A CHOICELESS VERSION OF THE HAHN-BANACH THEOREM FOR UNIFORMLY GÂTEAUX DIFFERENTIABLE NORMED SPACES

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Abstract. Denoting by AC(\(N\)) the countable axiom of choice, we show in ZF+AC(\(N\)) that the dual ball of a uniformly Gâteaux-differentiable Banach space is compact in the weak* topology. In ZF, we prove that this dual ball is (closely) convex-compact in the weak* topology. We deduce that uniformly Gâteaux-differentiable Banach spaces satisfy the continuous Hahn-Banach property in ZF. This enhances a result previously obtained in [1] for uniformly Fréchet differentiable Banach spaces.

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1. Introduction

1.1. Presentation of the results. We work in ZF, Zermelo-Fraenkel set-theory without the Axiom of Choice (for short AC). Given a real vector space $E$, a mapping $p : E \to \mathbb{R}$ is sub-linear if for every $x, y \in E$, and every $\lambda \in \mathbb{R}_+$, $p(x + y) \leq p(x) + p(y)$ (sub-addivity), and $p(\lambda x) = \lambda p(x)$ (positive homogeneity). Consider the “Hahn-Banach axiom”, a well known consequence of AC:

**HB:** Let $E$ be a (real) vector space. If $p : E \to \mathbb{R}$ is a sub-linear mapping, if $F$ is a vector subspace of $E$, if $f : F \to \mathbb{R}$ is a linear mapping such that $f \leq p|_F$, then there exists a linear mapping $g : E \to \mathbb{R}$ extending $f$ such that $g \leq p$.

It is known that HB is not provable in ZF (see [12], [8], [7]), and that HB does not imply AC (HB does not even imply that every sequence $(P_n)_{n \in \mathbb{N}}$ of pairs has a non-empty product -see [11], [8], [7]-). Given a real topological vector space $E$ (i.e. $E$ is a real vector space such that the “sum” $+: E \times E \to E$ and the external multiplicative law $\cdot : \mathbb{R} \times E \to E$ are continuous for the product topology), say that $E$ satisfies the Continuous Hahn-Banach property (for short CHB property) if “For every continuous sub-linear mapping $p : E \to \mathbb{R}$, for every vector subspace $F$ of $E$, if $f : F \to \mathbb{R}$ is a linear mapping such that $f \leq p|_F$, then there exists a linear mapping $g : E \to \mathbb{R}$ extending $f$ such that $g \leq p$.” Although statement HB is not provable in ZF, however, some real normed spaces satisfy (in ZF) the CHB property: for example normed spaces with a well-orderable dense subset (in particular separable normed spaces), but also Hilbert spaces, spaces $\ell^0(I)$ (see [6]), uniformly convex Banach spaces with a Gâteaux–differential norm ([4]), uniformly Fréchet differentiable Banach spaces (see [1]). The aim of this paper is to prove (in ZF) that every uniformly Gâteaux-differentiable Banach space $E$ satisfies the following effective CHB property (see Corollary 1 in Section 5.3): “There is a mapping $\Psi$ associating to every $(p, F, f)$ where $p : E \to \mathbb{R}$ is a continuous sub-linear mapping, $F$ is a vector subspace of $E$, and $f : F \to \mathbb{R}$ is a linear mapping such that $f \leq p|_F$, a linear mapping $g := \Psi(p, F, f) : E \to \mathbb{R}$ extending $f$ such that $g \leq p$.” This enhances the result that we previously obtained in [1] for uniformly Fréchet differentiable Banach spaces, and this answers Question 3 of [1]. Notice that given a set $I$, $\ell^0(I)$ has an equivalent norm which is uniformly Gâteaux–differentiable (see Section 5.4.1), but if $I$ is infinite, then $\ell^0(I)$ has no uniformly Fréchet differentiable equivalent norm (see Remark 4).

Consider the countable Axiom of Choice, which is not provable in ZF, and which does not imply AC (see [7], [8]):

**AC(\mathbb{N}):** If $(A_n)_{n \in \mathbb{N}}$ is a family of non-empty sets, then there exists a mapping $f : \mathbb{N} \to \bigcup_{n \in \mathbb{N}}A_n$ associating to every $k \in \mathbb{N}$ an element $f(k) \in A_k$.

In this paper, we first provide in ZF+AC(\mathbb{N}) a criterion of compactness for certain complete gauge spaces satisfying a “smallness” property with respect to an extra metric (see Theorem 1 in Section 2.3.4). This geometric idea leads us to the fact that (see Theorem 3 in Section 4.3), given a normed space $E$ with a weak* uniformly rotund dual ball, then, in ZF+AC(\mathbb{N}), the dual ball $B_{E'}$ of $E$ is weak* compact (and this is not provable in ZF -see Remark 3), and in ZF this dual
ball is (closely) weak* convex-compact (see Definition in Section 4.3). It follows that if a normed space $E$ is uniformly Gâteaux-differentiable, then its dual ball is weak* compact in $\text{ZF} + \text{AC}(\mathbb{N})$, and (closely) convex-compact in $\text{ZF}$ (see Theorem 4 in Section 5.2). Using a result of [6] linking the CHB property on the normed space $E$ and the weak* convex-compactness of the dual ball of $E$, we deduce our main result: “Uniformly Gâteaux-differentiable normed spaces satisfy the CHB property”.

1.2. Some weak forms of AC. We now recall some weak forms of the Axiom of Choice which will be used in this paper and the known links between them. For detailed references and much information on various weak forms of the Axiom of Choice, see [8] and [7].

1.2.1. DC, AC($\mathbb{N}$) and AC($\mathbb{N}$, fin). The axiom of Dependent Choices asserts that:

- **DC**: Given a non-empty set $X$ and a binary relation $R$ on $X$ such that $\forall x \in X \exists y \in X\, xRy$, there exists a sequence $(x_n)_{n \in \mathbb{N}}$ of $X$ such that for every $n \in \mathbb{N}$, $x_nRx_{n+1}$.

The countable Axiom of Choice for finite sets says that:

- **AC($\mathbb{N}$, fin)**: If $(A_n)_{n \in \mathbb{N}}$ is a family of finite non-empty sets, then there exists a mapping $f : \mathbb{N} \to \bigcup_{n \in \mathbb{N}} A_n$ associating to every $n \in \mathbb{N}$ an element $f(n) \in A_n$.

Of course, $\text{AC} \Rightarrow \text{DC} \Rightarrow \text{AC}(\mathbb{N}) \Rightarrow \text{AC}(\mathbb{N}, \text{fin})$. However, the converse statements are not provable in $\text{ZF}$, and $\text{AC}(\mathbb{N}, \text{fin})$ is not provable in $\text{ZF}$ (see references in [7]).

