Uniform Eberlein Compactness and the Axiom of Choice

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The Axiom of Choice

AC (Axiom of Choice)

Given a family \((A_i)_{i \in I}\) of non-empty sets, there exists a function \(f : I \rightarrow \bigcup_{i \in I} A_i\) such that \(\forall i \in I\) \(f(i) \in A_i\).

The function \(f\) is called a choice function for the family \((A_i)_{i \in I}\).
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The function \(f\) is called a *choice function* for the family \((A_i)_{i \in I}\).

We work in set-theory without the Axiom of Choice **ZF**: \(\emptyset\), extension, pair, union, power-set, infinity, regularity, replacement (and separation).
Examples of “choice functions” existing in ZF

1. \((A_i, \ast_i)_{i \in I}\) is a family of groups;
2. \((A_i)_{i \in I}\) is a family of subsets of \(\mathbb{Q}\) (or of any well-orderable set);
3. \((A_i)_{i \in I}\) is a family of closed subsets of \(\mathbb{R}\) (or of any conditionally complete linear order);
4. \((A_i)_{i \in I}\) is a family of closed subsets of \(\mathbb{R}^n\);
Examples of “choice functions” existing in $\textbf{ZF}$

Let $(A_i)_{i \in I}$ be a family of non-empty sets.

In the following cases, this family has a choice function (in $\textbf{ZF}$):
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\textbf{T} (Tychonov axiom)

\textit{If $(X_i)_{i \in I}$ is a family of compact spaces, then the product space $\prod_{i \in I} X_i$ is also compact.}
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**Theorem ( Kelley 1950 [7])**

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Theorem (Kelley 1950 [7])

$\mathbf{AC} \iff \mathbf{T}.$

Proof.

Let $\infty$ be some set $\not\in \bigcup_i X_i$. For each $i \in I$, let $\hat{X}_i := X_i \cup \{\infty\}$. Endow each $\hat{X}_i$ with the topology generated by $\{\infty\}$ and cofinite subsets of $\hat{X}_i$. Each space $\hat{X}_i$ is compact (and $T_1$). Using the axiom $\mathbf{T}$, the space $\prod_i \hat{X}_i$ is compact. For each $i \in I$, let $F_i := X_i \times \prod_{t \neq i} \hat{X}_t$. The family of closed sets $(F_i)_i$ satisfies the FIP so $\bigcap_i F_i \neq \emptyset$. 
Distinct consequences of the “Tychonov axiom” (1)

\[ T_2 \] “Tychonov for Hausdorff spaces”

If \((X_i)_{i \in I}\) is a family of Hausdorff compact spaces, then the product space \(\prod_{i \in I} X_i\) is compact.
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Every family of finite non-empty sets has a choice function.
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Every family of finite non-empty sets has a choice function.

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Every sequence $(A_n)_{n \in \mathbb{N}}$ of finite non-empty sets has a choice function.
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Every sequence \((A_n)_{n \in \mathbb{N}}\) of finite non-empty sets has a choice function.

Given a set \(I\), we also consider the following statement:

\[ \text{AC}^{\text{fin}(I)} \text{ “Choice in finite subsets of } I\text{”} \]
The set of finite non-empty subsets of \(I\) has a choice function.
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- $\text{AC} \implies \text{T}_2 \implies \text{AC}^{\text{fin}} \implies \text{AC}^{\text{fin}}_N$
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Known facts

- $\text{AC} \Rightarrow T_2 \Rightarrow \text{AC}^{\text{fin}} \Rightarrow \text{AC}^\mathbb{N}_{\text{fin}}$
- The converse implications are false:

- $T_2 \nRightarrow \text{AC}^{\text{fin}}$ (Halpern-Levy 67, [3])
- $\text{AC}^{\text{fin}} \nRightarrow T_2$
- $\text{AC}^{\text{fin}} \nRightarrow \text{AC}^\mathbb{N}_{\text{fin}}$
- $\text{AC}^\mathbb{N}_{\text{fin}}$ is not provable in ZF.

Proof.

