A CHOICELESS VERSION OF THE HAHN-BANACH THEOREM FOR UNIFORMLY GÂTEAUX DIFFERENTIABLE NORMED SPACES

MARIANNE MORILLON

ABSTRACT. Denoting by **AC**($\mathbb{N}$) the countable axiom of choice, we show in **ZF**+**AC**($\mathbb{N}$) that the dual ball of a uniformly Gâteaux-differentiable Banach space is compact in the weak* topology. In **ZF**, we prove that this dual ball is (closely) convex-compact in the weak* topology. We deduce that uniformly Gâteaux–differentiable Banach spaces satisfy the continuous Hahn-Banach property in **ZF**. This enhances a result previously obtained in [1] for uniformly Fréchet differentiable Banach spaces.

EQUIPE RÉUNIONNAISE DE MATHÉMATIQUES ET INFORMATIQUE THÉORIQUE (ERMIT)

http://laboratoires.univ-reunion.fr/ermit

CONTENTS

1. Introduction 2
1.1. Presentation of the results 2
1.2. Some weak forms of **AC** 3
2. A criterion of compactness 3
2.1. Filters 3
2.2. Gauge spaces 3
2.3. A criterion of compactness in **ZF**+**AC**($\mathbb{N}$) 4
3. The continuous dual of a normed space 5
3.1. Continuous dual and bidual of a normed space 5
3.2. The weak* topology on the continuous dual 6
3.3. Polyhedras 6
3.4. Separating a convex set and a strict polyhedra 6
4. W*UR dual norms 7
4.1. W*UR dual norm 7
4.2. Property of smallness 7
4.3. Compactness of W*UR dual balls 7
5. UG differentiability yields Hahn-Banach 8

Date: March 29, 2008.

2000 *Mathematics Subject Classification.* Primary 03E25, 54 ; Secondary 46B.

Key words and phrases. compactness, gauge space, uniformly Gâteaux differentiable normed space, weak* topology, axiom of choice.

5.1. Uniform Gâteaux differentiability 8
5.2. Compactness of the dual ball of a UG space 8
5.3. The *CHB* property for UG spaces 9
5.4. Examples of UG renormings and applications 9
6. Appendix on UG spaces 10
6.1. The dual ball of a UG space is W*UR 10
6.2. UG renormings 11
References 12

1. Introduction

1.1. **Presentation of the results.** We work in **ZF**, Zermelo-Fraenkel set-theory without the Axiom of Choice (for short **AC**). Given a real vector space E, a mapping $p : E \to \mathbb{R}$ is *sub-linear* if for every $x, y \in E$, and every $\lambda \in \mathbb{R}_+$, $p(x+y) \leq p(x) + p(y)$ (sub-addivity), and $p(\lambda.x) = \lambda p(x)$ (positive homogeneity). Consider the *"Hahn-Banach axiom"*, a well known consequence of **AC**:

> **HB**: *Let E be a (real) vector space. If $p : E \to \mathbb{R}$ is a sub-linear mapping, if F is a vector subspace of E, if $f : F \to \mathbb{R}$ is a linear mapping such that $f \leq p_{\restriction F}$, then there exists a linear mapping $g : E \to \mathbb{R}$ extending f such that $g \leq p$".*

It is known that **HB** is not provable in **ZF** (see [12], [8], [7]), and that **HB** does not imply **AC** (**HB** does not even imply that every sequence $(P_n)_{n\in\mathbb{N}}$ of pairs has a non-empty product -see [11], [8], [7]-). Given a real topological vector space E (*i.e.* E is a real vector space such that the "sum" $+ : E \times E \to E$ and the external multiplicative law $. : \mathbb{R} \times E \to E$ are continuous for the product topology), say that E satisfies the *Continuous Hahn-Banach property* (for short *CHB* property) if *"For every* continuous *sub-linear mapping $p : E \to \mathbb{R}$, for every vector subspace F of E, if $f : F \to \mathbb{R}$ is a linear mapping such that $f \leq p_{\restriction F}$, then there exists a linear mapping $g : E \to \mathbb{R}$ extending f such that $g \leq p$."* Although statement **HB** is not provable in **ZF**, however, some real normed spaces satisfy (in **ZF**) the *CHB* property: for example normed spaces with a well-orderable dense subset (in particular separable normed spaces), but also Hilbert spaces, spaces $\ell^0(I)$ (see [6]), uniformly convex Banach spaces with a Gâteaux–differential norm ([4]), uniformly Fréchet differentiable Banach spaces (see [1]). The aim of this paper is to prove (in **ZF**) that every uniformly Gâteaux-differentiable Banach space E satisfies the following *effective CHB* property (see Corollary 1 in Section 5.3): *"There is a mapping Ψ associating to every (p, F, f) where $p : E \to \mathbb{R}$ is a continuous sub-linear mapping, F is a vector subspace of E, and $f : F \to \mathbb{R}$ is a linear mapping such that $f \leq p_{\restriction F}$, a linear mapping $g := \Psi(p, F, f) : E \to \mathbb{R}$ extending f such that $g \leq p$."* This enhances the result that we previously obtained in [1] for uniformly Fréchet differentiable Banach spaces, and this answers Question 3 of [1]. Notice that given a set I, $\ell^0(I)$ has an equivalent norm which is uniformly Gâteaux–differentiable (see Section 5.4.1), but if I is infinite, then $\ell^0(I)$ has no uniformly Fréchet differentiable equivalent norm (see Remark 4).

Consider the *countable Axiom of Choice*, which is not provable in **ZF**, and which does not imply **AC** (see [7], [8]):

> **AC**($\mathbb{N}$): *If $(A_n)_{n\in\mathbb{N}}$ is a family of non-empty sets, then there exists a mapping $f : \mathbb{N} \to \cup_{n\in\mathbb{N}} A_n$ associating to every $k \in \mathbb{N}$ an element $f(k) \in A_k$.*

In this paper, we first provide in **ZF**+**AC**($\mathbb{N}$) a criterion of compactness for certain complete gauge spaces satisfying a "smallness" property with respect to an extra metric (see Theorem 1 in Section 2.3.4). This geometric idea leads us to the fact that (see Theorem 3 in Section 4.3), given a normed space E with a weak* uniformly rotund dual ball, then, in **ZF**+**AC**($\mathbb{N}$), the dual ball $B_{E'}$ of E is weak* compact (and this is not provable in **ZF** -see Remark 3), and in **ZF** this dual ball is (closely) weak* *convex-compact* (see Definition in Section 4.3). It follows that if a normed space E is uniformly Gâteaux-differentiable, then its dual ball is weak* compact in **ZF**+**AC**($\mathbb{N}$), and (closely) convex-compact in **ZF** (see Theorem 4 in Section 5.2). Using a result of [6] linking the *CHB* property on the normed space E and the weak* convex-compactness of the dual ball of E, we deduce our main result: "Uniformly Gâteaux-differentiable normed spaces satisfy the *CHB* property".

1.2. **Some weak forms of AC.** We now recall some weak forms of the Axiom of Choice which will be used in this paper and the known links between them. For detailed references and much information on various weak forms of the Axiom of Choice, see [8] and [7].