2. A criterion of compactness

2.1. Filters.

2.1.1. Filters in lattices of sets. Given a set $X$, a lattice of subsets of $X$ is a subset $\mathcal{L}$ of $\mathcal{P}(X)$ containing $\varnothing$ and $X$, which is closed by finite intersections and finite unions. A filter of the lattice $\mathcal{L}$ is a non-empty proper subset $\mathcal{F}$ of $\mathcal{L}$ such that for every $A, B \in \mathcal{L}$:

(i) $(A, B \in \mathcal{F}) \Rightarrow A \cap B \in \mathcal{F}$
(ii) $(A \in \mathcal{F}$ and $A \subseteq B) \Rightarrow B \in \mathcal{F}$

A subset $A$ of $\mathcal{L}$ is contained in a filter of $\mathcal{L}$ if and only if $A$ satisfies the finite intersection property (for short FIP), that is, every finite subset of $A$ has a non-empty intersection; in this case, the intersection of all filters of $\mathcal{L}$ containing $A$ is a filter that we denote by $\text{fil}(A)$ (which is the smallest filter of $\mathcal{L}$ containing $A$, alias the “filter generated by $A$”).

2.1.2. Stationary sets. Given a filter $\mathcal{F}$ of a lattice $\mathcal{L}$ of subsets of a set $X$, an element $S \in \mathcal{L}$ is $\mathcal{F}$-stationary if for every $A \in \mathcal{F}$, $A \cap S \neq \varnothing$. The set $S(\mathcal{F})$ of $\mathcal{F}$-stationary sets satisfies the following properties:

(i) If $\mathcal{A}$ is a chain of $\mathcal{L}$ and if $\mathcal{A} \subseteq S(\mathcal{F})$, then $\mathcal{A} \cup \mathcal{F}$ satisfies the FIP.
(ii) If $F_1, \ldots, F_m \in \mathcal{L}$ and $F_1 \cup \cdots \cup F_m \in S(\mathcal{F})$, then there exists some $i_0 \in \{1..m\}$ such that $F_{i_0}$ is $\mathcal{F}$-stationary.

2.2. Gauge spaces.

2.2.1. Pseudo-metrics. Given a set $X$, a pseudo-metric on $X$ is a mapping $d : X \times X \to \mathbb{R}_+$ satisfying the following properties for every $x, y, z \in X$:

(i) $d(x, x) = 0$
(ii) $d(x, y) = d(y, x)$
(iii) $d(x, z) \leq d(x, y) + d(y, z)$
Given $a \in X$ and some pseudo-metric $d : X \times X \rightarrow \mathbb{R}_+$, given real numbers $R, R'$ satisfying $R \leq R'$, we define large $d$-balls and large $d$-crowns as follows:

$$B_d(a, R) := \{ x \in X : d(a, x) \leq R \}$$
$$D_d(a, R, R') := \{ x \in X : R \leq d(a, x) \leq R' \}$$

In an analogous way, we also define strict $d$-balls with strict inequalities. If $A$ is a non-empty subset of $X$, we define the $d$-diameter of $A$:

$$\text{diam}_d(A) := \inf \{ d(x, y) : x, y \in A \}$$

2.2.2. Complete gauge spaces. A gauge space is a set $X$ endowed with a family $(d_i)_{i \in I}$ of pseudo-metrics on $X$. Every gauge space naturally defines a topology on $X$ generated by the various $d_i$-strict balls, $i \in I$. Say that a subset $A$ of non-empty subsets of the gauge space $(X, (d_i)_{i \in I})$ is Cauchy if for every $\varepsilon > 0$ and for every $i \in I$, there exists $F \in A$ such that $\text{diam}_{d_i}(F) < \varepsilon$. A gauge space $(X, (d_i)_{i \in I})$ is said to be complete if every Cauchy filter of the lattice of closed subsets of $X$ has a non-empty intersection.

**Example 1 (Closed subsets of $\mathbb{R}^I$).** Let $I$ be a set. For each $i \in I$, consider the canonical projection $p_i : \mathbb{R}^I \rightarrow \mathbb{R}$, and the pseudo-metric $d_i : X \times X \rightarrow \mathbb{R}$ associating to each $(x, y) \in X \times X$ the real number $|p_i(x) - p_i(y)|$. Then, the gauge space $(\mathbb{R}^I, (d_i)_{i \in I})$ is complete. It follows that for every closed subset $A$ of $\mathbb{R}^I$, the gauge space $(A, (d_i|_{A \times A})_{i \in I})$ is also complete.

2.3. A criterion of compactness in ZF+AC(\mathbb{N}).

2.3.1. Compactness.

**Definition 1** (C-compactness, closed C-compactness). Given a class $\mathcal{C}$ of subsets of a set $X$, say that a subset $A$ of $X$ is $\mathcal{C}$-compact if for every family $(C_i)_{i \in I}$ of $\mathcal{C}$ such that $(C_i \cap A)_{i \in I}$ satisfies the FIP, $A \cap \bigcap_{i \in I} C_i$ is non-empty; say that $A$ is closely $\mathcal{C}$-compact if there is a mapping associating to every family $(C_i)_{i \in I}$ of $\mathcal{C}$ such that $(C_i \cap A)_{i \in I}$ satisfies the FIP, an element of $A \cap \bigcap_{i \in I} C_i$.

Recall that a topological space $X$ is compact if $X$ is $\mathcal{C}$-compact, where $\mathcal{C}$ is the set of closed subsets of $X$ (or, equivalently, every filter of the lattice of closed sets of $X$ has a non-empty intersection).

2.3.2. Sub-basis of closed sets.

**Definition 2** (basis, sub-basis of closed subsets). A set $\mathcal{B}$ of closed subsets of a topological space $X$ is a basis of closed sets if every closed set of $X$ is an intersection of elements of $\mathcal{B}$. A set $\mathcal{S}$ of closed subsets of $X$ is a sub-basis of closed sets if the set $\mathcal{B}$ of finite unions of elements of $\mathcal{S}$ is a basis of closed subsets of $X$.

The following result is easy.

**Proposition 1.** Let $X$ be a topological space, and $\mathcal{L}$ be a lattice of closed subsets of $X$ which is also a basis of closed subsets of $X$. If every filter of $\mathcal{L}$ has a non-empty intersection, then $X$ is compact.

