UNIFORM SMOOTHNESS ENTAILS HAHN-BANACH

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Abstract. We show in set theory **ZF** (without the Axiom of Choice), that uniformly smooth normed spaces satisfy an effective and geometric form of the Hahn-Banach property. We also compare in **ZF** the two notions of Gâteaux differentiability and smoothness of a norm, and we obtain a new equivalent of the Hahn-Banach axiom.

1. Introduction

We work in Zermelo-Fraenkel set theory **ZF** (without the Axiom of Choice), and we denote by **ZFC** set theory with the Axiom of Choice. Our paper deals with the *rôle* of the Axiom of Choice in abstract functional analysis, and more particularly, with the necessity of using the Axiom of Choice when invoking some consequence of the following *Hahn-Banach axiom* **HB** :

(**HB**, **Hahn-Banach**). *For every real vector space E, for every sublinear mapping $p : E \to \mathbb{R}$, for every subspace F of E and every linear mapping $f : F \to \mathbb{R}$ which is dominated by p (i.e. satisfying $\forall x \in F\ f(x) \leq p(x)$), there exists a linear mapping $g : E \to \mathbb{R}$ which extends f and such that $g \leq p$.*

Here, a mapping $p : E \to \mathbb{R}$ is said to be *sublinear* if for every $x, y \in E$ and every $\lambda \in \mathbb{R}_+$, $p(x + y) \leq p(x) + p(y)$ and $p(\lambda x) = \lambda p(x)$.

Recall that **HB** is a consequence of the Axiom of Choice, but that **HB** is not provable in set theory **ZF** (see [7]). However, several classes of Banach spaces satisfy in **ZF** some classical geometric forms of the Hahn-Banach property : for example, see [6] for *separable* Banach spaces which are both uniformly convex and Gâteaux differentiable (in a *constructive* setting) ; see [4] for Hilbert spaces, normed spaces which have a dense well-orderable subset, and spaces $\ell^0(I)$ (see Notation 2) where I is *any* set ; see [3] for uniformly convex Gâteaux differentiable Banach spaces, and in particular for spaces $L^p(\mathcal{B}, \nu)$ where $1 < p < +\infty$, $\mathcal{B}$ is a boolean algebra, and $\nu : \mathcal{B} \to \mathbb{R}_+ \cup \{+\infty\}$ is a finitely additive measure... Denoting by ω the first infinite ordinal, consider the following *axiom of Dependent Choices* :

(**DC**, **Dependent Choices**). *For every binary relation R on a nonempty set E satisfying $\forall x \in E\ \exists y \in E\ xRy$, there exists a sequence $(x_n)_{n\in\omega}$ satisfying $\forall n \in \omega\ x_n R x_{n+1}$.*

In [3], it is also shown in (**ZF**+**DC**) that several geometric Hahn-Banach properties hold in Gâteaux differentiable spaces. Note that, obviously **DC** is a consequence of the Axiom of Choice, but **DC** does not imply **HB**, and **HB** does not imply **DC** (see [7]). For a recent account on links between numerous consequences of the Axiom of Choice, see [5].

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We adopt the following conventions about normed spaces : all vector spaces that we consider are vector spaces over the field $\mathbb{R}$ of *real* numbers. Given a normed space $(E, \|.\|)$, a real number $R \geq 0$ and a point $a \in E$, we denote by $B(a, R)$ the open ball $\{x \in E : \|x - a\| < R\}$, and we denote by $\Gamma(a, R)$ the closed ball $\{x \in E : \|x - a\| \leq R\}$. Let $\Gamma_E := \{x \in E : \|x\| \leq 1\}$ and let $S_E := \{x \in E : \|x\| = 1\}$ respectively denote the closed unit ball and the unit sphere of E. We denote by E' *continuous dual of* E, *i.e.* the vector space of all continuous linear mappings from E to $\mathbb{R}$; it is endowed with the *dual norm* : for every $f \in E'$, $\|f\| = \sup_{x \in \Gamma_E} f(x)$.

Definition 1 (Tangent form to a nonempty convex set). Let $(E, \|.\|)$ be a normed space, let C be a closed nonempty convex subset of E such that $0 \notin C$, and let $\rho := \inf_{z \in C} \|z\|$ be the distance between 0 and C. An element $f \in E'$ is called a *tangent form to C seen from* 0 if and only if $f[\Gamma(0, \rho)] \leq \rho \leq f[C]$.