1.2.1. **DC**, **AC**($\mathbb{N}$) *and* **AC**($\mathbb{N}$, **fin**). The axiom of *Dependent Choices* asserts that:

> **DC**: Given a non-empty set X and a binary relation R on X such that $\forall x \in X \exists y \in X\ xRy$, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of X such that for every $n \in \mathbb{N}$, $x_n R x_{n+1}$.

The *countable Axiom of Choice for finite sets* says that:

> **AC**($\mathbb{N}$, **fin**): *If $(A_n)_{n\in\mathbb{N}}$ is a family of finite non-empty sets, then there exists a mapping $f : \mathbb{N} \to \cup_{n\in\mathbb{N}} A_n$ associating to every $n \in \mathbb{N}$ an element $f(n) \in A_n$.*

Of course, **AC** $\Rightarrow$ **DC** $\Rightarrow$ **AC**($\mathbb{N}$) $\Rightarrow$ **AC**($\mathbb{N}$, **fin**). However, the converse statements are not provable in **ZF**, and **AC**($\mathbb{N}$, **fin**) is not provable in **ZF** (see references in [7]).

2. A criterion of compactness

2.1. Filters.

2.1.1. *Filters in lattices of sets.* Given a set X, a *lattice* of subsets of X is a subset $\mathcal{L}$ of $\mathcal{P}(X)$ containing $\varnothing$ and X, which is closed by finite intersections and finite unions. A *filter* of the lattice $\mathcal{L}$ is a non-empty proper subset $\mathcal{F}$ of $\mathcal{L}$ such that for every $A, B \in \mathcal{L}$:

(i) $(A, B \in \mathcal{F}) \Rightarrow A \cap B \in \mathcal{F}$
(ii) $(A \in \mathcal{F}$ and $A \subseteq B) \Rightarrow B \in \mathcal{F}$

A subset $\mathcal{A}$ of $\mathcal{L}$ is contained in a filter of $\mathcal{L}$ if and only if $\mathcal{A}$ satisfies the *finite intersection property* (for short *FIP*), that is, every finite subset of $\mathcal{A}$ has a non-empty intersection; in this case, the intersection of all filters of $\mathcal{L}$ containing $\mathcal{A}$ is a filter that we denote by $fil(\mathcal{A})$ (which is the smallest filter of $\mathcal{L}$ containing $\mathcal{A}$, *alias* the "filter generated by $\mathcal{A}$").

2.1.2. *Stationary sets.* Given a filter $\mathcal{F}$ of a lattice $\mathcal{L}$ of subsets of a set X, an element $S \in \mathcal{L}$ is $\mathcal{F}$-*stationary* if for every $A \in \mathcal{F}$, $A \cap S \neq \varnothing$. The set $\mathcal{S}(\mathcal{F})$ of $\mathcal{F}$-stationary sets satisfies the following properties:

(i) If $\mathcal{A}$ is a chain of $\mathcal{L}$ and if $\mathcal{A} \subseteq \mathcal{S}(\mathcal{F})$, then $\mathcal{A} \cup \mathcal{F}$ satisfies the *FIP*.
(ii) If $F_1, \dots, F_m \in \mathcal{L}$ and $F_1 \cup \dots \cup F_m \in \mathcal{S}(\mathcal{F})$, then there exists some $i_0 \in \{1..m\}$ such that F_{i_0} is $\mathcal{F}$-stationary.

2.2. Gauge spaces.

2.2.1. *Pseudo-metrics.* Given a set X, a *pseudo-metric* on X is a mapping $d : X \times X \to \mathbb{R}_+$ satisfying the following properties for every $x, y, z \in X$:

(i) $d(x, x) = 0$
(ii) $d(x, y) = d(y, x)$
(iii) $d(x, z) \leq d(x, y) + d(y, z)$

Given $a \in X$ and some pseudo-metric $d : X \times X \to \mathbb{R}_+$, given real numbers R, R' satisfying $R \leq R'$, we define *large d-balls* and *large d-crowns* as follows:

$$B_d(a, R) := \{x \in X : \ d(a, x) \leq R\}$$
$$D_d(a, R, R') := \{x \in X : \ R \leq d(a, x) \leq R'\}$$

In an analogous way, we also define *strict* d-balls with strict inequalities. If A is a non-empty subset of X, we define the *d-diameter* of A:

$$\mathrm{diam}_d(A) := \inf\{d(x, y) : x, y \in A\}$$

2.2.2. *Complete gauge spaces.* A *gauge space* is a set X endowed with a family $(d_i)_{i\in I}$ of pseudo-metrics on X. Every gauge space naturally defines a topology on X generated by the various d_i-strict balls, $i \in I$. Say that a subset $\mathcal{A}$ of non-empty subsets of the gauge space $(X, (d_i)_{i\in I})$ is *Cauchy* if for every $\varepsilon > 0$ and for every $i \in I$, there exists $F \in \mathcal{A}$ such that $\mathrm{diam}_{d_i}(F) < \varepsilon$. A gauge space $(X, (d_i)_{i\in I})$ is said to be *complete* if every Cauchy filter of the lattice of closed subsets of X has a non-empty intersection.

Example 1 (**Closed subsets of $\mathbb{R}^I$**)**.** Let I be a set. For each $i \in I$, consider the canonical projection $p_i : \mathbb{R}^I \to \mathbb{R}$, and the pseudo-metric $d_i : X \times X \to \mathbb{R}$ associating to each $(x, y) \in X \times X$ the real number $|p_i(x) - p_i(y)|$. Then, the gauge space $(\mathbb{R}^I, (d_i)_{i\in I})$ is complete. It follows that for every closed subset A of $\mathbb{R}^I$, the gauge space $(A, (d_{i\restriction A\times A})_{i\in I})$ is also complete.

2.3. A criterion of compactness in ZF+AC($\mathbb{N}$).

2.3.1. *Compactness.*

Definition 1 ($\mathcal{C}$-compactness, closed $\mathcal{C}$-compactness)**.** Given a class $\mathcal{C}$ of subsets of a set X, say that a subset A of X is *$\mathcal{C}$-compact* if for every family $(C_i)_{i\in I}$ of $\mathcal{C}$ such that $(C_i \cap A)_{i\in I}$ satisfies the *FIP*, $A \cap \cap_{i\in I} C_i$ is non-empty; say that $\mathcal{A}$ is *closely* $\mathcal{C}$-compact if there is a mapping associating to every family $(C_i)_{i\in I}$ of $\mathcal{C}$ such that $(C_i \cap A)_{i\in I}$ satisfies the *FIP*, an element of $A \cap \cap_{i\in I} C_i$.

Recall that a topological space X is *compact* if X is $\mathcal{C}$-compact, where $\mathcal{C}$ is the set of closed subsets of X (or, equivalently, every filter of the lattice of closed sets of X has a non-empty intersection).

2.3.2. *Sub-basis of closed sets.*

Definition 2 (basis, sub-basis of closed subsets)**.** A set $\mathcal{B}$ of closed subsets of a topological space X is a *basis of closed sets* if every closed set of X is an intersection of elements of $\mathcal{B}$. A set $\mathcal{S}$ of closed subsets of X is a *sub-basis of closed sets* if the set $\mathcal{B}$ of finite unions of elements of $\mathcal{S}$ is a basis of closed subsets of X.