2.3.3. Property of smallness.

**Definition 3** (smallness in thin crowns). Let $d, d'$ be two pseudo-metrics on a set $X$. Let $a \in X$. Say that a set $\mathcal{C}$ of subsets of $X$ satisfies the property of $d'$-smallness in thin $d$-crowns centered at $a$ if for every $R \in \mathbb{R}_+$, for every $\varepsilon > 0$ there exists $\eta \in [0, R]$ such that for every $C \in \mathcal{C}$,

$$C \subseteq D_d(a, R - \eta, R + \eta) \Rightarrow \text{diam}_{d'}(C) < \varepsilon$$

If $(X, (d_i)_{i \in I})$ is a gauge space, say that $\mathcal{C}$ satisfies the property of $(d_i)_{i \in I}$-smallness in thin $d$-crowns centered at $a$ if for every $i \in I$, $\mathcal{C}$ satisfies the property of $d_i$-smallness in thin $d$-crowns centered at $a$. 


2.3.4. Criterion of compactness.

**Theorem 1.** Let \((X, (d_i)_{i \in I})\) be a complete gauge space. Let \(T\) be a topology on \(X\) which is included in the associated gauge topology, and let \(C\) be a sub-basis of closed sets of \((X, T)\) which is closed by finite intersections. Let \(a \in X\). Let \(d : X \times X \to \mathbb{R}_+\) be a pseudo metric such that \(d\)-large balls centered at \(a\) belong to \(C\), and such that \(T\) is included in the topology \(T_d\) associated to \(d\). If \(C\) satisfies the property of \((d_i)_{i \in I}\)-smallness in thin \(d\)-crowns centered at \(a\), then:

(i) In \(\text{ZF} + \text{AC} (\mathbb{N})\), every large \(d\)-ball with center \(a\) (and thus, every \(d\)-bounded \(T\)-closed subset of \(X\)) is compact in \(T\).

(ii) In \(\text{ZF}\), every every large \(d\)-ball with center \(a\) (and thus, every \(d\)-bounded element of \(C\)) is \(C\)-compact. Moreover, if the gauge space \((X, (d_i)_{i \in I})\) is Hausdorff, then every \(d\)-bounded element of \(C\) is closely \(C\)-compact.

**Proof.** Let \(L\) be the lattice generated by \(C\). Let \(\rho > 0\) and let \(B\) be the large \(d\)-ball \(B_d(a, \rho)\).

(i) Let \(\mathcal{F}\) be a filter of \(L\) containing \(B\). Let us prove in \(\text{ZF} + \text{AC} (\mathbb{N})\) that \(\cap \mathcal{F}\) is non-empty (using Proposition 1, this will imply that \(B\) is \(T\)-compact). Let \(R := \inf \{r \in \mathbb{R}_+ : B_d(a, r) \in S (\mathcal{F})\}\). Since \(\{B_d(a, r) : r > R\}\) is a chain of \(\mathcal{F}\)-stationary sets of \(L\), the set \(\mathcal{F} \cup \{B_d(a, r) : r > R\}\) generates a filter \(\mathcal{G}\) of \(L\) (see Section 2.1.2-(i)). If \(R = 0\) then \(a \in \cap \mathcal{F}\) (because elements of \(\mathcal{F}\) are \(T_d\)-closed).

Assume that \(R > 0\). For every \(\varepsilon > 0\), there exists some element of \(\mathcal{G}\) which is included in the crown \(D_d(a, R - \varepsilon, R + \varepsilon)\); with \(\text{AC} (\mathbb{N})\), choose for every \(n \in \mathbb{N}\), a finite subset \(Z_n\) of \(C\) such that \(\cup Z_n \in \mathcal{G}\) and \(\cup Z_n \subseteq D_d(a, R - \varepsilon, R + \varepsilon)\). With \(\text{AC} (\mathbb{N}, \text{fin})\), the set \(\bigcup_{n \in \mathbb{N}} Z_n\) is countable. We define by induction a sequence \((C_n)_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} Z_n\) such that for every \(n \in \mathbb{N}\), \(\mathcal{G} \cup \{C_i : i < n\}\) generates a filter \(\mathcal{G}_n\) and \(C_n \in S (\mathcal{G}_n)\); given some \(n \in \mathbb{N}\), \(\cup Z_n \subseteq \mathcal{G} \subseteq \text{fil} (\mathcal{G}, (C_i)_{i < n}) \subseteq S (\text{fil} (\mathcal{G}, (C_i)_{i < n}))\); using Section 2.1.2-(i), it follows that there exists \(C_n \in Z_n\) satisfying \(C_n \in S (\text{fil} (\mathcal{G}, (C_i)_{i < n}))\). Since \(C\) satisfies the property of \((d_i)_{i \in I}\)-smallness in thin \(d\)-crowns centered at \(a\), the filter \(\mathcal{H} := \bigcup_{n \in \mathbb{N}} \mathcal{G}_n\) is Cauchy in the gauge space \((X, (d_i)_{i \in I})\). Since this gauge space is complete, and since its topology contains \(T\), \(\cap \mathcal{H}\) is non-empty. Thus \(\emptyset \neq \cap \mathcal{G} \subseteq \cap \mathcal{F}\).

(ii) Let \(A\) be subset of non-empty elements \(C\) which is closed by finite intersection and such that the ball \(B\) belongs to \(A\). Let us show (in \(\text{ZF}\)) that \(\cap A\) is non-empty. Let \(\mathcal{F}\) be the filter of \(L\) generated by \(A\). Let \(R := \inf \{r \in \mathbb{R}_+ : B_d(a, r) \in S (\mathcal{F})\}\). Denote by \(A'\) the set \(\{A \cap B_d(a, r) : A \in \mathcal{A} \text{ and } r > R\}\). If \(R = 0\) then \(a \in \cap \mathcal{A}\). If \(R > 0\), then for every \(\varepsilon > 0\), there exists some element of \(A'\) which is included in the crown \(D_d(a, R - \varepsilon, R + \varepsilon)\). Since \(C\) satisfies the property of \((d_i)_{i \in I}\)-smallness in thin \(d\)-crowns centered at \(a\), the family \(A'\) is Cauchy in the gauge space \((X, (d_i)_{i \in I})\); since this uniform space is complete and since \(A'\) satisfies the FIP, \(\cap A'\) is non-empty so \(\emptyset \neq \cap A' \subseteq \cap \mathcal{A}\). Moreover, if the gauge space \((X, (d_i)_{i \in I})\) is Hausdorff, then \(\cap A'\) is a singleton \(\{a\}\) which is \(\text{ZF}\)-definable from \((X, (d_i)_{i \in I})\), \(d\) and \(A\).

\(\square\)

3. The continuous dual of a normed space

All vector spaces that we consider are defined over the field \(\mathbb{R}\) of real numbers.

