{\alpha}} \frac{\boldsymbol{\ell}!}{(\boldsymbol{\mu} - \mathbf{1})^{\boldsymbol{\ell}}} \prod_{n=1}^N S(\alpha_n, \ell_n) \right) \right] \\ &= \frac{\boldsymbol{\mu}^{\mathbf{1}}}{(\mathbf{1} - \boldsymbol{\mu})^{\mathbf{1}}} \sum_{\boldsymbol{\ell} \in \mathbb{N}^N} Z_{\boldsymbol{\ell}}(-\mathbf{k}), \end{aligned}$$

where, for each $\boldsymbol{\ell} \in \mathbb{N}^N$,

$$Z_{\boldsymbol{\ell}}(-\mathbf{k}) \stackrel{\text{def}}{=} \sum_{\substack{\boldsymbol{\alpha} \in S_{\mathbf{k}} \\ \boldsymbol{\ell} \leq \boldsymbol{\alpha}}} \left((-1)^{|\boldsymbol{\alpha}|} a_{\boldsymbol{\alpha}} \frac{\boldsymbol{\ell}!}{(\boldsymbol{\mu} - \mathbf{1})^{\boldsymbol{\ell}}} \prod_{n=1}^N S(\alpha_n, \ell_n) \right).$$

The family $(Z_{\boldsymbol{\ell}}(-\mathbf{k}))_{\boldsymbol{\ell} \in \mathbb{N}^N}$ is nearly null, more precisely its support is included in $\{\boldsymbol{\ell} \in \mathbb{N}^N \mid \exists \boldsymbol{\alpha} \in S_{\mathbf{k}}, \boldsymbol{\alpha} \geq \boldsymbol{\ell}\}$, which is clearly a finite subset of \mathbb{N}^N . Moreover, since $\ell > k \Rightarrow S(k, \ell) = 0$, we see that

$$Z_{\boldsymbol{\ell}}(-\mathbf{k}) = \frac{\boldsymbol{\ell}!}{(\boldsymbol{\mu} - \mathbf{1})^{\boldsymbol{\ell}}} \sum_{\boldsymbol{\alpha} \in S_{\mathbf{k}}} \left((-1)^{|\boldsymbol{\alpha}|} a_{\boldsymbol{\alpha}} \prod_{n=1}^N S(\alpha_n, \ell_n) \right). \quad (1)$$

Finally, we note that

$$|Z_{\boldsymbol{\ell}}(-\mathbf{k})|_p \leq \prod_{n=1}^N \frac{|\ell_n!|_p}{|\mu_n - 1|_p^{\ell_n}}.$$

This will be needed in the following.

By using Lemma 5.5, we manipulate the sums as follows:

$$Z_{\boldsymbol{\ell}}(-\mathbf{k}) = \frac{\boldsymbol{\ell}!}{(\boldsymbol{\mu} - \mathbf{1})^{\boldsymbol{\ell}}} \sum_{\boldsymbol{\alpha} \in S_{\mathbf{k}}} \left\{ (-1)^{|\boldsymbol{\alpha}|} a_{\boldsymbol{\alpha}} \prod_{n=1}^N \left[\frac{1}{\ell_n!} \sum_{j_n=0}^{\ell_n} \left((-1)^{\ell_n - j_n} \binom{\ell_n}{j_n} j_n^{\alpha_n} \right) \right] \right\} \quad (2)$$

$$= (\mathbf{1} - \boldsymbol{\mu})^{-\boldsymbol{\ell}} \sum_{\boldsymbol{\alpha} \in S_{\mathbf{k}}} \left\{ (-1)^{|\boldsymbol{\alpha}|} a_{\boldsymbol{\alpha}} \prod_{n=1}^N \left[\sum_{j_n=0}^{\ell_n} \left((-1)^{j_n} \binom{\ell_n}{j_n} j_n^{\alpha_n} \right) \right] \right\} \quad (3)$$

$$= (\mathbf{1} - \boldsymbol{\mu})^{-\boldsymbol{\ell}} \sum_{\boldsymbol{\alpha} \in S_{\mathbf{k}}} \left\{ (-1)^{|\boldsymbol{\alpha}|} a_{\boldsymbol{\alpha}} \sum_{\mathbf{j} \in \prod_{n=1}^N \{0, \dots, \ell_n\}} \left[\prod_{n=1}^N \left((-1)^{j_n} \binom{\ell_n}{j_n} j_n^{\alpha_n} \right) \right] \right\} \quad (4)$$