The following result is easy.

Proposition 1. *Let X be a topological space, and $\mathcal{L}$ be a lattice of closed subsets of X which is also a basis of closed subsets of X. If every filter of $\mathcal{L}$ has a non-empty intersection, then X is compact.*

2.3.3. *Property of smallness.*

Definition 3 (smallness in thin crowns)**.** Let d, d' be two pseudo-metrics on a set X. Let $a \in X$. Say that a set $\mathcal{C}$ of subsets of X satisfies the *property of d'-smallness in thin d-crowns centered at a* if for every $R \in \mathbb{R}_+^*$, for every $\varepsilon > 0$ there exists $\eta \in]0, R[$ such that for every $C \in \mathcal{C}$,

$$C \subseteq D_d(a, R - \eta, R + \eta) \Rightarrow \mathrm{diam}_{d'}(C) < \varepsilon$$

If $(X, (d_i)_{i\in I})$ is a gauge space, say that $\mathcal{C}$ satisfies the *property of $(d_i)_{i\in I}$-smallness in thin d-crowns centered at a* if for every $i \in I$, $\mathcal{C}$ satisfies the property of d_i-smallness in thin d-crowns centered at a.

2.3.4. *Criterion of compactness.*

Theorem 1. *Let $(X,(d_i)_{i\in I})$ be a complete gauge space. Let $\mathcal{T}$ be a topology on X which is included in the associated gauge topology, and let $\mathcal{C}$ be a sub-basis of closed sets of $(X,\mathcal{T})$ which is closed by finite intersections. Let $a\in X$. Let $d: X\times X\to\mathbb{R}_+$ be a pseudo metric such that d-large balls centered at a belong to $\mathcal{C}$, and such that $\mathcal{T}$ is included in the topology $\mathcal{T}_d$ associated to d. If $\mathcal{C}$ satisfies the property of $(d_i)_{i\in I}$-smallness in thin d-crowns centered at a, then:*

(i) In $\mathbf{ZF}+\mathbf{AC}(\mathbb{N})$*, every large d-ball with center a (and thus, every d-bounded $\mathcal{T}$-closed subset of X) is compact in $\mathcal{T}$.*

(ii) In $\mathbf{ZF}$*, every every large d-ball with center a (and thus, every d-bounded element of $\mathcal{C}$) is $\mathcal{C}$-compact. Moreover, if the gauge space $(X,(d_i)_{i\in I})$ is Hausdorff, then every d-bounded element of $\mathcal{C}$ is closely $\mathcal{C}$-compact.*

Proof. Let $\mathcal{L}$ be the lattice generated by $\mathcal{C}$. Let $\rho>0$ and let B be the large d-ball $B_d(a,\rho)$.
(i) Let $\mathcal{F}$ be a filter of $\mathcal{L}$ containing B. Let us prove in $\mathbf{ZF}+\mathbf{AC}(\mathbb{N})$ that $\cap\mathcal{F}$ is non-empty (using Proposition 1, this will imply that B is $\mathcal{T}$-compact). Let $R:=\inf\{r\in\mathbb{R}_+:\ B_d(a,r)\in\mathcal{S}(\mathcal{F})\}$. Since $\{B_d(a,r):\ r>R\}$ is a chain of $\mathcal{F}$-stationary sets of $\mathcal{L}$, the set $\mathcal{F}\cup\{B_d(a,r):\ r>R\}$ generates a filter $\mathcal{G}$ of $\mathcal{L}$ (see Section 2.1.2-(i)). If $R=0$ then $a\in\cap\mathcal{F}$ (because elements of $\mathcal{F}$ are $\mathcal{T}_d$-closed). Assume that $R>0$. For every $\varepsilon>0$, there exists some element of $\mathcal{G}$ which is included in the crown $D_d(a,R-\varepsilon,R+\varepsilon)$; with $\mathbf{AC}(\mathbb{N})$, choose for every $n\in\mathbb{N}$, a finite subset $\mathcal{Z}_n$ of $\mathcal{C}$ such that $\cup\mathcal{Z}_n\in\mathcal{G}$ and $\cup\mathcal{Z}_n\subseteq D_d(a,R-\frac{1}{n+1},R+\frac{1}{n+1})$. With $\mathbf{AC}(\mathbb{N},\mathbf{fin})$, the set $\cup_{n\in\mathbb{N}}\mathcal{Z}_n$ is countable. We define by induction a sequence $(C_n)_{n\in\mathbb{N}}\in\prod_{n\in\mathbb{N}}\mathcal{Z}_n$ such that for every $n\in\mathbb{N}$, $\mathcal{G}\cup\{C_i:\ i<n\}$ generates a filter $\mathcal{G}_n$ and $C_n\in\mathcal{S}(\mathcal{G}_n)$: given some $n\in\mathbb{N}$, $\cup\mathcal{Z}_n\in\mathcal{G}\subseteq fil(\mathcal{G},(C_i)_{i<n})\subseteq\mathcal{S}(fil(\mathcal{G},(C_i)_{i<n}))$; using Section 2.1.2-(i), it follows that there exists $C_n\in\mathcal{Z}_n$ satisfying $C_n\in\mathcal{S}(fil(\mathcal{G},(C_i)_{i<n}))$. Since $\mathcal{C}$ satisfies the property of $(d_i)_{i\in I}$-smallness in thin d-crowns centered at a, the filter $\mathcal{H}:=\cup_{n\in\mathbb{N}}\mathcal{G}_n$ is Cauchy in the gauge space $(X,(d_i)_{i\in I})$. Since this gauge space is complete, and since its topology contains $\mathcal{T}$, $\cap\mathcal{H}$ is non-empty. Thus $\varnothing\neq\cap\mathcal{G}\subseteq\cap\mathcal{F}$.
(ii) Let $\mathcal{A}$ be subset of non-empty elements $\mathcal{C}$ which is closed by finite intersection and such that the ball B belongs to $\mathcal{A}$. Let us show (in $\mathbf{ZF}$) that $\cap\mathcal{A}$ is non-empty. Let $\mathcal{F}$ be the filter of $\mathcal{L}$ generated by $\mathcal{A}$. Let $R:=\inf\{r\in\mathbb{R}_+:\ B_d(a,r)\in\mathcal{S}(\mathcal{F})\}$. Denote by $\mathcal{A}'$ the set $\{A\cap B_d(a,r):\ A\in\mathcal{A}\text{ and }r>R\}$. If $R=0$ then $a\in\cap\mathcal{A}$. If $R>0$, then for every $\varepsilon>0$, there exists some element of $\mathcal{A}'$ which is included in the crown $D_d(a,R-\varepsilon,R+\varepsilon)$. Since $\mathcal{C}$ satisfies the property of $(d_i)_{i\in I}$-smallness in thin d-crowns centered at a, the family $\mathcal{A}'$ is Cauchy in the gauge space $(X,(d_i)_{i\in I})$; since this uniform space is complete and since $\mathcal{A}'$ satisfies the *FIP*, $\cap\mathcal{A}'$ is non-empty so $\varnothing\neq\cap\mathcal{A}'\subseteq\cap\mathcal{A}$. Moreover, if the gauge space $(X,(d_i)_{i\in I})$ is Hausdorff, then $\cap\mathcal{A}'$ is a singleton $\{a\}$ which is $\mathbf{ZF}$-definable from $(X,(d_i)_{i\in I})$, d and $\mathcal{A}$. $\square$