3.1. Continuous dual and bidual of a normed space. Given a normed space \(E\) endowed with a norm \(|x|\), we denote by \(B_E\) the large unit ball \(\{x \in E : |x| \leq 1\}\), and by \(S_E\) the unit sphere of \(E\). The topology on \(E\) associated to the norm is called the **strong topology**. The vector space \(E'\) endowed with the dual norm \(|x|^*\) is the **continuous dual** of the normed space \(E\). We also denote by \(E''\) the **continuous bidual** of \(E\). i.e. the continuous dual of \(E'\). Let \(\text{can} : E \to E''\) be the **canonical mapping** associating to every \(x \in E\) the "evaluating mapping" \(\tilde{x} : E' \to \mathbb{R}\), satisfying for every \(f \in E'\) the equality \(\tilde{x}(f) = f(x)\). A Banach space is a normed space which is (Cauchy)-complete for the metric associated to the norm (i.e. every Cauchy filter of closed sets has a non-empty intersection). The weak topology \(\sigma(E, E')\) on the normed space \(E\) is the weakest topology on \(E\) such that elements \(f \in E'\) are continuous.
Remark 1. In a model of \( \text{ZF} \) where \( \text{HB} \) fails, there exists a non null (infinite dimensional) normed space \( E \) such that \( E' = \{0\} \) (see [6, Lemma 5] or [9]). In such a model of \( \text{ZF} \), the weak topology on \( E \) is trivial with only two open sets.

3.2. The weak* topology on the continuous dual. Given a normed space \( E \) and a vector subspace \( F \) of \( E' \), we denote by \( \sigma(E,F) \) the weakest topology on \( E \) such that every \( f \in F \) is continuous on \( E \) endowed with this topology. Then \( E \) endowed with this topology is a locally convex topological vector space. Given a normed space \( E \), the weak* topology on \( E' \) is the topology \( \sigma(E',\text{can}[E]) \) where \( \text{can} : E \to E'' \) is the canonical mapping. This topology is also denoted by \( \sigma(E',E) \). Notice that the locally convex tvs \( (E',\sigma(E',E)) \) is Hausdorff (in \( \text{ZF} \)). Moreover, strongly closed balls of \( E' \) are weak* closed (in \( \text{ZF} \)). For every \( h \in E \), consider the pseudo-metric \( d_h : E' \times E' \to \mathbb{R} \) associating to every \( (f,g) \in E' \times E' \) the real number \( |f(h) - g(h)| \). The topology associated to the gauge space \( (E',\{d_h\}_{h \in E}) \) is the weak* topology \( \sigma(E',E) \) on \( E' \). This system \( (E',\{d_h\}_{h \in E}) \) is the canonical gauge space associated to the weak* topology on \( E' \).

Proposition 2. Let \( E \) be a normed space. The closed unit ball \( B_{E'} \) of \( E' \) is complete in the canonical gauge space \( (E',\{d_h\}_{h \in E}) \) associated to the weak* topology \( \sigma(E',E) \) on \( E' \).

Proof. Consider the mapping \( h : B_{E'} \to [-1,1]^{B_E} \) associating to each \( f \in B_{E'} \) the family \( \{f(x)\}_{x \in B_E} \). The range \( R := h[B_{E'}] \) of \( h \) is a closed subset of \([-1,1]^{B_E} \), whence it is a complete gauge subspace of \([-1,1]^{B_E} \). Moreover \( h : B_{E'} \to h[B_{E'}] \) is an isomorphism of gauge spaces, thus it follows that the gauge space \( B_{E'} \) is also complete. \( \square \)

3.3. Polyhedras. Given a vector space \( E \), and a vector subspace \( F \) of the algebraic dual \( E^* \) of \( E \), a strict \( F \)-hemi-space (resp. large \( F \)-hemi-space) of \( E \) is a subset of \( E \) of the form \( \{f < \lambda\} \) (resp. \( \{f \leq \lambda\} \)) where \( f \in F \setminus \{0\} \) and \( \lambda \in \mathbb{R} \). A strict \( F \)-polyhedra (resp. large \( F \)-polyhedra) of \( E \) is a finite intersection of strict (resp. large) \( F \)-hemi-spaces. If \( E \) is a topological vector space and if \( F = E' \), then \( F \)-hemi-spaces (resp. \( F \)-polyhedras) are also called hemi-spaces of \( E \) (resp. polyhedras of \( E \)). If \( E \) is a normed space, we distinguish on \( E' \):

- polyhedras of \( E' \) endowed with the strong topology (they also are the polyhedras of \( E' \) endowed with the weak topology)
- polyhedras of \( E' \) endowed with the weak* topology, which we also call \( \ast \)-polyhedras.

Recall the following Proposition, which is choiceless:

Proposition 3. Given a Banach space \( E \), weak* continuous linear mappings \( \phi : E' \to \mathbb{R} \) are evaluation mappings \( \check{a} : E' \to \mathbb{R} \) for \( a \in E \).

Proof. See for example [2]. \( \square \)

3.4. Separating a convex set and a strict polyhedra.

Theorem 2. Let \( E \) be a topological vector space. If \( C \) is a convex subset of \( E \), and if \( P \) is a strict polyhedra of \( E \) disjoint from \( C \), then there exists \( f \in E' \) such that \( f[P] < f[C] \).

Proof. Let \( f_1, \ldots, f_m \in E' \setminus \{0\} \) and \( \alpha_1, \ldots, \alpha_m \in \mathbb{R} \) such that \( P = \bigcap_{i=1}^{m} \{x \in E : f_i(x) < \alpha_i\} \). Let \( F = (f_i)_{1 \leq i \leq m} : E \to \mathbb{R}^m \). Let \( V := \ker(F) = \bigcap_{i=1}^{m} \ker(f_i) \) and let \( \text{can} : E \to E/V \) be the canonical mapping: the mapping \( \text{can} \) is open. Since the vector subspace \( V \) is closed, the quotient \( E/V \) is Hausdorff; moreover, the vector space \( E/V \) is finite dimensional (because \( E/V \) is isomorphic to the finite dimensional space \( F[E] \)). It follows that the tvs \( E/V \) satisfies the various classical geometrical Hahn-Banach properties. Let \( K := \text{can}[C] \) and \( U := \text{can}[P] \). The convex subsets \( K \) and \( U \) are disjoint in \( E/V \) (by definition of the mappings \( f_i \)) and \( U \) is open in \( E/V \) (because \( \text{can} \) is open). Using a geometric form of Hahn-Banach in \( E/V \), there exists \( g \in (E/V)' \) such that \( g[U] < g[K] \). Let \( f := g \circ \text{can} \). Then \( f[P] < f[C] \). \( \square \)
Example 2. Let $E$ be a normed space, let $C$ be a convex subset of $E'$. If $P$ is a strict *-polyhedron of $E'$ which is disjoint from $C$, then there exists $a \in E$ such that $\tilde{a}[C] < \tilde{a}[P]$.