For $T_2 \Rightarrow \text{AC}^{\text{fin}}$, use Kelley's argument.

For the "non implications", see References in Jech [5] or Howard and Rubin [4].

Our aim is to prove that $\text{AC}^{\text{fin}} \mathbb{N}$ (resp. $\text{AC}^\mathbb{N}_{\text{fin}}$) is equivalent to the following statement:

"The closed unit ball of a Hilbert space with a hilbertian basis is compact (resp. closely compact) in the weak topology."
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ZF-provable consequences of Tychonov (1)

Say that a compact topological space $X$ is closely compact if there exists a function $\Phi$ which associates to every non-empty closed subset $F$ of $X$ an element $\Phi(F) \in F$.

Examples
1. A complete linearly ordered set endowed with the order topology is closely compact (and Hausdorff).
2. Given a set $X$, the "one-point compactification" $\hat{X} := X \cup \{\infty\}$ of the discrete space $X$ is compact (and Hausdorff). (The open subsets of $\hat{X}$ are subsets of $X$ and cofinite subsets of $\hat{X}$ containing $\infty$.)

Moreover, $\text{AC fin}(X) \iff \hat{X}$ is closely compact.
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ZF-provable consequences of Tychonov (2)

Theorem (F-M 1998, [2])

For every ordinal $\alpha$, and every family $(X_i, \Phi_i)_{i \in \alpha}$ of $T_1$ closely compact spaces, the space $\prod_{i \in \alpha} X_i$ is closely compact.

Proof.

Proof by transfinite recursion on $\alpha$.

In particular, $[0, 1)^\alpha$ (and so $[0, 1)^\mathbb{N}$) is closely compact.

Corollary $\text{AC}_\text{fin}(I) \iff \text{"The space } \prod_{n \in \mathbb{N}} \sigma_n(I) \text{ is closely compact."}$.

Proof.

With $\text{AC}_\text{fin}(I)$, $\sigma_1(I)$ is closely compact: let $\Phi$ be a witness of this close compactness. Thus, each space $(\sigma_1(I))_n$ is closely compact with a witness definable from $\Phi$. So each space $\sigma_n(I)$ (continuous image of $(\sigma_1(I))_n$ by $\bigcup_n$) is closely compact with a witness definable from $\Phi$. 


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**Proof.**

Proof by transfinite recursion on $\alpha$. In particular, $[0, 1]^\alpha$ (and so $[0, 1]^\mathbb{N}$) is closely compact.

**Corollary**

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The space $B^+(I)$

Given $n \in \mathbb{N}$, consider the following closed subsets of $[-1, 1]^I$: 

\begin{align*}
B_p(I) := \{ & x = (x_i)_{i \in I} \in [-1, 1]^I : \sum_{i} |x_i|^p \leq 1 \} \\
B^+ (I) := \{ & x = (x_i)_{i \in I} \in [0, 1]^I : \sum_{i} x_i \leq 1 \} \\
\sigma_n(I) := \{ & F \subseteq I : F \text{ has at most } n \text{ elements} \} 
\end{align*}

Remarks

1. $B^+ (I)$ is compact (resp. closely compact) iff $B_1(I)$ is compact (resp. closely compact).

2. With $T_2$, the space $B^+ (I)$ is compact.

3. $\sigma_1(I)$ is the one-point compactification of the discrete space $I$.

4. If $B^+(I)$ is closely compact, then $AC_{fin}(I)$ holds.

5. The function $\bigcup_n : \sigma_1(I) \rightarrow \sigma_n(I)$ which maps each $(F_i)_{1 \leq i \leq n}$ to $\bigcup_{1 \leq i \leq n} F_i$ is continuous and onto (and has a section if $AC_{fin}(I)$ holds).
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- $B_p(I) := \{ x = (x_i)_{i \in I} \in [-1, 1]^I : \sum_i |x_i|^p \leq 1 \}$ for $1 \leq p < +\infty$;
- $B^+(I) := \{ x = (x_i)_{i \in I} \in [0, 1]^I : \sum_i x_i \leq 1 \}$;
- $\sigma_n(I) := \{ F \subseteq I : F \text{ has at most } n \text{ elements} \}$. 