3. The continuous dual of a normed space

All vector spaces that we consider are defined over the field $\mathbb{R}$ *of* real *numbers.*

3.1. **Continuous dual and bidual of a normed space.** Given a normed space E endowed with a norm $\|.\|$, we denote by B_E the large unit ball $\{x\in E:\|x\|\leq 1\}$, and by S_E the unit sphere of E. The topology on E associated to the norm is called the *strong topology*. The vector space E' endowed with the *dual norm* $\|.\|^*$ is the *continuous dual* of the normed space E. We also denote by E'' the *continuous bidual* of E *i.e.* the continuous dual of E'. Let $can: E\to E''$ be the *canonical mapping* associating to every $x\in E$ the "evaluating mapping" $\tilde{x}: E'\to\mathbb{R}$, satisfying for every $f\in E'$ the equality $\tilde{x}(f)=f(x)$. A *Banach* space is a normed space which is (Cauchy)-complete for the metric associated to the norm (*i.e.* every Cauchy filter of closed sets has a non-empty intersection). The *weak topology* $\sigma(E,E')$ on the normed space E is the weakest topology on E such that elements $f\in E'$ are continuous.

Remark 1. In a model of **ZF** where **HB** fails, there exists a non null (infinite dimensional) normed space E such that $E' = \{0\}$ (see [6, Lemma 5] or [9]). In such a model of **ZF**, the weak topology on E is trivial with only two open sets.

3.2. **The weak* topology on the continuous dual.** Given a normed space E and a vector subspace F of E', we denote by $\sigma(E, F)$ the weakest topology on E such that every $f \in F$ is continuous on E endowed with this topology. Then E endowed with this topology is a locally convex topological vector space. Given a normed space E, the *weak* topology on* E' is the topology $\sigma(E', can[E])$ where $can : E \to E''$ is the canonical mapping. This topology is also denoted by $\sigma(E', E)$. Notice that the locally convex *tvs* $(E', \sigma(E', E))$ is Hausdorff (in **ZF**). Moreover, strongly closed balls of E' are weak* closed (in **ZF**). For every $h \in E$, consider the pseudo-metric $d_h : E' \times E' \to \mathbb{R}$ associating to every $(f, g) \in E' \times E'$ the real number $|f(h) - g(h)|$. The topology associated to the gauge space $(E', (d_h)_{h \in E})$ is the weak* topology $\sigma(E', E)$ on E'. This system $(E', (d_h)_{h \in E})$ is the *canonical gauge space* associated to the weak* topology on E'.

Proposition 2. *Let E be a normed space. The closed unit ball $B_{E'}$ of E' is complete in the canonical gauge space $(E', (d_h)_{h \in E})$ associated to the weak* topology $\sigma(E', E)$ on E'.*

Proof. Consider the mapping $h : B_{E'} \to [-1, 1]^{B_E}$ associating to each $f \in B_{E'}$ the family $(f(x))_{x \in B_E}$. The range $R := h[B_{E'}]$ of h is a closed subset of $[-1, 1]^{B_E}$, whence it is a complete gauge subspace of $[-1, 1]^{B_E}$. Moreover $h : B_{E'} \to h[B_{E'}]$ is an isomorphism of gauge spaces, thus it follows that the gauge space $B_{E'}$ is also complete. □

3.3. **Polyhedras.** Given a vector space E, and a vector subspace F of the algebraic dual E^* of E, a *strict F-hemi-space (resp. large F-hemi-space)* of E is a subset of E of the form $(f < \lambda)$ (resp. $(f \leq \lambda)$) where $f \in F \backslash \{0\}$ and $\lambda \in \mathbb{R}$. A *strict F-polyhedra (*resp. *large F-polyhedra)* of E is a finite intersection of strict (*resp.* large) F-hemi-spaces. If E is a topological vector space and if $F = E'$, then F-hemi-spaces (*resp.* F-polyhedras) are also called hemi-spaces of E (*resp. polyhedras of* E). If E is a normed space, we distinguish on E':

- polyhedras of E' endowed with the strong topology (they also are the polyhedras of E' endowed with the weak topology)
- polyhedras of E' endowed with the weak* topology, which we also call **-polyhedras.*

Recall the following Proposition, which is choiceless:

Proposition 3. *Given a Banach space E, weak* continuous linear mappings $\phi : E' \to \mathbb{R}$ are evaluation mappings $\tilde{a} : E' \to \mathbb{R}$ for $a \in E$.*

Proof. See for example [2]. □

3.4. **Separating a convex set and a strict polyhedra.**

Theorem 2. *Let E be a topological vector space. If C is a convex subset of E, and if P is a strict polyhedra of E disjoint from C, then there exists $f \in E'$ such that $f[P] < f[C]$.*

Proof. Let $f_1, ..., f_m \in E' \backslash \{0\}$ and $\alpha_1, ..., \alpha_m \in \mathbb{R}$ such that $P = \bigcap_{i=1}^m \{x \in E : \ f_i(x) < \alpha_i\}$. Let $F = (f_i)_{1 \leq i \leq m} : E \to \mathbb{R}^m$. Let $V := Ker(F) = \bigcap_{i=1}^m Ker(f_i)$ and let $can : E \twoheadrightarrow E/V$ be the canonical mapping: the mapping can is open. Since the vector subspace V is closed, the quotient *tvs* E/V is Hausdorff; moreover, the vector space E/V is finite dimensional (because E/V is isomorphic the finite dimensional space $F[E]$). It follows that the *tvs* E/V satisfies the various classical geometrical Hahn-Banach properties. Let $K := can[C]$ and $U := can[P]$. The convex subsets K and U are disjoint in E/V (by definition of the mappings f_i) and U is open in E/V (because can is open). Using a geometric form of Hahn-Banach in E/V, there exists $g \in (E/V)'$ such that $g[U] < g[K]$. Let $f := g \circ can$. Then $f[P] < f[C]$. □

Example 2. Let E be a normed space, let C be a convex subset of E'. If P is a strict *-polyhedra of E' which is disjoint from C, then there exists $a \in E$ such that $\tilde{a}[C] < \tilde{a}[P]$.

Proof. Apply Theorem 2 to the (Hausdorff) topological vector space E' endowed with the $\sigma(E', E)$-topology. $\square$

4. W*UR dual norms

4.1. **W*UR dual norm.**

Notation 1. Given a normed space E, for every $a \in S_E$ and $\eta \in [0, 1]$, we denote by $C_{a,\eta}$ the following weak* closed convex subset of the dual ball $B_{E'}$:

$$C_{a,\eta} := \{f \in B_{E'} : f(a) \geq 1 - \eta\}$$

Given a normed space E, say that the dual norm $\|.\|^*$ on E' is *weak* uniformly rotund* (for short *W*UR*) when for every $h \in E$,

$$\lim_{\eta -> 0} \sup_{f,g \in C_{a,\eta}} |f(h) - g(h)| = 0 \quad \text{uniformly for } a \in S_E$$

This means that for every $h \in E$, and every $\varepsilon > 0$, there exists some $\eta > 0$ such that for every $a \in S_E$, $\mathrm{diam}_{d_h}(C_{a,\eta}) < \varepsilon$.