Remark 1. In the conditions of Definition 1, $\sup_{\Gamma(0,\rho)} f = \inf_C f = \rho$ hence $\|f\| = 1$.

Remark 2. In the conditions of Definition 1, one can prove in (**ZF+HB**) that there exists at least one tangent form to C seen from 0.

Remark 3. In the conditions of Definition 1, if E is *finite-dimensional*, one can prove in **ZF** (see Lemma 6) that there exists at least one tangent form to C seen from 0.

Remark 4. More generally, given a nonempty closed convex subset C of a normed space E, a point $a \in E \backslash C$, an *affine* mapping $f : E \to \mathbb{R}$, say that f is a *tangent form to C seen from* a if $f(a) = 0$ and $f[\Gamma(a, \rho)] \leq \rho \leq f[C]$, where ρ is the distance between a and C. (Recall that a mapping $f : E \to \mathbb{R}$ is said to be *affine* if there exists a (unique) linear mapping $g : E \to \mathbb{R}$ and a (unique) real number C such that $f = g + C$).

In this paper, our aim is to prove in **ZF** (see Theorem 1), that given a uniformly smooth normed space $(E, \|.\|)$, for every nonempty closed convex subset C of E, and for every $a \in E \backslash C$, there exists a unique tangent form to C seen from a ; moreover, this tangent form is *definable* from $(E, \|.\|)$, C and a. Our proof relies on the following geometric fact (see Lemma 3) : given a convex set C which is contained in a *thin* crown of a uniformly smooth normed space, for every $x, y \in C$, the two Gâteaux differentials at point x and y are *close* to one another. As a consequence, it will follow that every uniformly smooth normed space E satisfies the two following equivalent (see [3]) properties :

> **Effective Mazur property :** *There is a mapping Φ which, to every ordered pair (C, a) where C is a nonempty closed convex subset of E and $a \in E \backslash C$, associates a tangent form to C seen from a.*
>
> **Effective continuous Hahn-Banach property :** *There is a function Ψ such that, for every* continuous *sublinear mapping $p : E \to \mathbb{R}$, for every vector subspace F of E, and for every linear mapping $f : F \to \mathbb{R}$ satisfying $f \leq p_{\upharpoonright F}$, the triple (p, F, f) belongs to dom(Ψ) and $\Psi(p, F, f)$ is a linear mapping from E to $\mathbb{R}$ extending f and satisfying $\Psi(p, F, f) \leq p$.*

Notice that **HB** is equivalent to its *"multiple form"* :

> *Given a family $(E_i)_{i \in I}$ of real vector spaces, and a family $(p_i)_{i \in I}$ of sublinear mappings $p_i : E_i \to \mathbb{R}$, for every family $(F_i)_{i \in I}$ of subspaces $F_i \subseteq E_i$ and every*

family $(f_i)_{i\in I}$ of linear mappings $f_i : F_i \to \mathbb{R}$ satisfying $\forall x \in F_i\ f_i(x) \leq p_i(x)$), there exists a family $(g_i)_{i\in I}$ of linear mappings $g_i : E_i \to \mathbb{R}$ such that for every $i \in I$, g_i extends f_i and $g_i \leq p_i$.

It follows that in (**ZF**+**HB**), every normed space satisfies the effective continuous Hahn-Banach property, hence it also satisfies the Effective Mazur property.

Remark 5. Although in **ZFC** every uniformly smooth normed space is *superreflexive* (see [1]), hence it has an equivalent norm which is both uniformly convex and uniformly smooth (see [2], Proposition 5.2, page 159), we cannot use these results because the classical proofs depend on the Axiom of Choice : for example, the proof of the non-existence of bounded infinite trees in a uniformly smooth Banach space E in [1] (Proposition 4 p.232-235) relies on the weak compactness of the closed unit ball of E, and this weak compactness cannot be proved in **ZF**, even in the case of Hilbert spaces (see [4]).

In Section 5, we will also compare two classical notions of differentiability for a norm : in (**ZF**+**HB**), it is well known (see [1], Proposition 2 p.179) that given a normed space E, smoothness of the norm at a point $a \in E\backslash\{0\}$ is equivalent to Gâteaux differentiability of the norm at this point, but this equivalence is not provable in **ZF** (see Sections 5.1 and 5.2) : this leads us to a new equivalent of the Hahn-Banach axiom (see Proposition 2).