Proof. Apply Theorem 2 to the (Hausdorff) topological vector space $E'$ endowed with the $\sigma(E', E)$-topology. \hfill $\square$

4. W*UR dual norms

4.1. W*UR dual norm.

Notation 1. Given a normed space $E$, for every $a \in S_E$ and $\eta \in [0, 1]$, we denote by $C_{a, \eta}$ the following weak* closed convex subset of the dual ball $B_{E'}$:

$$C_{a, \eta} := \{ f \in B_{E'} : f(a) \geq 1 - \eta \}$$

Given a normed space $E$, say that the dual norm $\|\|^{*}$ on $E'$ is weak* uniformly rotund (for short W*UR) when for every $h \in E$,

$$\lim_{\eta \to 0} \limsup_{f, g \in C_{a, \eta}} |f(h) - g(h)| = 0 \quad \text{uniformly for } a \in S_E$$

This means that for every $h \in E$, and every $\varepsilon > 0$, there exists some $\eta > 0$ such that for every $a \in S_E$, $\text{diam}_{E'}(C_{a, \eta}) < \varepsilon$.

Remark 2. In $\text{ZF} + \text{AC}(\mathbb{N})$, the above Definition is equivalent to the Definition given in [3, Def. 6.1 p. 61] relying on sequences of $E'$.

4.2. Property of smallness.

Proposition 4. Let $E$ be a normed space. Let $d$ be the distance given by the dual norm on $E'$. Let $(d_{h})_{h \in E}$ be the canonical gauge space of the weak* topology on $E'$. Let $\mathcal{P}$ be the set of large *-polyhedra of $E'$. Let $\mathcal{P}_h := \{ P \cap B(0, r) : P \in \mathcal{P} \text{ and } r \in \mathbb{R}_+ \}$. If $E$ has a W*UR dual norm, then $\mathcal{P}_h$ has the property of $(d_{h})_{h \in E}$-smallness in thin $d$-crowns centered at $0_{E'}$.

Proof. Let $h \in E$. Let $\varepsilon > 0$. Since the dual norm of $E'$ is W*UR, let $\eta > 0$ such that for every $a \in S_E$, $\text{diam}_{E'}(C_{a, \eta}) \leq \varepsilon$. If $P \in \mathcal{P}$ and if $P \cap B_d(0, 1 - \eta) = \emptyset$, then, using a $\text{ZF}$-provable consequence of HB (see Example 2 in Section 3.4), there exists $a \in S_E$ such that $\tilde{a}[B(0, 1 - \eta)] < \tilde{a}[P]$. It follows that $(P \cap B_{E'}) \subseteq C_{a, \eta}$, thus $\text{diam}_{E'}(P \cap B_{E'}) \leq \varepsilon$. \hfill $\square$

4.3. Compactness of $\text{W*UR}$ dual balls. Given a topological vector space $E$, say that a subset $A$ of $E$ is convex-compact if $A$ is $C$-compact where $C$ is the set of closed convex subsets of $E$.

Theorem 3. Let $(E, \|\|)$ be a normed space. Assume that the dual norm on $E'$ is weak* uniformly rotund.

(i) The axiom AC($\mathbb{N}$) implies that the closed unit ball of $E'$ is compact in the weak* topology.

(ii) The closed unit ball of $E'$ is convex-compact in the weak* topology.

Proof. Let $\mathcal{P}$ be the set of large *-polyhedra of $E'$. Let $d$ be the metric associated to the norm of $E'$. Let $\mathcal{C} := \{ P \cap B_d(0, r) : P \in \mathcal{P} \text{ and } 0 \leq r \leq 1 \}$. Then $\mathcal{C}$ is a sub-basis of the weak* topology on $B_{E'}$, which is closed by finite intersection. Let $(d_{h})_{h \in E}$ be the canonical gauge space of the weak* topology on $E'$: the subset $B_{E'}$ of the gauge space $(E', (d_{h})_{h \in E})$ is complete (see Proposition 2). Large $d$-balls centered at $0_{E'}$ belong to $\mathcal{C}$ and, by Proposition 4, the class $\mathcal{C}$ satisfies the property of $(d_{h})_{h \in E}$-smallness in thin $d$-crowns of $B_{E'}$ centered at $0_{E'}$.

(i) In $\text{ZF} + \text{AC}(\mathbb{N})$, we apply Theorem 1-(i) to the class $\mathcal{C}$, and it follows that $B_{E'}$ is weak* compact.

(ii) Using Theorem 1-(ii), and since the gauge space $(E', (d_{h})_{h \in E})$ is Hausdorff, the closed unit ball of $E'$ is convex-compact. \hfill $\square$

Remark 3. Consider the following two statements:
A\textsubscript{w*ur} Given a normed space such that the dual norm is weak* uniformly rotund, then the dual ball is weak* compact.

AH: The closed unit ball of a Hilbert space is weakly compact.

It is easy to see that A\textsubscript{w*ur} implies AH. Moreover it is known (see [6]) that AH implies AC(\mathbb{N}, \text{fin}). Since AC(\mathbb{N}, \text{fin}) is not provable in ZF, it follows that A\textsubscript{w*ur} is not provable in ZF.

Question 1. Does AH imply A\textsubscript{w*ur}?

5. UG differentiability yields Hahn-Banach

5.1. Uniform Gâteaux differentiability. Recall that (see [3, Def 6.5 p. 63]) a normed space \((E, \|\cdot\|)\) is uniformly Gâteaux-differentiable (for short UG) when it is Gâteaux-differentiable, and, for every \(h \in E\),

\[
\lim_{t \to 0} \left( \frac{\|a + th\| - \|a\|}{t} - G(a, h) \right) = 0 \quad \text{uniformly for } a \in S_E
\]

Remark 4. A norm which is uniformly Fréchet differentiable (see [3, Def. 1.9 p.8]) is uniformly Gâteaux differentiable but the converse statement is false. Indeed, given a uniformly Fréchet differentiable Banach space \(E\), the canonical mapping \(\text{can} : E \to E''\) is isometric and onto. On the other hand, if \(I\) is a set, the continuous bidual space of \(\ell^0(I)\) is \(\ell^\infty(I)\) and, if \(I\) is infinite, the canonical mapping \(\text{can} : \ell^0(I) \to \ell^\infty(I)\) is isometric but it is not onto (because \(1_I \notin \text{can}[\ell^0(I)] = \ell^0(I)\)). It follows that if \(I\) is infinite, the space \(\ell^0(I)\) does not have a uniformly Fréchet differentiable equivalent norm.