Remark 2. In **ZF**+**AC**($\mathbb{N}$), the above Definition is equivalent to the Definition given in [3, Def. 6.1 p. 61] relying on sequences of E'.

4.2. **Property of smallness.**

Proposition 4. *Let E be a normed space. Let d be the distance given by the dual norm on E'. Let $(d_h)_{h\in E}$ be the canonical gauge space of the weak* topology on E'. Let $\mathcal{P}$ be the set of large *-polyhedras of E'. Let $\mathcal{P}_b := \{P \cap B(0, r) : P \in \mathcal{P}$ and $r \in \mathbb{R}_+\}$. If E has a W*UR dual norm, then $\mathcal{P}_b$ has the property of $(d_h)_{h\in E}$-smallness in thin d-crowns centered at $0_{E'}$.*

Proof. Let $h \in E$. Let $\varepsilon > 0$. Since the dual norm of E' is W*UR, let $\eta > 0$ such that for every $a \in S_E$, $\mathrm{diam}_{d_h}(C_{a,\eta}) \leq \varepsilon$. If $P \in \mathcal{P}$ and if $P \cap B_d(0, 1 - \eta) = \varnothing$, then, using a **ZF**-provable consequence of **HB** (see Example 2 in Section 3.4), there exists $a \in S_E$ such that $\tilde{a}[B(0, 1 - \eta)] < \tilde{a}[P]$. It follows that $(P \cap B_{E'}) \subseteq C_{a,\eta}$, thus $\mathrm{diam}_{d_h}(P \cap B_{E'}) \leq \varepsilon$. $\square$

4.3. **Compactness of W*UR dual balls.**

Given a topological vector space E, say that a subset A of E is *convex-compact* if A is $\mathcal{C}$-compact where $\mathcal{C}$ is the set of closed convex subsets of E.

Theorem 3. *Let $(E, \|.\|)$ be a normed space. Assume that the dual norm on E' is weak* uniformly rotund.*

(i) The axiom **AC**($\mathbb{N}$) *implies that the closed unit ball of E' is compact in the weak* topology.*
(ii) The closed unit ball of E' is closely convex-compact in the weak topology.*

Proof. Let $\mathcal{P}$ be the set of large *-polyhedras of E'. Let d be the metric associated to the norm of E'. Let $\mathcal{C} := \{P \cap B_d(0, r) : P \in \mathcal{P}$ and $0 \leq r \leq 1\}$. Then $\mathcal{C}$ is a sub-basis of the weak* topology on $B_{E'}$, which is closed by finite intersection. Let $(d_h)_{h\in E}$ be the canonical gauge space of the weak* topology on E': the subset $B_{E'}$ of the gauge space $(E', (d_h)_{h\in E})$ is complete (see Proposition 2). Large d-balls centered at $0_{E'}$ belong to $\mathcal{C}$ and, by Proposition 4, the class $\mathcal{C}$ satisfies the property of $(d_h)_{h\in E}$-smallness in thin d-crowns of $B_{E'}$ centered at $0_{E'}$.
(i) In **ZF**+**AC**($\mathbb{N}$), we apply Theorem 1-(i) to the class $\mathcal{C}$, and it follows that $B_{E'}$ is weak* compact.
(ii) Using Theorem 1-(ii), and since the gauge space $(E', (d_h)_{h\in E})$ is Hausdorff, the closed unit ball of E' is closely convex-compact. $\square$

Remark 3. Consider the two following statements:

$\mathbf{A}_{w*ur}$ *Given a normed space such that the dual norm is weak* uniformly rotund, then the dual ball is weak* compact.*

AH: *The closed unit ball of a Hilbert space is weakly compact.*

It is easy to see that $\mathbf{A}_{w*ur}$ implies **AH**. Moreover it is known (see [6]) that **AH** implies $\mathbf{AC}(\mathbb{N}, \mathbf{fin})$. Since $\mathbf{AC}(\mathbb{N}, \mathbf{fin})$ is not provable in **ZF**, it follows that $\mathbf{A}_{w*ur}$ is not provable in **ZF**.

Question 1. Does **AH** imply $\mathbf{A}_{w*ur}$?

5. UG differentiability yields Hahn-Banach

5.1. **Uniform Gâteaux differentiability.** Recall that (see [3, Def 6.5 p. 63]) a normed space $(E, \|.\|)$ is *uniformly Gâteaux-differentiable* (for short *UG*) when it is Gâteaux-differentiable, and, for every $h \in E$,

$$\lim_{t->0} \big(\frac{\|a+th\| - \|a\|}{t} - G(a,h)\big) = 0 \quad \text{uniformly for } a \in S_E$$

Remark 4. A norm which is uniformly Fréchet differentiable (see [3, Def. 1.9 p.8]) is uniformly Gâteaux differentiable but the converse statement is false. Indeed, given a *uniformly Fréchet differentiable* Banach space E, the canonical mapping $can : E \to E''$ is isometric and onto. On the other hand, if I is a set, the continuous bidual space of $\ell^0(I)$ is $\ell^\infty(I)$ and, if I is infinite, the canonical mapping $can : \ell^0(I) \to \ell^\infty(I)$ is isometric but it is not onto (because $1_I \notin can[\ell^0(I)] = \ell^0(I)$). It follows that if I is infinite, the space $\ell^0(I)$ does not have a uniformly Fréchet differentiable equivalent norm.

Proposition 5 ([3])**.** *If a normed space E is uniformly Gâteaux-differentiable, then the dual norm of E' is W*UR.*

Proof. Slightly change the proof in [3, p.63-64] to get a proof in **ZF** (for sake of completeness, we give a proof in Section 6.1, see the Appendix). □

Remark 5. The converse statement holds in **ZF+HB** (see [3, p.63-64]).

5.2. **Compactness of the dual ball of a UG space.**

Theorem 4. *Let E be a uniformly Gâteaux differentiable Banach space.*

(i) The axiom $\mathbf{AC}(\mathbb{N})$ *implies that the closed unit ball of E' (and thus every bounded weak* closed subset of E') is compact in the weak* topology.*
(ii) The closed unit ball of E' is closely convex-compact in the weak topology.*

Proof. Use Theorem 3 and Proposition 5. □

Remark 6. It is known that, in **ZF**+$\mathbf{AC}(\mathbb{N})$ (see [10]), every bounded weakly closed subset of a Hilbert space is weakly compact. It is also known that, in **ZFC** (see [5]), the closed dual ball of a uniformly Gâteaux-differentiable normed space E is homeomorphic with a bounded weakly closed subset of Hilbert space. However, the proof in [5] relies on much axiom of choice; in particular, it does not seem to be valid in **ZF**+$\mathbf{AC}(\mathbb{N})$. Thus, we cannot use these two results to deduce in **ZF**+$\mathbf{AC}(\mathbb{N})$ Theorem 4 from [10] and [5].