Remarks

1. $B^+(I)$ is compact (resp. closely compact) iff $B_1(I)$ is compact (resp. closely compact).
2. With $T_2$, the space $B^+(I)$ is compact.
3. $\sigma_1(I)$ is the one-point compactification of the discrete space $I$.
4. If $B^+(I)$ is closely compact, then $AC_{\text{fin}}(I)$ holds.
5. The function $\bigcup_n : \sigma_1(I) \to \sigma_n(I)$ which maps each $(F_i)_{1 \leq i \leq n}$ to $\bigcup_{1 \leq i \leq n} F_i$ is continuous and onto (and has a section if $AC_{\text{fin}}(I)$ holds).
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Given $n \in \mathbb{N}$, consider the following closed subsets of $[-1, 1]^I$:

$B_p(I) := \{ x = (x_i)_{i \in I} \in [-1, 1]^I : \sum_i |x_i|^p \leq 1 \}$ for $1 \leq p < +\infty$;

$B^+(I) := \{ x = (x_i)_{i \in I} \in [0, 1]^I : \sum_i x_i \leq 1 \}$;

$\sigma_n(I) := \{ F \subseteq I : F \text{ has at most } n \text{ elements} \}$.

Remarks

1. $B_+(I)$ is compact (resp. closely compact) iff $B_1(I)$ is compact (resp. closely compact) iff $B_2(I)$ is compact (resp. closely compact).

2. With $T_2$, the space $B^+(I)$ is compact.

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4. If $B^+(I)$ is closely compact, then $AC^{\text{fin}(I)}$ holds.
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5. The function $\cup_n : \sigma_1(I)^n \rightarrow \sigma_n(I)$ which maps each $(F_i)_{1 \leq i \leq n}$ to $\cup_{1 \leq i \leq n} F_i$ is continuous and onto (and has a section if $\text{AC}^{\text{fin}(I)}$ holds).
Close compactness of $B^+(I)$ (1)

Our aim is to prove the following result:

Theorem, M.M., preprint [9]

Let $I$ be a set. Then $\text{AC}_{\text{fin}}(I) \iff \text{"B}^+(I)\text{ is closely compact}."

In particular, $B^+(\mathbb{R})$ is closely compact. This enhances results in (M.M. 2008 [8]).

Consequence: $\text{AC}_{\text{fin}}$ is equivalent to the following statement:

"The closed unit ball of a Hilbert space with a hilbertian basis is weakly compact."
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Consequence: $\text{AC}^{\text{fin}}$ is equivalent to the following statement: *“The closed unit ball of a Hilbert space with a hilbertian basis is weakly compact.”*
The space $B^+(I)$ is a continuous image of a closed subset of the product space $\prod_{n \in \mathbb{N}} \sigma_{2^{n+1}}(I)$.

Proof. (sketch) Let $\phi: \{0, 1\}^\mathbb{N} \rightarrow [0, 1]^I$ be the function $(\varepsilon_n)_{n \in \mathbb{N}} \mapsto \sum_{n \in \mathbb{N}} \varepsilon_n 2^{n+1}$. Then $\phi$ is continuous, onto (with a definable section). Let $\psi_I := \phi_I: (\{0, 1\}^\mathbb{N})^I \rightarrow [0, 1]^I$; then $\psi_I$ is continuous and onto (with a definable section). Let $Z := \psi_I^{-1}(B^+(I))$: then $Z$ is a closed subset of $(\{0, 1\}^\mathbb{N})^I$ and one easily checks that $Z \subseteq \prod_{n \in \mathbb{N}} \sigma_{2^{n+1}}(I)$. And $\psi_I[Z] = B^+(I)$.

Proof of the main Theorem
If $\text{AC} \text{ fin}(I)$ holds, then $B^+(I)$ is closely compact.