$$= (\mathbf{1} - \boldsymbol{\mu})^{-\boldsymbol{\ell}} \sum_{\mathbf{j} \in \prod_{n=1}^N \{0, \dots, \ell_n\}} \left\{ \prod_{n=1}^N \left[(-1)^{j_n} \binom{\ell_n}{j_n} \right] \sum_{\boldsymbol{\alpha} \in S_{\mathbf{k}}} \left[(-1)^{|\boldsymbol{\alpha}|} a_{\boldsymbol{\alpha}} \prod_{n=1}^N j_n^{\alpha_n} \right] \right\} \quad (5)$$

$$= (\mathbf{1} - \boldsymbol{\mu})^{-\boldsymbol{\ell}} \sum_{\mathbf{j} \in \prod_{n=1}^N \{0, \dots, \ell_n\}} \left\{ \prod_{n=1}^N \left[(-1)^{j_n} \binom{\ell_n}{j_n} \right] \sum_{\boldsymbol{\alpha} \in S_{\mathbf{k}}} \left[a_{\boldsymbol{\alpha}} \prod_{n=1}^N (-j_n)^{\alpha_n} \right] \right\} \quad (6)$$

$$= (\mathbf{1} - \boldsymbol{\mu})^{-\boldsymbol{\ell}} \sum_{\mathbf{j} \in \prod_{n=1}^N \{0, \dots, \ell_n\}} \left\{ (-1)^{|\mathbf{j}|} \binom{\boldsymbol{\ell}}{\mathbf{j}} Q(-\mathbf{j}) \prod_{t=1}^T P_t(-\mathbf{j})^{k_t} \right\}. \quad (7)$$

For a unit $x \in \mathbb{Z}_p$, we denote its Teichmüller representative by $w(x)$ and set $\langle x \rangle = x/w(x)$. Since each $P_t(-\mathbf{j})$ is a unit in \mathbb{Z}_p , it follows that if $r_t \in \{0, \dots, p-2\}$ satisfies $k_t \equiv r_t \pmod{p-1}$, then $P_t(-\mathbf{j})^{k_t} = w(P_t(-\mathbf{j}))^{r_t} \langle P_t(-\mathbf{j}) \rangle^{k_t}$. Setting $\mathbf{r} = (r_1, \dots, r_T) \in \{0, \dots, p-2\}^T$, we now define the function $Z_{\boldsymbol{\ell}}^{(\mathbf{r})} : \mathbb{Z}_p^T \rightarrow \mathbb{C}_p$ by

$$Z_{\boldsymbol{\ell}}^{(\mathbf{r})}(s_1, \dots, s_T) = (\mathbf{1} - \boldsymbol{\mu})^{-\boldsymbol{\ell}} \sum_{\mathbf{j} \in \prod_{n=1}^N \{0, \dots, \ell_n\}} (-1)^{|\mathbf{j}|} \binom{\boldsymbol{\ell}}{\mathbf{j}} Q(-\mathbf{j}) \prod_{t=1}^T w\left(P_t(-\mathbf{j})\right)^{r_t} \langle P_t(-\mathbf{j}) \rangle^{-s_t}.$$

Thus, $Z_\ell^{(\mathbf{r})}(-\mathbf{k}) = Z_\ell(-\mathbf{k})$. By our previous observation, we then have the bound

$$|Z_\ell^{(\mathbf{r})}(-\mathbf{k})|_p \leq \prod_{n=1}^N \frac{|\ell_n!|_p}{|\mu_n - 1|_p^{\ell_n}}.$$

Since $Z_\ell^{(\mathbf{r})}$ is continuous, and the set $\prod_{t=1}^T \{-r_t + (p-1)\mathbb{N}\}$ is dense in \mathbb{Z}_p^T , we deduce that

$$|Z_\ell^{(\mathbf{r})}(\mathbf{s})|_p \leq \prod_{n=1}^N \frac{|\ell_n!|_p}{|\mu_n - 1|_p^{\ell_n}} \quad \forall \mathbf{s} \in \mathbb{Z}_p^T.$$

To finish the argument, we define $Z_p^{(\mathbf{r})}(Q, P_1, \dots, P_T, \boldsymbol{\mu}, \cdot)$ as an *a priori* formal series:

$$Z_p^{(\mathbf{r})}(Q, P_1, \dots, P_T, \boldsymbol{\mu}, \mathbf{s}) = \frac{\boldsymbol{\mu}^{\mathbf{1}}}{(\mathbf{1} - \boldsymbol{\mu})^{\mathbf{1}}} \sum_{\ell \in \mathbb{N}^N} Z_\ell^{(\mathbf{r})}(\mathbf{s}).$$

