Uniform Eberlein Compactness and the Axiom of Choice

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The Axiom of Choice

AC (Axiom of Choice)

Given a family $(A_i)_{i\in I}$ of non-empty sets, there exists a function $f : I \to \cup_{i\in I} A_i$ such that $\forall i \in I\ f(i) \in A_i$.

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We work in set-theory without the Axiom of Choice **ZF**:

$\varnothing$, extension, pair, union, power-set, infinity, regularity, replacement (and separation).

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Proof.

Let ∞ be some set $\notin \cup_i X_i$. For each $i \in I$, let $\hat{X}_i := X_i \cup \{\infty\}$. Endow each $\hat{X}_i$ with the topology generated by $\{\infty\}$ and cofinite subsets of $\hat{X}_i$. Each space $\hat{X}_i$ is compact (and T_1). Using the axiom **T**, the space $\prod_i X_i$ is compact. For each $i \in I$, let $F_i := X_i \times \prod_{t \neq i} \hat{X}_t$. The family of closed sets $(F_i)_i$ satisfies the *FIP* so $\cap_i F_i \neq \varnothing$. $\square$

Distinct consequences of the "Tychonov axiom" (1)

$\mathbf{T}_2$ "Tychonov for Hausdorff spaces"

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Given a set I, we also consider the following statement:

$\mathbf{AC}^{fin(I)}$ "Choice in finite subsets of I"

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Our aim is to prove that $\mathbf{AC}^{\mathbf{fin}}_{\mathbb{N}}$ (*resp.* $\mathbf{AC}^{\mathbf{fin}}$) is equivalent to the following statement: *"The closed unit ball of a Hilbert space with a hilbertian basis is compact* (resp. *closely compact) in the weak topology."*

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Theorem (F-M 1998, [2])

For every ordinal α, and every family $(X_i, \Phi_i)_{i\in\alpha}$ of T_1 closely compact spaces, the space $\prod_{i\in\alpha} X_i$ is closely compact.

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$\mathbf{AC}^{fin(I)} \Leftrightarrow$ *"The space $\prod_{n\in\mathbb{N}} \sigma_n(I)$ is closely compact."*.

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Proof.

With $\mathbf{AC}^{fin(I)}$, $\sigma_1(I)$ is closely compact: let Φ be a witness of this close compactness. Thus, each space $\left(\sigma_1(I)\right)^n$ is closely compact with a witness definable frome Φ. So each space $\sigma_n(I)$ (continuous image of $\left(\sigma_1(I)\right)^n$ by $\cup_n$) is closely compact with a witness definable frome Φ. □

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5. The function $\cup_n : \sigma_1(I)^n \to \sigma_n(I)$ which maps each $(F_i)_{1 \leq i \leq n}$ to $\cup_{1 \leq i \leq n} F_i$ is continuous and onto (and has a section if $\mathbf{AC}^{fin(I)}$ holds).

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Consequence: $\mathbf{AC}^{fin}$ is equivalent to the following statement: *"The closed unit ball of a Hilbert space with a hilbertian basis is weakly compact."*

Close compactness of $B^+(I)$ (2)

Theorem (Benyamini, Rudin et Wage, 1977 [1])

The space $B^+(I)$ is a continuous image of a closed subset of the product space $\prod_{n\in\mathbb{N}} \sigma_{2^{n+1}}(I)$.

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Proof.

(sketch) Let $\phi : \{0,1\}^{\mathbb{N}} \to [0,1]$ be the function $(\varepsilon_n)_{n\in\mathbb{N}} \mapsto \sum_{n\in\mathbb{N}} \frac{\varepsilon_n}{2^{n+1}}$. Then ϕ is continuous, onto (with a definable section). Let $\psi_I := \phi^I : \left(\{0,1\}^{\mathbb{N}}\right)^I \to [0,1]^I$: then ψ_I is continuous and onto (with a definable section). Let $Z := \psi_I^{-1}[B^+(I)]$: then Z is a closed subset of $\left(\{0,1\}^{\mathbb{N}}\right)^I$ and one easily checks that $Z \subseteq \prod_{n\in\mathbb{N}} \sigma_{2^{n+1}}(I)$. And $\psi_I[Z] = B^+(I)$. $\square$

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Close compactness of $B^+(I)$ (2)

Theorem (Benyamini, Rudin et Wage, 1977 [1])

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Proof.

With $\mathbf{AC}^{fin(I)}$, $\prod_{n\in\mathbb{N}} \sigma_n(I)$ is closely compact (Corollary of p. 8). Thus the continuous image $\prod_{n\in\mathbb{N}} \sigma_{2^{n+1}}(I)$ is also closely compact. We end with Benyamini, Rudin et Wage's result. $\square$

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Lemma: $T_\omega^{fin(I)} \Rightarrow$ "$\sigma_1(I)^{\mathbb{N}}$ is compact".

Let $P := \sigma_1(I)^{\mathbb{N}}$. let $\mathcal{L}$ be the lattice of subsets of P generated by closed subsets of the form $F_i \times \sigma_1(I)^{\mathbb{N}\setminus\{i\}}$. Let $\mathcal{F}$ be a filter of $\mathcal{L}$.

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Proposition

If $\sigma_1(I)^{\mathbb{N}}$ is compact, then $\prod_{n\in\mathbb{N}} \sigma_{2^{n+1}}(I)$ -and thus $B^+(I)$- is compact.

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Proposition

$\mathbf{AC}_{\mathbb{N}}^{fin} \Leftrightarrow \mathbf{T}_{\mathbb{N}}^{fin}$.

Some questions

Given a set I, *AETF*:

"$B^+(I)$ is compact"; "The subset $B_1(I) := \{x = (x_i)_{i \in I} \in [-1,1]^I : \sum_i |x_i| \le 1\}$ of $[0,1]^I$ is compact"; "The closed unit ball $B_2(I)$ of the Hilbert space $\ell^2(I)$ is compact in the weak topology."

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Does $\mathbf{AC^{fin}}$ imply **AH**?

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AH: *The closed unit ball of a Hilbert space is weakly compact.*
BH: *"Every Hilbert space has a Hilbertian basis."*
Then **AH** $\Rightarrow$ $\mathbf{AC^{fin}}$ and $\mathbf{AC^{fin}}$+**BH** $\Rightarrow$ **AH**.

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Some questions

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Is **BH** provable in **ZF**? in **ZF**+$\mathbf{AC}^{\mathbf{fin}}$? What about the existence of Markhushevich bases in WCG Banach spaces?

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