Proof. With $\text{AC} \text{ fin}(I)$, $\prod_{n \in \mathbb{N}} \sigma_n(I)$ is closely compact (Corollary of p. 8). Thus the continuous image $\prod_{n \in \mathbb{N}} \sigma_{2^{n+1}}(I)$ is also closely compact. We end with Benyamini, Rudin et Wage’s result.
Close compactness of $B^+(I)$ (2)

Theorem (Benyamini, Rudin et Wage, 1977 [1])

The space $B^+(I)$ is a continuous image of a closed subset of the product space $\prod_{n \in \mathbb{N}} \sigma_{2^n+1}(I)$.

Proof.

(sketch) Let $\phi : \{0, 1\}^\mathbb{N} \rightarrow [0, 1]$ be the function $(\varepsilon_n)_{n \in \mathbb{N}} \mapsto \sum_{n \in \mathbb{N}} \frac{\varepsilon_n}{2^{n+1}}$. Then $\phi$ is continuous, onto (with a definable section). Let $\psi_I := \phi^I : (\{0, 1\}^\mathbb{N})^I \rightarrow [0, 1]^I$: then $\psi_I$ is continuous and onto (with a definable section). Let $Z := \psi_I^{-1}[B^+(I)]$: then $Z$ is a closed subset of $(\{0, 1\}^\mathbb{N})^I$ and one easily checks that $Z \subseteq \prod_{n \in \mathbb{N}} \sigma_{2^n+1}(I)$. And $\psi_I[Z] = B^+(I)$. \qed
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If $\text{AC}^{\text{fin}(I)}$ holds, then $B_+(I)$ is closely compact.
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We end with Benyamini, Rudin et Wage’s result.
Compactness of $B^+(I)$ (1)

Given a set $I$, consider the following axiom:

$T^\text{fin}(I)_\omega$: If $(F_n)_{n \in \mathbb{N}}$ is a sequence of finite discrete subsets of $I$, then $\prod_{n \in \mathbb{N}} F_n$ is compact.
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Theorem (M 2008 -preprint-)

“$B^+(I)$ is compact” $\iff T^\text{fin}(I)_\omega$. 

Proof. 

$\Rightarrow$ Let $(F_n)_{n \in \mathbb{N}}$ be a sequence of finite subsets of $I$. Let $D := \bigcup_{n \in \mathbb{N}} F_n$. We show that $D$ is countable (thus $\prod_{n \in \mathbb{N}} F_n$ is compact).

We may assume that the $F_n$ are pairwise disjoint.

Since $B^+(I)$ is compact, $B^+(D)$ is also compact.

For every $n \in \mathbb{N}$, let $\varepsilon_n : |F_n| \to [0,1]$ be an increasing function such that $\sum_{n \in \mathbb{N}} \sum_{0 < i < |F_n|} \varepsilon_n(i) = 1$;

let $\tilde{F}_n := \{ x \in B^+(D) : x \upharpoonright F_n \text{ is a bijection from } F_n \text{ to } \text{rg}(\varepsilon_n) \}$.

Each $\tilde{F}_n$ is a closed subset of $B^+(D)$ and the sequence $(\tilde{F}_n)_{n \in \mathbb{N}}$ satisfies the FIP.

Thus $Z := \bigcap_{n \in \mathbb{N}} \tilde{F}_n$ is non-empty.

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Given a set $I$, consider the following axiom:

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Each $f|_{F_n}$ defines a well order on $F_n$, and thus $D$ is countable.
Compactness of $B^+(I)$ (2)