Proposition 5 ([3]). If a normed space \(E\) is uniformly Gâteaux-differentiable, then the dual norm of \(E'\) is W*UR.

Proof. Slightly change the proof in [3, p.63-64] to get a proof in ZF (for sake of completeness, we give a proof in Section 6.1, see the Appendix). \(\square\)

Remark 5. The converse statement holds in ZF+HB (see [3, p.63-64]).

5.2. Compactness of the dual ball of a UG space.

Theorem 4. Let \(E\) be a uniformly Gâteaux differentiable Banach space.

(i) The axiom AC(\mathbb{N}) implies that the closed unit ball of \(E'\) (and thus every bounded weak* closed subset of \(E'\)) is compact in the weak* topology.

(ii) The closed unit ball of \(E'\) is closely convex-compact in the weak* topology.

Proof. Use Theorem 3 and Proposition 5. \(\square\)

Remark 6. It is known that, in ZF+AC(\mathbb{N}) (see [10]), every bounded weakly closed subset of a Hilbert space is weakly compact. It is also known that, in ZFC (see [5]), the closed dual ball of a uniformly Gâteaux-differentiable normed space \(E\) is homeomorphic with a bounded weakly closed subset of Hilbert space. However, the proof in [5] relies on much axiom of choice; in particular, it does not seem to be valid in ZF+AC(\mathbb{N}). Thus, we cannot use these two results to deduce in ZF+AC(\mathbb{N}) Theorem 4 from [10] and [5].

Question 2. Is the following statement provable in ZF+AC(\mathbb{N})? in ZF+DC? Does it imply some weak form of the Axiom of Choice?

The closed dual ball of a uniformly Gâteaux differentiable normed space is homeomorphic with a bounded closed subset of a Hilbert space.
5.3. The CHB property for UG spaces. Say that a normed space $E$ satisfies the finite extension property (for short FEP) if for every finite dimensional subspace $F$ of $E$, and every linear mapping $f : F \to \mathbb{R}$, there exists $g \in E'$ extending $f$ such that $\|g\| \leq \|f\|$. Say that the normed space $E$ satisfies the effective finite extension property if there is a mapping associating to finite dimensional subspace $F$ of $E$ and every linear mapping $f : F \to \mathbb{R}$, a linear mapping $g : E \to \mathbb{R}$ extending $f$ such that $\|g\| \leq \|f\|$.

**Proposition 6.** Every Gâteaux-differentiable normed space satisfies the effective finite extension property.

**Proof.** Given a Gâteaux-differentiable normed space $E$, a proper finite dimensional subset $F$ of $E$, and a non-null linear mapping $f : F \to \mathbb{R}$, consider some point $a$ in the sphere of $F$ at which $f$ reaches its least upper bound on the closed unit ball $B_F$. Since the subspace $F$ is Gâteaux-differentiable, there exists a unique linear mapping $u \in S_{F'}$ such that $u(a) = 1$, namely $u = G(a,.|F)$. It follows that $f := \|f\|G(a,.|F)$ thus the linear continuous mapping $g := \|f\|G(a,.)$ extends $f$. \[ \square \]

**Corollary 1.** Uniformly Gâteaux differentiable normed spaces satisfy the effective continuous Hahn-Banach property.

**Proof.** Since $E$ is uniformly Gâteaux-differentiable, $B_{F'}$ is closely convex-compact in the weak* topology (see Theorem 4). Moreover, the space $E$ satisfies the FEP (see Proposition 6). It follows that $E$ satisfies the effective CHB property (see [6, Theorem 6 p.13]). \[ \square \]

**Question 3.** Recall that in [4], we showed in $\text{ZF} + \text{DC}$ that Gâteaux differentiable Banach spaces satisfy the CHB property. Is the CHB property for Gâteaux differentiable Banach spaces provable in $\text{ZF} + \text{AC}(\mathbb{N})$? In $\text{ZF}$? What about the CHB property for Fréchet-differentiable Banach spaces?

5.4. Examples of UG renormings and applications.

5.4.1. UG renormings of $\ell^0(I)$.

**Example 3.** The Banach space $\ell^0(I)$ has an equivalent norm which is uniformly Gâteaux differentiable. It follows that:

(i) The space $\ell^0(I)$ satisfies the effective CHB property.

(ii) Every bounded weak* closed convex subset of $\ell^1(I)$ is weak* convex-compact.

(iii) In $\text{ZF} + \text{AC}(\mathbb{N})$, every bounded weak* closed subset of $\ell^1(I)$ is weak* compact.

**Proof.** The canonical mapping $\ell^2(I) \to \ell^0(I)$ is continuous and dense, so Proposition 9 (see Appendix) implies that $\ell^0(I)$ has a UG renorming. \[ \square \]

**Remark 7.** Results (i) and (ii) were already obtained in [6].

5.4.2. Spaces $L_1(E, \mathcal{B}, m)$. Given a set $E$, and boolean algebra $(\mathcal{B}, \cap, \cup, \emptyset, E)$ of subsets of $E$, a measure on $\mathcal{B}$ is a mapping $m : \mathcal{B} \to [0, +\infty]$ satisfying $m(\emptyset) = 0$ and, for every $x,y \in \mathcal{B}$, $x \cap y = 0 \Rightarrow m(x \cup y) = m(x) + m(y)$. We denote by $\mathcal{E}$ the set of “simple functions” i.e. the (real) vector subspace of $\mathbb{R}^E$ generated by indicators of elements of $\mathcal{B}$. We define as usual the $m$-integral $\int_E f dm$ of a positive simple function $f$, and the the $m$-integral of simple real functions $f$ such that $\int_E |f| dm < +\infty$. For every real number $p \in [1, +\infty[$, we denote by $\mathcal{E}_p$ the vector subspace $\{ f \in \mathcal{E} : \int_E |f|^p dm < +\infty \}$ of $\mathcal{E}$. Then the mapping $\mathcal{N}_p : \mathcal{E}_p \to \mathbb{R}_+$ associating to every $f \in \mathcal{E}_p$ the real number $(\int_E |f|^p dm)^{1/p}$ is a semi-norm. We denote by $L_p(E, \mathcal{B}, m)$ the Banach space which is the separated Cauchy completion of the semi-normed space $(\mathcal{E}_p, \mathcal{N}_p)$ (which can be built in $\text{ZF}$). We also define $\mathcal{E}_\infty$ as follows: given $f = \sum_{1 \leq i \leq n} \lambda_i 1_{a_i} \in \mathcal{E}$ such that $(a_i)_{1 \leq i \leq n}$ is a partition of $E$ in elements of $\mathcal{B}$, define $\mathcal{N}_\infty(f) := 0$ if $m(E) = 0$, and else, $\mathcal{N}_\infty(f) := \max\{ |\lambda_i| :
1 ≤ i ≤ n and m(ai) > 0). Then E∞ := \{f ∈ E : \mathcal{N}_∞(f) < +∞\} is a vector subspace of E, and (E∞, \mathcal{N}_∞) is a semi-normed space. We denote by L∞(E, B, m) the separated Cauchy completion of this semi-normed space.