Our paper is organized as follows : in Section 2 we present various notions of differentiability for a norm ; in Section 3 we prove our main result Theorem 1, namely a choiceless Hahn-Banach theorem in uniformly smooth spaces ; in Section 4 we give several examples which show how Functional Analysis looks like without the Hahn-Banach axiom ; then finally, in Section 5 we compare the two notions of smoothness and Gâteaux differentiability for a norm.

2. Various notions of differentiability of the norm

In this Section, we recall the following notions of differentiability for a norm $\|.\|$ on a vector space E : smoothness, Gâteaux differentiability, Fréchet differentiability and uniform smoothness (these notions are stated from the weakest to the strongest).

2.1. Smoothness of the norm.

A continuous linear mapping $f \in S_{E'}$ is said to be *norming* at a point $a \in E\backslash\{0\}$ if and only if $f(a) = \|a\|$. The normed space $(E, \|.\|)$ is said to be *smooth at point* $a \in E\backslash\{0\}$ if and only if there *exists* a *unique* norming mapping at this point a. The space $(E, \|.\|)$ is said to be *smooth* (see [1] page 177) if and only if it is smooth at every point $a \in E\backslash\{0\}$.

Note, for every $a \in E\backslash\{0\}$, the existence of a norming linear mapping at point a is provable in (**ZF** + **HB**) ; so in (**ZF** + **HB**), smoothness at a point is equivalent to *uniqueness* of a norming linear mapping at this point.

2.2. Gâteaux differentiability of the norm.

Here are some classic facts about *Gâteaux differentiability* for a norm (see [1] pages 178-179). For every $a \in E\backslash\{0\}$ and every $h \in E$, the convexity of the norm implies that the function $\tau_a^h : t \mapsto \frac{\|a+th\|-\|a\|}{t}$ is non-decreasing, thus it has a limit when $t \to 0^-$ (*resp.* when $t \to 0^+$) and the following inequality holds :

$$\lim_{t\to 0^-} \frac{\|a+th\|-\|a\|}{t} \le \lim_{t\to 0^+} \frac{\|a+th\|-\|a\|}{t} \tag{1}$$

Let $G^-(a,h) := \lim_{t\to 0^-} \frac{\|a+th\|-\|a\|}{t}$ and $G^+(a,h) := \lim_{t\to 0^+} \frac{\|a+th\|-\|a\|}{t}$. The sublinearity of the norm implies that the mapping $G^+(a,.)$ is sublinear and satisfies

$$\forall h \in E,\ G^+(a,h) \le \|h\| \tag{2}$$

Moreover, since $\|x\| = \|-x\|$ holds for every $x \in E$,

$$\forall h \in E,\ \ G^+(a,-h) = -G^-(a,h) \tag{3}$$

Statement (3) implies that the mapping $G^-(a,.)$ is *superlinear*, *i.e.* the two following conditions are satisfied :

$$\forall h_1, h_2 \in E,\ \ G^-(a, h_1+h_2) \ge G^-(a,h_1) + G^-(a,h_2)$$

$$\forall \lambda \in \mathbb{R}_+\ \forall x \in E,\ \ G^-(a,\lambda x) = \lambda G^-(a,x)$$

Now, the norm $\|.\|$ is *Gâteaux differentiable* at a point $a \in E\backslash\{0\}$ if and only if, for every $h \in E$, $\lim_{t\to 0, t\neq 0} \frac{\|a+th\|-\|a\|}{t}$ exists in $\mathbb{R}$, *i.e.* $G^+(a,h) = G^-(a,h)$; in this case, for every $h \in E$, we denote by $G(a,h)$ the real number $\lim_{t\to 0,t\neq 0} \frac{\|a+th\|-\|a\|}{t}$; then the mapping $G(a,.)$ is linear (because $G^+(a,.)$ is sublinear and $G^-(a,.)$ is superlinear), it is continuous with norm ≤ 1 (because of (2)), and in fact $\|G(a,.)\| = 1$ (because $G(a,a) = \|a\|$). The normed space $(E, \|.\|)$ is *Gâteaux differentiable* if and only if its norm is Gâteaux differentiable at every point $a \in E\backslash\{0\}$.

In Subsection 5.1 we will show that in **ZF**, Gâteaux differentiability at a given point implies smoothness of the norm at this point ; the converse is provable in (**ZF**+**HB**), but it is not provable in **ZF** (see Proposition 2).