Lemma: $T^\text{fin}(I)_\omega \Rightarrow \text{"}\sigma_1(I)^\mathbb{N} \text{ is compact}\text{"}$. Let $P := \sigma_1(I)^\mathbb{N}$. Let $\mathcal{L}$ be the lattice of subsets of $P$ generated by closed subsets of the form $F_i \times \sigma_1(I)^\mathbb{N}\setminus\{i\}$. Let $\mathcal{F}$ be a filter of $\mathcal{L}$. Since each closed subset of $\sigma_1(I)^\mathbb{N}$ is finite or contains $\infty$, we can build by recursion a sequence $(F_n)_{n \in \mathbb{N}}$ of finite subsets of $\sigma_1(I)^\mathbb{N}$ such that for every $n \in \mathbb{N}$, $\prod_{i < n} F_i \times \sigma_1(I)^\mathbb{N}\setminus\{i\}$ is $\mathcal{F}$-stationary (i.e. meets every element of $\mathcal{F}$). Then, using $T^\text{fin}(I)_\omega$, the space $F := \prod_{i \in \mathbb{N}} F_i$ is compact and non-empty. Now $\{F \cap Z : Z \in \mathcal{F}\}$ is a family of closed subsets of $F$ satisfying the FIP, thus $\bigcap \mathcal{F} \neq \emptyset$. Proposition If $\sigma_1(I)^\mathbb{N}$ is compact, then $\prod_{n \in \mathbb{N}} \sigma_2(n+1)^\mathbb{N}$ and thus $B^+(I)$ is compact.
Compactness of $B^+(I)$ (2)

Lemma: $T_{\omega}^{\text{fin}(I)} \Rightarrow \text{“}\sigma_1(I)^\mathbb{N}\text{ is compact”}$.

Let $P := \sigma_1(I)^\mathbb{N}$. Let $\mathcal{L}$ be the lattice of subsets of $P$ generated by closed subsets of the form $F_i \times \sigma_1(I)^\mathbb{N}\{i\}$. Let $\mathcal{F}$ be a filter of $\mathcal{L}$. 
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Lemma: $T^{\text{fin}(I)}_\omega \Rightarrow \text{"}\sigma_1(I)^\mathbb{N}\text{ is compact"}.$

Let $P := \sigma_1(I)^\mathbb{N}$. Let $\mathcal{L}$ be the lattice of subsets of $P$ generated by closed subsets of the form $F_i \times \sigma_1(I)^\mathbb{N}\setminus\{i\}$. Let $\mathcal{F}$ be a filter of $\mathcal{L}$. Since each closed subset of $\sigma_1(I)$ is finite or contains $\infty$, we can build by recursion a sequence $(F_n)_{n \in \mathbb{N}}$ of finite subsets of $\sigma_1(I)$ such that for every $n \in \mathbb{N}$, $\prod_{i < n} F_i \times \sigma_1(I)^\mathbb{N}\setminus n$ is $\mathcal{F}$-stationnar (i.e. meets every element of $\mathcal{F}$).
Lemma: $T^{\text{fin}(I)}_\omega \Rightarrow \text{“}\sigma_1(I)^\mathbb{N} \text{ is compact”}$. 

Let $P := \sigma_1(I)^\mathbb{N}$. Let $\mathcal{L}$ be the lattice of subsets of $P$ generated by closed subsets of the form $F_i \times \sigma_1(I)^\mathbb{N}\{i\}$. Let $\mathcal{F}$ be a filter of $\mathcal{L}$. Since each closed subset of $\sigma_1(I)$ is finite or contains $\infty$, we can build by recursion a sequence $(F_n)_{n \in \mathbb{N}}$ of finite subsets of $\sigma_1(I)$ such that for every $n \in \mathbb{N}$, $\prod_{i \leq n} F_i \times \sigma_1(I)^\mathbb{N}\{n\}$ is $\mathcal{F}$-stationnar (i.e. meets every element of $\mathcal{F}$). Then, using $T^{\text{fin}(I)}_\omega$, the space $F := \prod_{i \in \mathbb{N}} F_i$ is compact and non-empty.
Compactness of $B^+(I)$ (2)

Lemma: $T^{\text{fin}(I)}_\omega \Rightarrow \text{"}\sigma_1(I)^N\text{ is compact".}$