**Example 4.** Let (E, B, m) be a measured boolean algebra of sets. Assume that m(1_B) < +∞, where 1_B is the unit of B. Then, the Banach space L1(B, µ) has an equivalent norm which is uniformly Gâteaux differentiable. It follows that:

(i) The space L1(B, m) satisfies the effective CHB property.
(ii) Every bounded weak* closed convex subset of L∞(B, m) is convex-compact in the weak* topology.
(iii) In ZF + AC(\mathbb{N}), every bounded weak* closed subset of L∞(B, m) is weak* compact.

**Proof.** Since m(1_B) < +∞, the canonical mapping L2(B, m) → L1(B, m) is continuous and dense, so Proposition 9 (see Appendix) implies that L1(B, m) has a UG renorming. The continuous dual of L1(B, m) is L∞(B, m), so (ii) and (iii) follow from Theorem 4.

**Remark 8.** The previous Corollary cannot be extended to the case where m(1_B) = +∞ because ℓ1(\mathbb{R}) does not have an equivalent Gâteaux differentiable norm.

## 6. Appendix on UG Spaces

### 6.1. The dual ball of a UG space is W*UR.

#### 6.1.1. An easy criterion for UG differentiability.

**Proposition 7** ([3]). Let (E, \|\|) be a normed space. The following conditions are equivalent:

(i) E is uniformly Gâteaux-differentiable.
(ii) For every h ∈ SE, \lim_{t \to 0} \frac{\|a + th\| + \|a - th\| - 2}{t} = 0 uniformly for a ∈ SE.

**Proof.** See [3, Lemma 6.6 p. 63] for a proof which is valid in ZF. □

#### 6.1.2. Proof of Proposition 5: “If a normed space E is uniformly Gâteaux-differentiable, then the dual norm of E' is W*UR.”

**Proof.** Let h ∈ SE. Let ε ∈ ]0, 1[. Since E is uniformly Gâteaux-differentiable, Proposition 7 implies some η1 ∈ ]0, 1[ such that for every a ∈ SE:

|t| ≤ η1 \Rightarrow \frac{\|a + th\| + \|a - th\| - 2}{|t|} < ε

Then

|t| ≤ η1 \Rightarrow \|a + th\| + \|a - th\| ≤ 2 + ε|t|

Let η := εη1. Let a ∈ SE and f, g ∈ C_{a, η}. Then for every real number t satisfying 0 < |t| ≤ η1,

f(a + th) + g(a - th) ≤ \|a + th\| + \|a - th\| ≤ 2 + ε|t|

In particular, for t = η1 we have:

f(η1h) + g(-η1h) ≤ 2 + η1ε - f(a) - g(a) ≤ 3η1ε

thus f(h) - g(h) ≤ 3ε. It follows that sup\{f, g ∈ C_{a, η} : \|f(h) - g(h)\| ≤ 3ε. □
6.2. **UG renormings.** The following Proposition is a slightly different version of [3, Prop. 6.2 p. 61]. Indeed, we do not use sequences but quantitative \( \varepsilon-\eta \) conditions:

**Proposition 8.** Let \( E \) be a Banach space. The following conditions are equivalent:

(i) The dual norm on \( E' \) is W*UR.

(ii) For every \( h \in S_E \), and every \( \varepsilon > 0 \), there exists \( \eta > 0 \) satisfying for every \( f, g \in B_{E'} \),

\[
(2\|f\|^2 + 2\|g\|^2 - \|f + g\|^2 \leq \eta) \Rightarrow \|f(h) - g(h)\| \leq \varepsilon
\]

**Proof.** (i) \( \Rightarrow \) (ii) Let \( h \in S_E \) and \( \varepsilon > 0 \). Since the dual norm on \( E' \) is W*UR, let \( \eta_1 > 0 \) such that for every \( f, g \in B_{E'} \), and every \( a \in S_E \), \( f, g \in C_{a,\eta_1} \Rightarrow \|f(h) - g(h)\| < \varepsilon \). Let \( \eta \in [0, \frac{\varepsilon^2}{4}] \) such that

\[
\frac{\varepsilon^2}{4} - \sqrt{\eta} > 0, \quad 1 - \frac{2\sqrt{\eta}}{\varepsilon} > 1 - \eta, \quad \text{and} \quad (1 - \frac{2\sqrt{\eta}}{\varepsilon})(1 - \frac{2\sqrt{\eta} + 4\sqrt{\eta}}{\varepsilon}) > 1 - \eta_1.
\]

Assume that \( f, g \in B_{E'} \) satisfy:

\[
2\|f\|^2 + 2\|g\|^2 - \|f + g\|^2 \leq \eta
\]

Equation (2) implies that

\[
\|f\| \leq \|g\| + \sqrt{\eta} \quad \text{and} \quad \|g\| \leq \|f\| + \sqrt{\eta}
\]

and thus

\[
\frac{f + g}{2} \leq \|f\| + \frac{\sqrt{\eta}}{2} \quad \text{and} \quad \|f + g\| \leq \|f\| + \sqrt{\eta} \leq \frac{\varepsilon^2}{4} + \frac{\sqrt{\eta}}{2}
\]

- If \( \|f\| \leq \frac{\varepsilon}{2} \) and \( \|g\| \leq \frac{\varepsilon}{2} \), then \( \|f(h) - g(h)\| \leq \|f(h) + g(h)\| \leq \varepsilon \).