Question 2. Is the following statement provable in **ZF**+$\mathbf{AC}(\mathbb{N})$? in **ZF+DC**? Does it imply some weak form of the Axiom of Choice?

The closed dual ball of a uniformly Gâteaux differentiable normed space is homeomorphic with a bounded closed subset of a Hilbert space.

5.3. **The *CHB* property for UG spaces.** Say that a normed space E satisfies the *finite extension property* (for short *FEP*) if for every finite dimensional subspace F of E, and every linear mapping $f : F \to \mathbb{R}$, there exists $g \in E'$ extending f such that $\|g\| \leq \|f\|$. Say that the normed space E satisfies the *effective finite extension property* if there is a mapping associating to finite dimensional subspace F of E and every linear mapping $f : F \to \mathbb{R}$, a linear mapping $g : E \to \mathbb{R}$ extending f such that $\|g\| \leq \|f\|$.

Proposition 6. *Every Gâteaux-differentiable normed space satisfies the effective finite extension property.*

Proof. Given a Gâteaux-differentiable normed space E, a proper finite dimensional subset F of E, and a non-null linear mapping $f : F \to \mathbb{R}$, consider some point a in the sphere of F at which f reaches its least upper bound on the closed unit ball B_F. Since the subspace F is Gâteaux-differentiable, there exists a unique linear mapping $u \in S_{F'}$ such that $u(a) = 1$, namely $u = G(a, .)_{\restriction F}$. It follows that $f := \|f\| G(a, .)_{\restriction F}$ thus the linear continuous mapping $g := \|f\| G(a, .)$ extends f. □

Corollary 1. *Uniformly Gâteaux differentiable normed spaces satisfy the effective continuous Hahn-Banach property.*

Proof. Since E is uniformly Gâteaux-differentiable, $B_{E'}$ is closely convex-compact in the weak* topology (see Theorem 4). Moreover, the space E satisfies the *FEP* (see Proposition 6). It follows that E satisfies the effective *CHB* property (see [6, Theorem 6 p.13]). □

Question 3. Recall that in [4], we showed in **ZF+DC** that Gâteaux differentiable Banach spaces satisfy the *CHB* property. Is the *CHB* property for Gâteaux differentiable Banach spaces provable in $\mathbf{ZF} + \mathbf{AC}(\mathbb{N})$? in **ZF**? What about the *CHB* property for *Fréchet-differentiable* Banach spaces?

5.4. **Examples of UG renormings and applications.**

5.4.1. *UG renormings of* $\ell^0(I)$.

Example 3. The Banach space $\ell^0(I)$ has an equivalent norm which is uniformly Gâteaux differentiable. It follows that:

(i) The space $\ell^0(I)$ satisfies the effective *CHB* property.
(ii) Every bounded weak* closed convex subset of $\ell^1(I)$ is weak* convex-compact.
(iii) In **ZF**+$\mathbf{AC}(\mathbb{N})$, every bounded weak* closed subset of $\ell^1(I)$ is weak* compact.

Proof. The canonical mapping $\ell^2(I) \to \ell^0(I)$ is continuous and dense, so Proposition 9 (see Appendix) implies that $\ell^0(I)$ has a UG renorming. □

Remark 7. Results (i) and (ii) were already obtained in [6].

5.4.2. *Spaces* $L_1(E, \mathcal{B}, m)$. Given a set E, and boolean algebra $(\mathcal{B}, \cap, \cup, \varnothing, E)$ of subsets of E, a *measure* on $\mathcal{B}$ is a mapping $m : \mathcal{B} \to [0, +\infty]$ satisfying $m(\varnothing = 0$ and, for every $x, y \in \mathcal{B}$, $x \cap y = 0 \Rightarrow m(x \vee y) = m(x) + m(y)$. We denote by $\mathcal{E}$ the set of "simple functions" *i.e.* the (real) vector subspace of $\mathbb{R}^E$ generated by indicators of elements of $\mathcal{B}$. We define as usual the m-integral $\int_E f.dm$ of a *positive* simple function f, and the the m-integral of simple real functions f such that $\int_E |f|.dm < +\infty$. For every real number $p \in [1, +\infty[$, we denote by $\mathcal{E}_p$ the vector subspace $\{f \in \mathcal{E} : \int_E |f|^p.dm < +\infty\}$ of $\mathcal{E}$. Then the mapping $\mathcal{N}_p : \mathcal{E}_p \to \mathbb{R}_+$ associating to every $f \in \mathcal{E}_p$ the real number $(\int_E |f|^p.dm)^{1/p}$ is a semi-norm. We denote by $L_p(E, \mathcal{B}, m)$ the Banach space which is the separated Cauchy completion of the semi-normed space $(\mathcal{E}_p, \mathcal{N}_p)$ (which can be built in **ZF**). We also define $\mathcal{E}_\infty$ as follows: given $f = \sum_{1 \leq i \leq n} \lambda_i 1_{a_i} \in \mathcal{E}$ such that $(a_i)_{1 \leq i \leq n}$ is a partition of E in elements of $\mathcal{B}$, define $\mathcal{N}_\infty(f) := 0$ if $m(E) = 0$, and else, $\mathcal{N}_\infty(f) := \max\{|\lambda_i| :$

$1 \leq i \leq n$ and $m(a_i) > 0\}$. Then $\mathcal{E}_\infty := \{f \in \mathcal{E} : \mathcal{N}_\infty(f) < +\infty\}$ is a vector subspace of $\mathcal{E}$, and $(\mathcal{E}_\infty, \mathcal{N}_\infty)$ is a semi-normed space. We denote by $L_\infty(E, \mathcal{B}, m)$ the separated Cauchy completion of this semi-normed space.

Example 4. Let $(E, \mathcal{B}, m)$ be a measured boolean algebra of sets. Assume that $m(1_\mathcal{B}) < +\infty$, where $1_\mathcal{B}$ is the unit of $\mathcal{B}$. Then, the Banach space $L_1(\mathcal{B}, \mu)$ has an equivalent norm which is uniformly Gâteaux differentiable. It follows that:

(i) The space $L_1(\mathcal{B}, m)$ satisfies the effective *CHB* property.
(ii) Every bounded weak* closed convex subset of $L_\infty(\mathcal{B}, m)$ is convex-compact in the weak* topology.
(iii) In **ZF**+**AC**($\mathbb{N}$), every bounded weak* closed subset of $L_\infty(\mathcal{B}, m)$ is weak* compact.

Proof. Since $m(1_\mathcal{B}) < +\infty$, the canonical mapping $L_2(\mathcal{B}, m) \to L_1(\mathcal{B}, m)$ is continuous and dense, so Proposition 9 (see Appendix) implies that $L_1(\mathcal{B}, m)$ has a UG renorming. The continuous dual of $L_1(\mathcal{B}, m)$ is $L_\infty(\mathcal{B}, m)$, so (ii) and (iii) follow from Theorem 4. □

Remark 8. The previous Corollary cannot be extended to the case where $m(1_\mathcal{B}) = +\infty$ because $\ell^1(\mathbb{R})$ does not have an equivalent Gâteaux differentiable norm.