Let $P := \sigma_1(I)^N$. Let $\mathcal{L}$ be the lattice of subsets of $P$ generated by closed subsets of the form $F_i \times \sigma_1(I)^N \setminus \{i\}$. Let $\mathcal{F}$ be a filter of $\mathcal{L}$. Since each closed subset of $\sigma_1(I)$ is finite or contains $\infty$, we can build by recursion a sequence $(F_n)_{n \in \mathbb{N}}$ of finite subsets of $\sigma_1(I)$ such that for every $n \in \mathbb{N}$, $\prod_{i < n} F_i \times \sigma_1(I)^N \setminus n$ is $\mathcal{F}$-stationnary (i.e. meets every element of $\mathcal{F}$). Then, using $T^{\text{fin}(I)}_\omega$, the space $F := \prod_{i \in \mathbb{N}} F_i$ is compact and non-empty. Now $\{F \cap Z : Z \in \mathcal{F}\}$ is a family of closed subsets of $F$ satisfying the FIP, thus $\cap \mathcal{F} \neq \emptyset$. 
Compactness of $B^+(I)$ (2)

Lemma: $T_{\omega}^{\text{fin}(I)} \Rightarrow \text{“} \sigma_1(I)^\mathbb{N} \text{ is compact”}$.

Let $P := \sigma_1(I)^\mathbb{N}$. Let $\mathcal{L}$ be the lattice of subsets of $P$ generated by closed subsets of the form $F_i \times \sigma_1(I)^\mathbb{N}\setminus\{i\}$. Let $\mathcal{F}$ be a filter of $\mathcal{L}$. Since each closed subset of $\sigma_1(I)$ is finite or contains $\infty$, we can build by recursion a sequence $(F_n)_{n \in \mathbb{N}}$ of finite subsets of $\sigma_1(I)$ such that for every $n \in \mathbb{N}$, $\prod_{i < n} F_i \times \sigma_1(I)^\mathbb{N}\setminus n$ is $\mathcal{F}$-stationnary (i.e. meets every element of $\mathcal{F}$). Then, using $T_{\omega}^{\text{fin}(I)}$, the space $F := \prod_{i \in \mathbb{N}} F_i$ is compact and non-empty. Now $\{F \cap Z : Z \in \mathcal{F}\}$ is a family of closed subsets of $F$ satisfying the FIP, thus $\cap \mathcal{F} \neq \emptyset$.

Proposition

If $\sigma_1(I)^\mathbb{N}$ is compact, then $\prod_{n \in \mathbb{N}} \sigma_2^{n+1}(I)$ -and thus $B^+(I)$- is compact.
Compactness of $B^+ (I)$ (3)

Using Kelley’s argument, $T^\text{fin}(I)$ implies the following statement:
Compactness of $B^+(I)$ (3)

Using Kelley's argument, $T_{\omega}^{\text{fin}(I)}$ implies the following statement: $\text{AC}_{\omega}^{\text{fin}(I)}$: "Every sequence $(F_n)_{n \in \mathbb{N}}$ of non-empty finite subsets of $I$ has a non-empty product."

Consequence

One cannot prove in $\text{ZF}$ that $B^+(\mathcal{P}(\mathbb{R}))$ is compact.
Compactness of $B^+(I)$ (3)

Using Kelley’s argument, $T_{\omega}^{\text{fin}(I)}$ implies the following statement: $\text{AC}_{\omega}^{\text{fin}(I)}$: “Every sequence $(F_n)_{n \in \mathbb{N}}$ of non-empty finite subsets of $I$ has a non-empty product.”

Consequence
One cannot prove in $\text{ZF}$ that $B^+(\mathcal{P}(\mathbb{R}))$ is compact.

Proof.
Consider a model of $\text{ZF}$ with a sequence $(P_n)_{n \in \mathbb{N}}$ of pairs of subsets of $\mathbb{R}$ such that $\prod_{n \in \mathbb{N}} P_n$ is empty.
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Consider a model of $\text{ZF}$ with a sequence $(P_n)_{n \in \mathbb{N}}$ of pairs of subsets of $\mathbb{R}$ such that $\prod_{n \in \mathbb{N}} P_n$ is empty. Let $I := \bigcup_{n \in \mathbb{N}} P_n$. 
Compactness of $B^+(I)$ (3)

Using Kelley’s argument, $T^\text{fin}(I)$ implies the following statement:

$\text{AC}^\text{fin}(I)$: “Every sequence $(F_n)_{n \in \mathbb{N}}$ of non-empty finite subsets of $I$ has a non-empty product.”