- Else, we may assume \( w.l.o.g. \) that \( \|f\| \geq \frac{\varepsilon}{2} \) (whence \( \|g\| \geq \frac{\varepsilon}{2} - \sqrt{\eta} > 0 \)). Since \( \|f\| > \frac{\varepsilon}{2} \), by (2),

\[
\|f + g\|^2 \geq \frac{\|f\|^2 + \|g\|^2}{2} - \frac{\eta}{4},
\]

and with (3),

\[
\|f + g\|^2 \geq \frac{\|f\|^2 + \|g\|^2}{2} - \frac{\eta}{4} \geq \|f\|^2 - \sqrt{\eta} = \|f\|^2(1 - \frac{\sqrt{\eta}}{\|f\|^2}).
\]

Since \( \|f\| \geq \frac{\varepsilon}{2} \), we deduce that

\[
\|f + g\|^2 \geq \frac{\|f\|^2(1 - \frac{2\sqrt{\eta}}{\varepsilon})}{\|f\|^2} \geq 2\|f\|(1 - \frac{2\sqrt{\eta}}{\varepsilon^2})
\]

We deduce that

\[
\|f + g\| \geq \|f\|(1 - \frac{2\sqrt{\eta}}{\varepsilon^2}) - \|f\| + \sqrt{\eta}
\]

\[
= \|f\| - \frac{4\|f\|\sqrt{\eta}}{\varepsilon^2} - \sqrt{\eta} \geq \|f\| - \frac{4\sqrt{\eta}}{\varepsilon^2} - \sqrt{\eta}
\]

\[
= \|f\|(1 - \sqrt{\eta}(1 + \frac{4\sqrt{\eta}}{\varepsilon^2} - \frac{1}{\|f\|})) \geq \|f\|(1 - \sqrt{\eta}(1 + \frac{4\sqrt{\eta}}{\varepsilon^2} - \frac{1}{\|f\|}))
\]

Let \( a \in S_E \) such that \( \frac{f + g}{2}(a) + \eta \geq \|\frac{f + g}{2}\| \); then \( f(a) \geq \|f + g\| - g(a) - 2\eta \geq \|f + g\| - \|g\| - 2\eta \geq \|f\|(1 - \frac{2\sqrt{\eta}}{\varepsilon}(1 + \frac{4\sqrt{\eta}}{\varepsilon}) - \frac{4\eta}{\varepsilon}) \). By definition of \( \eta \) it follows that \( \frac{f}{\|f\|} \in C_{a,\eta} \). On the other hand,

\[
g(a) \geq \|f + g\| - \|f\| - 2\eta \geq \|f + g\| - \|f\| - 2\eta
\]

\[
\|f\| - \frac{4\sqrt{\eta}}{\varepsilon^2} - 2\eta = \|f\|(1 - \frac{2\eta + 4\sqrt{\eta}}{\|f\|}) \geq \|f\|(1 - \frac{2\eta + 4\sqrt{\eta}}{\|f\|}) \geq (\|g\| - \sqrt{\eta})(1 - \frac{2\eta + 4\sqrt{\eta}}{\varepsilon})
\]

thus \( \frac{g(a)}{\|g\|} \geq (1 - \frac{\sqrt{\eta}}{\|g\|})(1 - \frac{2\eta + 4\sqrt{\eta}}{\varepsilon}) \geq (1 - \frac{\sqrt{\eta}}{\|f\| \sqrt{2\sqrt{\eta}}})(1 - \frac{2\eta + 4\sqrt{\eta}}{\varepsilon}) \). By definition of \( \eta \), \( \frac{\varepsilon}{\|g\|} \in C_{a,\eta} \).

Thus, using (2), \( \|f(h) - g(h)\| \leq \varepsilon \).
(ii) \(\Rightarrow\) (i) Let \(h \in E, \varepsilon > 0\) and \(\eta > 0\) such that every \(f, g \in E'\) satisfy (1). Let \(\lambda := \frac{2}{\varepsilon}\). Given some \(a\) in the unit sphere of \(E\), then for every \(f, g \in C_{a, \lambda}\),

\[
\frac{\|f\|^2 + \|g\|^2}{2} - \frac{f + g}{2}^2 \leq 1 - (1 - \lambda)^2 = 2\lambda - \lambda^2 \leq 2\lambda
\]

thus \(2\|f\|^2 + 2\|g\|^2 - \|f + g\|^2 \leq 8\lambda = \eta\), thus using (1), \(|f(h) - g(h)| < \varepsilon\).

\[\square\]

**Proposition 9** ([3]). Let \(E\) be a Banach space. If there exists a uniformly Gâteaux-differentiable normed space (for example a Hilbert space) \(H\), and a continuous linear mapping \(T : H \to E\) such that the vector subspace \(\text{Im}(T)\) is dense in \(E\), then \(E\) admits an equivalent norm which has a W*UR dual norm (and thus \(E\) has an equivalent norm which is uniformly Gâteaux-differentiable).

**Proof.** See [3, p. 65-66]. For sake of completeness, we give the proof which is valid in \(ZF\). We denote by \(T' : F' \to E'\) the transpose of \(T\) associating to each \(g \in F'\) the mapping \(g \circ T\). Without loss of generality, we may assume that \(\|T\| \leq 1\), thus \(\|T'\| \leq 1\). Consider the norm \(\|\|\cdot\|\) on \(E'\) such that for every \(f \in E'\):

\[
\|f\| := \|f\|_{E'}^2 + \|T'(f)\|_{H'}^2.
\]

This norm on \(E'\) is equivalent to the dual norm of \(E'\) because \(T'\) is continuous. Moreover, this equivalent norm \(\|\|\cdot\|\) is a dual norm because its closed unit ball is weak* closed in \(E'\) (the mapping \(\|\|\cdot\|\) is weak* lower continuous). We now show that this dual norm \(\|\|\cdot\|\) is W*UR. We use Proposition 8. Let \(k \in E\) and \(\varepsilon > 0\). We seek for some real number \(\eta > 0\) such that for every \(f, g \in B_{E'}\), satisfying \(2\|f\|^2 + 2\|g\|^2 - \|f + g\|^2 < \eta\), \(|f(k) - g(k)| \leq \varepsilon\). Since \(\text{Im}(T)\) is dense in \(E\), we may assume that \(k = T(h)\) where \(h \in H\). Since \(H\) is W*UR, let \(\eta > 0\) such that for every \(u, v \in B_{H'}\),

\[
(2\|u\|^2 + 2\|v\|^2 - \|u + v\|^2 < \eta) \Rightarrow |u(h) - v(h)| \leq \varepsilon
\]

Let \(f, g \in B_{E'}\) such that

\[
2\|f\|^2 + 2\|g\|^2 - \|f + g\|^2 < \eta
\]

Then the two following inequalities hold:

\[
2\|f\|^2 + 2\|g\|^2 - \|f + g\|^2 < \eta \quad \text{and} \quad 2\|T'(f)\|^2 + 2\|T'(g)\|^2 - \|T'(f + g)\|^2 < \eta
\]

The second inequality and (7) imply that

\[
|T'(f)(h) - T'(g)(h)| \leq \varepsilon \quad \text{i.e.} \quad |f(k) - g(k)| \leq \varepsilon
\]

\[\square\]

**References**


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