6. Appendix on UG spaces

6.1. **The dual ball of a UG space is W*UR.**

6.1.1. *An easy criterion for UG differentiability.*

Proposition 7 ([3])**.** *Let $(E, \|.\|)$ be a normed space. The following conditions are equivalent:*

(i) E is uniformly Gâteaux-differentiable.
(ii) For every $h \in S_E$, $\lim_{t->0} \frac{\|a+th\|+\|a-th\|-2}{t} = 0$ uniformly for $a \in S_E$

Proof. See [3, Lemma 6.6 p. 63] for a proof which is valid in **ZF**. □

6.1.2. *Proof of Proposition 5: "If a normed space E is uniformly Gâteaux-differentiable, then the dual norm of E' is W*UR.".*

Proof. Let $h \in S_E$. Let $\varepsilon \in]0, 1[$. Since E is uniformly Gâteaux-differentiable, Proposition 7 implies some $\eta_1 \in]0, 1[$ such that for every $a \in S_E$:

$$|t| \leq \eta_1 \Rightarrow \frac{\|a + th\| + \|a - th\| - 2}{|t|} < \varepsilon$$

Then

$$|t| \leq \eta_1 \Rightarrow \|a + th\| + \|a - th\| \leq 2 + \varepsilon|t|$$

Let $\eta := \varepsilon\eta_1$. Let $a \in S_E$ and $f, g \in C_{a,\eta}$. Then for every real number t satisfying $0 < |t| \leq \eta_1$,

$$f(a + th) + g(a - th) \leq \|a + th\| + \|a - th\| \leq 2 + \varepsilon|t|$$

In particular, for $t = \eta_1$ we have:

$$f(\eta_1 h) + g(-\eta_1 h) \leq 2 + \eta_1\varepsilon - f(a) - g(a) \leq 3\eta_1\varepsilon$$

thus $f(h) - g(h) \leq 3\varepsilon$. It follows that $\sup_{f,g\in C_{a,\eta}} \|f(h) - g(h)\| \leq 3\varepsilon$. □

6.2. **UG renormings.** The following Proposition is a slightly different version of [3, Prop. 6.2 p. 61]. Indeed, we do not use sequences but quantitative ε-η conditions:

Proposition 8. *Let E be a Banach space. The following conditions are equivalent :*

*(i) The dual norm on E' is W*UR.*

(ii) For every $h \in S_E$, and every $\varepsilon > 0$, there exists $\eta > 0$ satisfying for every $f, g \in B_{E'}$,

$$(2\|f\|^2 + 2\|g\|^2 - \|f+g\|^2 \leq \eta) \quad \Rightarrow |f(h) - g(h)| \leq \varepsilon \tag{1}$$

Proof. (i) $\Rightarrow$ (ii) Let $h \in S_E$ and $\varepsilon > 0$. Since the dual norm on E' is W*UR, let $\eta_1 > 0$ such that for every $f, g \in B_{E'}$, and every $a \in S_E$, $f, g \in C_{a,\eta_1} \Rightarrow |f(h) - g(h)| < \varepsilon$. Let $\eta \in]0, \frac{\varepsilon^2}{4}[$ such that $\frac{\varepsilon}{2} - \sqrt{\eta} > 0$, $1 - \frac{2\sqrt{\eta}}{\varepsilon}(1 + \frac{4}{\varepsilon^2}) - 4\frac{\eta}{\varepsilon} > 1 - \eta_1$ and $(1 - \frac{\sqrt{\eta}}{\varepsilon/2 - \sqrt{\eta}})(1 - 2\frac{2\eta + 4\frac{\sqrt{\eta}}{\varepsilon}}{\varepsilon}) > 1 - \eta_1$.

Assume that $f, g \in B_{E'}$ satisfy:

$$2\|f\|^2 + 2\|g\|^2 - \|f+g\|^2 \leq \eta \tag{2}$$

Equation (2) implies that

$$\|f\| \leq \|g\| + \sqrt{\eta} \text{ and } \|g\| \leq \|f\| + \sqrt{\eta} \tag{3}$$

and thus

$$\|\frac{f+g}{2}\| \leq \|f\| + \frac{\sqrt{\eta}}{2} \text{ and } \|\frac{f+g}{2}\| \leq \|g\| + \frac{\sqrt{\eta}}{2} \tag{4}$$

-If $\|f\| \leq \frac{\varepsilon}{2}$ and $\|g\| \leq \frac{\varepsilon}{2}$, then $|f(h) - g(h)| \leq |f(h)| + |g(h)| \leq \varepsilon$.

-Else, we may assume *w.l.g.* that $\|f\| > \frac{\varepsilon}{2}$ (whence $\|g\| \geq \frac{\varepsilon}{2} - \sqrt{\eta} > 0$). Since $\|f\| > \frac{\varepsilon}{2}$, by (2), $\|\frac{f+g}{2}\|^2 \geq \frac{\|f\|^2 + \|g\|^2}{2} - \frac{\eta}{4}$, and with (3), $\|\frac{f+g}{2}\|^2 \geq \frac{\|f\|^2 + (\|f\| - \sqrt{\eta})^2}{2} - \frac{\eta}{4} \geq \|f\|^2 - \sqrt{\eta} = \|f\|^2(1 - \frac{\sqrt{\eta}}{\|f\|^2})$. Since $\|f\| \geq \frac{\varepsilon}{2}$, we deduce that $\|\frac{f+g}{2}\|^2 \geq \|f\|^2(1 - \frac{4\sqrt{\eta}}{\varepsilon^2})$. It follows that

$$\|f+g\| \geq 2\|f\|\sqrt{1 - \frac{4\sqrt{\eta}}{\varepsilon^2}} \geq 2\|f\|(1 - 2\frac{\sqrt{\eta}}{\varepsilon^2})$$

We deduce that

$$\begin{aligned} \|f+g\| - \|g\| &\geq 2\|f\|(1 - 2\frac{\sqrt{\eta}}{\varepsilon^2}) - \|f\| - \sqrt{\eta} \\ &= \|f\| - \frac{4\|f\|\sqrt{\eta}}{\varepsilon^2} - \sqrt{\eta} \geq \|f\| - \frac{4\sqrt{\eta}}{\varepsilon^2} - \sqrt{\eta} \\ &= \|f\|(1 - \sqrt{\eta}(1 + \frac{4}{\varepsilon^2})\frac{1}{\|f\|}) \geq \|f\|(1 - \sqrt{\eta}(1 + \frac{4}{\varepsilon^2})\frac{2}{\varepsilon}) \end{aligned} \tag{5}$$