Consequence

One cannot prove in ZF that $B^+(\mathcal{P}(\mathbb{R}))$ is compact.

Proof.

Consider a model of ZF with a sequence $(P_n)_{n \in \mathbb{N}}$ of pairs of subsets of $\mathbb{R}$ such that $\prod_{n \in \mathbb{N}} P_n$ is empty. Let $I := \bigcup_{n \in \mathbb{N}} P_n$. Then $\text{AC}^\text{fin}(I)$ does not hold, thus $B^+(I)$ is not compact.
Compactness of $B^+(I)$ (3)

Using Kelley's argument, $T_\omega^{\text{fin}(I)}$ implies the following statement:

$AC_\omega^{\text{fin}(I)}$: “Every sequence $(F_n)_{n \in \mathbb{N}}$ of non-empty finite subsets of $I$ has a non-empty product.”

Consequence

One cannot prove in $ZF$ that $B^+(\mathcal{P}(\mathbb{R}))$ is compact.

Proof.

Consider a model of $ZF$ with a sequence $(P_n)_{n \in \mathbb{N}}$ of pairs of subsets of $\mathbb{R}$ such that $\prod_{n \in \mathbb{N}} P_n$ is empty. Let $I := \bigcup_{n \in \mathbb{N}} P_n$. Then $AC^{\text{fin}(I)}$ does not hold, thus $B^+(I)$ is not compact. But $I \subseteq \mathcal{P}(\mathbb{R})$, so $B^+(\mathcal{P}(\mathbb{R}))$ is not compact.

Proposition

$AC_\mathbb{N}^{\text{fin}} \iff T_\mathbb{N}^{\text{fin}}$. 
Some questions

Given a set $I$, $AETF$:

"$B^+(I)$ is compact"; "The subset $B_1(I) := \{x = (x_i)_{i \in I} \in [-1, 1]^I : \sum_i |x_i| \leq 1\}$ of $[0, 1]^I$ is compact"; "The closed unit ball $B_2(I)$ of the Hilbert space $\ell^2(I)$ is compact in the weak topology."
Some questions

Given a set $I$, \textit{AETF}:

“$B^+(I)$ is compact”; “The subset $B_1(I) := \{x = (x_i)_{i \in I} \in [-1, 1]^I : \sum_i |x_i| \leq 1\}$ of $[0, 1]^I$ is compact”; “The closed unit ball $B_2(I)$ of the Hilbert space $\ell^2(I)$ is compact in the weak topology.”
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Consider the following statements:

AH: *The closed unit ball of a Hilbert space is weakly compact.*
Some questions

Given a set $I$, $AETF$:

“$B^+(I)$ is compact”; “The subset $B_1(I) := \{ x = (x_i)_{i \in I} \in [-1, 1]^I : \sum_i |x_i| \leq 1 \}$ of $[0, 1]^I$ is compact”; “The closed unit ball $B_2(I)$ of the Hilbert space $\ell^2(I)$ is compact in the weak topology.”

Consider the following statements:

$AH$: The closed unit ball of a Hilbert space is weakly compact.

$BH$: “Every Hilbert space has a Hilbertian basis.”
Some questions

Given a set $I$, AETF:

“$B^+(I)$ is compact”; “The subset

$B_1(I) := \{ x = (x_i)_{i \in I} \in [-1, 1]^I : \sum_i |x_i| \leq 1 \}$ of $[0, 1]^I$ is compact”; “The closed unit ball $B_2(I)$ of the Hilbert space $\ell^2(I)$ is compact in the weak topology.”