2.3. **Fréchet differentiability.** Given a normed space $(E, \|.\|)$, the norm $\|.\|$ is said to be *Fréchet differentiable* at a point $a \in E\backslash\{0\}$ if it is Gâteaux differentiable and

$$\lim_{t\to 0,\ t\neq 0} \frac{\|a+th\|-\|a\|}{t} \text{ is uniform in } h \in S_E \tag{4}$$

Since the function τ_a^h is non-decreasing on $\mathbb{R}_+^*$, the norm $\|.\|$ is Fréchet differentiable at point a if and only if

$$\lim_{t\to 0^+, t\neq 0} \left(\frac{\|a+th\|-\|a\|}{t} - \frac{\|a-th\|-\|a\|}{-t}\right) = 0 \text{ , uniformly in } h \in S_E \tag{5}$$

The normed space $(E, \|.\|)$ is said to be *Fréchet differentiable* if its norm is Fréchet differentiable at every point $a \in E\backslash\{0\}$.

2.4. **Uniform smoothness.** Let $(E, \|.\|)$ be a normed space such that $E \neq \{0\}$. The normed space $(E, \|.\|)$ is said to be *uniformly Fréchet differentiable* (see [2] Definition 1.9 p.8) if

$$\lim_{t\to 0,\, t\neq 0} \frac{\|a+th\| - \|a\|}{t} \text{ exists for each } a \in S_E \text{ and each } h \in S_E, \tag{6}$$

and is uniform in $(a, h) \in S_E \times S_E$.

Notice that $(E, \|.\|)$ is uniformly Fréchet differentiable if and only if

$$\lim_{t\to 0^+, t\neq 0} \left(\frac{\|a+th\| - \|a\|}{t} - \frac{\|a-th\| - \|a\|}{-t}\right) = 0 \text{ , uniformly in } a, h \in S_E \tag{7}$$

Now, consider the following function ρ_E, which is called the *modulus of smoothness of* the normed space E (see [1] page 204) :

$$\rho_E \;: t \to \sup_{\|a\|=\|b\|=1} \{\frac{\|a+tb\| + \|a-tb\|}{2} - 1\}$$

The space E is said to be *uniformly smooth* if

$$\lim_{t\to 0^+} \frac{\rho_E(t)}{t} = 0 \tag{8}$$

This is equivalent to the following condition :

$$\lim_{h\to 0,\, h\neq 0} \frac{\|a+h\| + \|a-h\| - 2\,\|a\|}{\|h\|} = 0 \text{ , uniformly in } a \in S_E \tag{9}$$

Since (7) and (9) are equivalent, uniform Fréchet differentiability and uniform smoothness are equivalent.

It is easy to prove that every finite-dimensional normed space E which is Gâteaux differentiable is uniformly smooth (because Γ_E is compact), but in general, Gâteaux differentiability, Fréchet differentiability and uniform smoothness are three distinct notions.

2.5. **Šmulian tests.** We recall some necessary conditions for uniform smoothness of the norm.

Definition 2 (**Šmulian test of uniform smoothness).** A mapping η from $\mathbb{R}_+^*$ to $\mathbb{R}_+^*$ is said to be a *Šmulian test* of uniform smoothness for a normed space $(E, \|.\|)$ if and only if, for every $\varepsilon > 0$, and for every $f, g \in S_{E'}$:

$$[\exists a \in S_E,\ \big(f(a) > 1 - \eta(\varepsilon) \text{ and } g(a) > 1 - \eta(\varepsilon)\big)] \Rightarrow \|f - g\| < \varepsilon$$

Proposition 1. *Let $(E, \|.\|)$ be a uniformly smooth normed space with modulus of smoothness ρ, and let $\delta : \mathbb{R}_+^* \to \mathbb{R}_+^*$ be the mapping $\varepsilon \mapsto \sup\{t \in]0,1] \;:\; \frac{\rho(t)}{t} \leq \frac{\varepsilon}{4}\}$. Then the mapping $\eta : \varepsilon \mapsto \frac{\varepsilon\delta(\varepsilon)}{4}$ is a Šmulian test of uniform smoothness for E.*

Proof. We follow the idea in the proof of Theorem 1.4 page 3 in [2], but, in order to work in **ZF**, we avoid the use of sequences.