One now shows that the series converges p -adically on \mathbb{Z}_p^T . Using the upper bound for $Z_\ell(-\mathbf{k})$ noted above, it therefore suffices to show the following for any n :

$$\frac{|\ell!|_p}{|\mu_n - 1|_p^\ell} \xrightarrow{\ell \rightarrow +\infty} 0.$$

Given $\ell \in \mathbb{N}$, we denote by $S_p(\ell)$ the sum of the digits for ℓ written in base p . It is well known that for $\ell \in \mathbb{N}$ we have

$$v_p(\ell!) = (\ell - S_p(\ell))/(p-1).$$

If c denotes the number of digits of ℓ in base p , then $S_p(\ell) \leq c(p-1)$ and $\ell \geq p^{c-1}$; from this we deduce $S_p(\ell) \ll \log \ell$. Since

$$v_p\left(\frac{\ell!}{(\mu_n - 1)^\ell}\right) = \frac{\ell - S_p(\ell)}{p-1} - \ell v_p(\mu_n - 1) = \left(\frac{1}{p-1} - v_p(\mu_n - 1)\right)\ell - \frac{S_p(\ell)}{p-1},$$

the two bounds $1/(p-1) - v_p(\mu_n - 1) > 0$ and $S_p(\ell) \ll \log \ell$ now imply

$$v_p(\ell!/(\mu_n - 1)^\ell) \xrightarrow{\ell \rightarrow +\infty} +\infty.$$

Thus, $\sum_\ell Z_\ell^{(\mathbf{r})}(\mathbf{s})$ converges p -adically on \mathbb{Z}_p^T . This shows that the function $Z_p^{(\mathbf{r})}(Q, P_1, \dots, P_T, \boldsymbol{\mu}, \mathbf{s})$, p -adically interpolates the function $-\mathbf{k} \mapsto Z(Q, P_1, \dots, P_T, \boldsymbol{\mu}, -\mathbf{k})$ when $\mathbf{k} \equiv \mathbf{r} \pmod{p-1}$, and completes the proof of Lemma 5.9 and, therefore, the proof of Theorem C. \square

6. The case of characters

Let $Q, P_1, \dots, P_T \in \mathbb{R}[X_1, \dots, X_N]$ and χ_1, \dots, χ_N be functions from \mathbb{N}^* into \mathbb{C} . To these data we can associate the following multivariable zeta series:

$$Z(Q; P_1, \dots, P_T; \chi_1, \dots, \chi_N; s_1, \dots, s_T) = \sum_{m_1 \geq 1, \dots, m_N \geq 1} \frac{(\prod_{n=1}^N \chi_n(m_n)) Q(m_1, \dots, m_N)}{\prod_{t=1}^T P_t(m_1, \dots, m_N)^{s_t}}$$

where $(s_1, \dots, s_T) \in \mathbb{C}^T$.

Thanks to the following easy lemma (proven in [Kow04, ch. I]), under a suitable hypothesis, such functions are linear combinations of functions of the type $Z(Q; P_1, \dots, P_T; \mu_1, \dots, \mu_N; \cdot)$.

LEMMA 6.1. *Let $\chi: \mathbb{N}^* \rightarrow \mathbb{C}$, that is D -periodic and whose mean value is null (that is, $\sum_{m=1}^D \chi(m) = 0$). For all $d \in \{1, \dots, D-1\}$, we set $\mu_d = \exp(2i\pi(d/D))$. Then there exists a_1, \dots, a_{D-1} such that for all $m \in \mathbb{N}^*$ we have $\chi(m) = \sum_{d=1}^{D-1} a_d \mu_d^m$.*

Combining the preceding lemma and Theorem A, we obtain the following.

THEOREM 6.2. *Let $Q, P_1, \dots, P_T \in \mathbb{R}[X_1, \dots, X_N]$ and $\chi_1, \dots, \chi_N: \mathbb{N}^* \rightarrow \mathbb{C}$ periodic of null mean value. We assume that P_1, \dots, P_T satisfy HDF and that*

$$\prod_{t=1}^T P_t(\mathbf{x}) \xrightarrow[\substack{|\mathbf{x}| \rightarrow +\infty \\ \mathbf{x} \in J^N}]{} +\infty.$$

Then $Z(Q; P_1, \dots, P_T; \chi_1, \dots, \chi_N; \cdot)$ extends to \mathbb{C}^T as an entire function.

It is now very easy to copy the Exchange Lemma for the series $Z(Q; P_1, \dots, P_T; \chi_1, \dots, \chi_N; \cdot)$.