Consider the following statements:

**AH**: *The closed unit ball of a Hilbert space is weakly compact.*

**BH**: “*Every Hilbert space has a Hilbertian basis.*”

Then $\textbf{AH} \Rightarrow \textbf{AC}^{\text{fin}}$ and $\textbf{AC}^{\text{fin}} + \textbf{BH} \Rightarrow \textbf{AH}$.
Some questions

Given a set \( I \), **AETF**: 

“\( B^+(I) \) is compact”; “The subset \( B_1(I) := \{ x = (x_i)_{i \in I} \in [-1, 1]^I : \sum_i |x_i| \leq 1 \} \) of \([0, 1]^I\) is compact”; “The closed unit ball \( B_2(I) \) of the Hilbert space \( \ell^2(I) \) is compact in the weak topology.”

Consider the following statements:

**AH**: *The closed unit ball of a Hilbert space is weakly compact.*

**BH**: “Every Hilbert space has a Hilbertian basis.”

Then \( \text{AH} \Rightarrow \text{AC}^{\text{fin}} \) and \( \text{AC}^{\text{fin}} + \text{BH} \Rightarrow \text{AH} \).

Questions

Does \( \text{AC}^{\text{fin}} \) imply \( \text{AH} \)?
Some questions

Given a set $I$, $AETF$:

“$B^+(I)$ is compact”; “The subset

$B_1(I) := \{ x = (x_i)_{i \in I} \in [-1, 1]^I : \sum_i |x_i| \leq 1 \}$ of $[0, 1]^I$ is compact”; “The closed unit ball $B_2(I)$ of the Hilbert space $\ell^2(I)$ is compact in the weak topology.”

Consider the following statements:

$AH$: *The closed unit ball of a Hilbert space is weakly compact.*

$BH$: *“Every Hilbert space has a Hilbertian basis.”*

Then $AH \Rightarrow AC_{\text{fin}}$ and $AC_{\text{fin}} + BH \Rightarrow AH$.

Questions

Does $AC_{\text{fin}}$ imply $AH$?
Some questions

Given a set $I$, $AETF$:

“$B^+(I)$ is compact”; “The subset $B_1(I) := \{ x = (x_i)_{i \in I} \in [-1, 1]^I : \sum_i |x_i| \leq 1 \}$ of $[0, 1]^I$ is compact”; “The closed unit ball $B_2(I)$ of the Hilbert space $l^2(I)$ is compact in the weak topology.”

Consider the following statements:

$\textbf{AH}$: The closed unit ball of a Hilbert space is weakly compact.

$\textbf{BH}$: “Every Hilbert space has a Hilbertian basis.”

Then $\textbf{AH} \Rightarrow \textbf{AC}^{\text{fin}}$ and $\textbf{AC}^{\text{fin}} + \textbf{BH} \Rightarrow \textbf{AH}$.

Questions

Does $\textbf{AC}^{\text{fin}}$ imply $\textbf{AH}$?

Does $\textbf{BH}$ imply $\textbf{AC}$ or some classical consequence of $\textbf{AC}$?
Some questions

Given a set $I$, $\text{AETF}$:

"$B^+(I)$ is compact"; "The subset
$B_1(I) := \{x = (x_i)_{i \in I} \in [-1, 1]^I : \sum_i |x_i| \leq 1\}$ of $[0, 1]^I$ is compact"; "The closed unit ball $B_2(I)$ of the Hilbert space $\ell^2(I)$ is compact in the weak topology."

Consider the following statements:

$\text{AH}$: *The closed unit ball of a Hilbert space is weakly compact.*

$\text{BH}$: *“Every Hilbert space has a Hilbertian basis.”*

Then $\text{AH} \Rightarrow \text{AC}^{\text{fin}}$ and $\text{AC}^{\text{fin}} + \text{BH} \Rightarrow \text{AH}$.

Questions

Does $\text{AC}^{\text{fin}}$ imply $\text{AH}$?

Does $\text{BH}$ imply $\text{AC}$ or some classical consequence of $\text{AC}$?

Is $\text{BH}$ provable in $\text{ZF}$? in $\text{ZF} + \text{AC}^{\text{fin}}$? What about the existence of Markhushevich bases in WCG Banach spaces?