Let $a \in S_E$ such that $\frac{f+g}{2}(a) + \eta \geq \|\frac{f+g}{2}\|$; then $f(a) \geq \|f+g\| - g(a) - 2\eta \geq \|f+g\| - \|g\| - 2\eta \geq \|f\|(1 - \frac{2\sqrt{\eta}}{\varepsilon}(1 + \frac{4}{\varepsilon^2}) - 4\frac{\eta}{\varepsilon})$. By definition of η it follows that $\frac{f}{\|f\|} \in C_{a,\eta_1}$. On the other hand,

$$\begin{aligned} g(a) &\geq \|f+g\| - f(a) - 2\eta \geq \|f+g\| - \|f\| - 2\eta \\ &\geq \|f\| - 4\frac{\sqrt{\eta}}{\varepsilon^2} - 2\eta = \|f\|(1 - \frac{2\eta + 4\frac{\sqrt{\eta}}{\varepsilon}}{\|f\|}) \geq \\ &\|f\|(1 - 2\frac{2\eta + 4\frac{\sqrt{\eta}}{\varepsilon}}{\varepsilon}) \geq (\|g\| - \sqrt{\eta})(1 - 2\frac{2\eta + 4\frac{\sqrt{\eta}}{\varepsilon}}{\varepsilon}) \end{aligned} \tag{6}$$

thus $\frac{g(a)}{\|g\|} \geq (1 - \frac{\sqrt{\eta}}{\|g\|})(1 - 2\frac{2\eta + 4\frac{\sqrt{\eta}}{\varepsilon}}{\varepsilon}) \geq (1 - \frac{\sqrt{\eta}}{\varepsilon/2 - \sqrt{\eta}})(1 - 2\frac{2\eta + 4\frac{\sqrt{\eta}}{\varepsilon}}{\varepsilon})$. By definition of η, $\frac{g}{\|g\|} \in C_{a,\eta_1}$. Thus, using (2), $|f(h) - g(h)| \leq \varepsilon$.

(ii) $\Rightarrow$ (i) Let $h \in E$, $\varepsilon > 0$ and $\eta > 0$ such that every $f, g \in E'$ satisfy (1). Let $\lambda := \frac{\eta}{8}$. Given some a in the unit sphere of E, then for every $f, g \in C_{a,\lambda}$,

$$\frac{\|f\|^2 + \|g\|^2}{2} - \|\frac{f+g}{2}\|^2 \leq 1 - (1-\lambda)^2) = 2\lambda - \lambda^2 \leq 2\lambda$$

thus $2\|f\|^2 + 2\|g\|^2 - \|f+g\|^2 \leq 8\lambda = \eta$, thus using (1), $|f(h) - g(h)| < \varepsilon$. □

Proposition 9 ([3])**.** *Let E be a Banach space. If there exists a uniformly Gâteaux-differentiable normed space (for example a Hilbert space) H, and a continuous linear mapping $T : H \to E$ such that the vector subspace $Im(T)$ is dense in E, then E admits an equivalent norm which has a W*UR dual norm (and thus E has an equivalent norm which is uniformly Gâteaux-differentiable).*

Proof. See [3, p. 65-66]. For sake of completeness, we give the proof which is valid in **ZF**. We denote by $T' : F' \to E'$ the *transpose* of T associating to each $g \in F'$ the mapping $g \circ T$. Without loss of generality, we may assume that $\|T\| \leq 1$, thus $\|T'\| \leq 1$. Consider the norm $|||.|||$ on E' such that for every $f \in E'$:

$$|||f|||^2 := \|f\|_{E'}^2 + \left\|T'(f)\right\|_{H'}^2$$

This norm on E' is equivalent to the dual norm of E' because T' is continuous. Moreover, this equivalent norm $|||.|||$ is a dual norm because its closed unit ball is weak* closed in E' (the mapping $|||.|||$ is weak* lower continuous). We now show that this dual norm $|||.|||$ is W*UR. We use Proposition 8. Let $k \in E$ and $\varepsilon > 0$. We seek for some real number $\eta > 0$ such that for every $f, g \in B_{E'}$, satisfying $2|||f|||^2 + 2|||g|||^2 - |||f+g|||^2 < \eta$, $|f(k) - g(k)| \leq \varepsilon$. Since $Im(T)$ is dense in E, we may assume that $k = T(h)$ where $h \in H$. Since H is W*UR, let $\eta > 0$ such that for every $u, v \in B_{H'}$,

$$(2\|u\|^2 + 2\|v\|^2 - \|u+v\|^2 < \eta) \Rightarrow |u(h) - v(h)| \leq \varepsilon \tag{7}$$

Let $f, g \in B_{E'}$ such that

$$2|||f|||^2 + 2|||g|||^2 - |||f+g|||^2 < \eta$$

Then the two following inequalities hold:

$$2\|f\|^2 + 2\|g\|^2 - \|f+g\|^2 < \eta \text{ and } 2\left\|T'(f)\right\|^2 + 2\left\|T'(g)\right\|^2 - \left\|T'(f+g)\right\|^2 < \eta$$

The second inequality and (7) imply that

$$|T'(f)(h) - T'(g)(h)| \leq \varepsilon \text{ i.e. } |f(k) - g(k)| \leq \varepsilon$$

□

References

[1] E. Albius and M. Morillon. Uniform smoothness entails Hahn-Banach. *Quaestiones Mathematicae*, 24:425–439, 2001.

[2] H. Brézis. *Analyse fonctionnelle. Théorie et applications.* Masson, 1983.

[3] R. Deville, G. Godefroy, and V. Zizler. *Smoothness and renormings in Banach spaces*, volume 64 of *Pitman Monographs and Surveys in Pure and Applied Mathematics.* Longman Scientific & Technical, Harlow, 1993.

[4] J. Dodu and M. Morillon. The Hahn-Banach property and the axiom of choice. *Math. Log. Q.*, 45(3):299–314, 1999.

[5] M. Fabian, G. Godefroy, and V. Zizler. The structure of uniformly Gateaux smooth Banach spaces. *Israel J. Math.*, 124:243–252, 2001.

[6] Fossy, J. and Morillon, M. The Baire category property and some notions of compactness. *J. Lond. Math. Soc., II. Ser.*, 57(1):1–19, 1998.

[7] P. Howard and J. E. Rubin. *Consequences of the Axiom of Choice.*, volume 59. American Mathematical Society, Providence, RI, 1998.

[8] T. J. Jech. *The Axiom of Choice.* North-Holland Publishing Co., Amsterdam, 1973.

[9] W. A. J. Luxemburg and M. Väth. The existence of non-trivial bounded functionals implies the Hahn-Banach extension theorem. *Z. Anal. Anwendungen*, 20(2):267–279, 2001.

[10] Morillon, M. Countable choice and compactness. *submitted*, 2007.
[11] D. Pincus. Independence of the prime ideal theorem from the Hahn Banach theorem. *Bull. Amer. Math. Soc.*, 78:766–770, 1972.
[12] D. Pincus. The strength of the Hahn-Banach theorem. In *Victoria Symposium on Nonstandard Analysis (Univ. Victoria, Victoria, B.C., 1972)*, pages 203–248. Lecture Notes in Math., Vol. 369. Springer, Berlin, 1974.

ERMIT, Département de Mathématiques et Informatique, Université de La Réunion, 15 avenue René Cassin - BP 7151 - 97715 Saint-Denis Messag. Cedex 9 FRANCE

E-mail address, Marianne Morillon: mar@univ-reunion.fr

URL: http://personnel.univ-reunion.fr/mar