Let us recall the following usual notation.

DEFINITION 6.3. For $\chi: \mathbb{N}^* \rightarrow \mathbb{C}$, we set $L(s, \chi) = \sum_{m=1}^{+\infty} (\chi(m)/m^s)$.

Then, exactly as was done in § 4, using the Exchange Lemma, we obtain the following.

THEOREM 6.4. *Let $Q, P_1, \dots, P_T \in \mathbb{R}[X_1, \dots, X_N]$. We assume that P_1, \dots, P_T satisfy HDF and that*

$$\prod_{t=1}^T P_t(\mathbf{x}) \xrightarrow[\substack{|\mathbf{x}| \rightarrow +\infty \\ \mathbf{x} \in J^N}]{} +\infty.$$

Let $\mathbf{k} = (k_1, \dots, k_T) \in \mathbb{N}^T$ and write

$$Q \prod_{t=1}^T P_t^{k_t} = \sum_{\alpha \in S} a_\alpha \mathbf{X}^\alpha.$$

Let $\chi_1, \dots, \chi_N: \mathbb{N}^* \rightarrow \mathbb{C}$ periodic of null mean value. Then

$$Z(Q; P_1, \dots, P_T; \chi_1, \dots, \chi_N; -\mathbf{k}) = \sum_{\alpha \in S} a_\alpha \prod_{n=1}^N L(-\alpha_n, \chi_n).$$

To make the p -adic interpolation, we need the following lemma (this is an exercise in [Kob77, ch. 3]).

LEMMA 6.5. *We assume that $\mu \in \mathbb{C}_p$ is a primitive root of unity of order ℓ .*

- (a) *If ℓ is not a power of p , then $|\mu - 1|_p = 1$.*
- (b) *If $\ell = p^h$, then $|\mu - 1|_p = p^{-1/p^{h-1}(p-1)}$.*

Now, using the expression of the function $Z(Q; P_1, \dots, P_T; \chi_1, \dots, \chi_N; \cdot)$ in terms of functions $Z(Q; P_1, \dots, P_T; \cdot)$, Theorem C, and Lemma 6.5(a), we obtain the following.

THEOREM 6.6. *Let p be a prime number. We fix a field morphism from \mathbb{C} into \mathbb{C}_p (left implicit in the discussion and by means of which we calculate $|x|_p$ for any $x \in \mathbb{C}$). Let $Q, P_1, \dots, P_T \in \mathbb{Z}[X_1, \dots, X_N]$ and $\chi_1, \dots, \chi_N: \mathbb{N}^* \rightarrow \mathbb{C}$ be periodic of null mean value. We assume that the periods are not divisible by p . We assume that P_1, \dots, P_T satisfy HDF, and that*

$$\prod_{t=1}^T P_t(\mathbf{x}) \xrightarrow[\substack{|\mathbf{x}| \rightarrow +\infty \\ \mathbf{x} \in J^N}]{} +\infty.$$

We set

$$\tilde{Z}(Q; P_1, \dots, P_T; \chi_1, \dots, \chi_N; \mathbf{s}) = \sum_{\substack{\mathbf{m} \in \mathbb{N}^{*N} \\ \forall t \in \{1, \dots, T\}, p \nmid P_t(\mathbf{m})}} \prod_{n=1}^N \chi_n(m_n) Q(\mathbf{m}) \prod_{t=1}^T P_t(\mathbf{m})^{-s_t}.$$

Let $\mathbf{r} \in \{0, \dots, p-2\}^T$. Then there exists $\tilde{Z}_p^{\mathbf{r}}(Q, P_1; \dots, P_T; \chi_1, \dots, \chi_N; \cdot): \mathbb{Z}_p^T \rightarrow \mathbb{C}_p$ continuous such that for all $\mathbf{k} \in \mathbb{N}^T$ satisfying $k_t \equiv r_t \pmod{p-1}$ for all $t \in \{1, \dots, T\}$, we have

$$\tilde{Z}_p^{\mathbf{r}}(Q; P_1, \dots, P_T; \chi_1, \dots, \chi_N; -\mathbf{k}) = \tilde{Z}(Q; P_1, \dots, P_T; \chi_1, \dots, \chi_N; -\mathbf{k}).$$

Remark 6.7. If some of the periods of the χ_n are divisible by p , one needs to look at the μ_d whose coefficient a_d in Lemma 6.1 is non-zero. Depending on their p -adic absolute value (calculated in Lemma 6.5), we then may or may not be able to make the p -adic interpolation.

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