Let $\varepsilon > 0$. Clearly the function $t \mapsto \rho(t)$ is convex on $\mathbb{R}_+^*$ hence the function $t \mapsto \frac{\rho(t)}{t}$ is non-decreasing on $\mathbb{R}_+^*$, thus for every real number t, $0 < t < \delta(\varepsilon) \Rightarrow \frac{\rho(t)}{t} \leq \frac{\varepsilon}{4}$. It follows that, for every $h \in E$, for every $a \in S_E$,

$$\|h\| \leq \delta(\varepsilon) \ \Rightarrow\ \|a+h\| + \|a-h\| - 2 \leq \frac{\varepsilon}{2}\|h\|$$

We deduce that for every $f, g \in \Gamma_{E'}$, for every $a \in S_E$ and for every $h \in E$:

$$\|h\| \leq \delta(\varepsilon) \Rightarrow f(a+h) + g(a-h) \leq \|a+h\| + \|a-h\| \leq 2 + \frac{\varepsilon}{2}\|h\|$$

whence

$$\|h\| \leq \delta(\varepsilon) \Rightarrow (f-g)(h) \leq 2 - f(a) - g(a) + \frac{\varepsilon}{2}\|h\|$$

So, for every $f, g \in \Gamma_{E'}$, for every $a \in S_E$ such that $f(a) > 1 - \eta(\varepsilon)$ and $g(a) > 1 - \eta(\varepsilon)$:

$$\forall h \in E,\ \big(\|h\| \leq \delta(\varepsilon) \Rightarrow (f-g)(h) \leq 2\eta(\varepsilon) + \frac{\varepsilon}{2}\|h\| \leq \varepsilon\delta(\varepsilon)\big)$$

Thus $\|f-g\| \leq \varepsilon$. □

Remark 6. The following statement is provable in **ZF** :

> *Let E be a normed space which has a Šmulian test of uniform smoothness. If for every $a \in S_E$, for every real number $\delta \in]0,1[$, there exists $f \in S_{E'}$ satisfying $f(a) \geq \delta$, then E is uniformly smooth.*

Proof. We adapt the proof of Theorem 1.4 p.3-4 in [2], avoiding the use of sequences. Suppose that E is not uniformly smooth ; consider some $\alpha > 0$ such that $\inf_{t>0} \frac{\rho(t)}{t} > \alpha$; then

$$\forall t > 0\ \exists (a,b) \in S_E \times S_E\ \ \frac{\|a+tb\| + \|a-tb\|}{2} \geq 1 + \alpha t \tag{10}$$

Let $\eta > 0$. Let $\varepsilon \in \mathbb{R}$ such that $0 < \varepsilon < \frac{\eta}{2+\frac{\alpha}{2}}$. Using (10), let $a, b \in S_E$ satisying $\|a+\varepsilon b\| + \|a-\varepsilon b\| \geq 2 + 2\alpha\varepsilon$. Let $f, g \in S_{E'}$ satisfying $f(a+\varepsilon b) \geq \|a+\varepsilon b\| - \frac{\alpha\varepsilon}{2}$ and $g(a-\varepsilon b) \geq \|a-\varepsilon b\| - \frac{\alpha\varepsilon}{2}$. Then

$$f(a) = f(a+\varepsilon b) - \varepsilon f(b) \geq \|a+\varepsilon b\| - \frac{\alpha\varepsilon}{2} - \varepsilon \geq 1 - \varepsilon - \frac{\alpha\varepsilon}{2} - \varepsilon = 1 - \varepsilon(2 + \frac{\alpha}{2}) \geq 1 - \eta$$

and likewise, $g(a) \geq 1 - \eta$. Besides

$$f(a+\varepsilon b) + g(a-\varepsilon b) \geq \|a+\varepsilon b\| + \|a-\varepsilon b\| - \alpha\varepsilon \geq 2 + \alpha\varepsilon$$

hence

$$(f-g)(\varepsilon b) = f(a+\varepsilon b) + g(a-\varepsilon b) - f(a) - g(a) \geq 2 + \alpha\varepsilon - 2 = \alpha\varepsilon$$

thus $\|f-g\| \geq \alpha$. So E cannot have any Šmulian test. □

Remark 7. It follows from Remark 6 that in **ZF**, every Gâteaux differentiable normed space which has a Šmulian test is uniformly smooth. Since every uniformly smooth normed space has a Šmulian test (see Proposition 1) and since uniform smoothness implies Gâteaux differentiability, the two following properties are equivalent : "E is uniformly smooth.", "E is Gâteaux differentiable and E admits a Šmulyan test."

Using Remark 6, the following statement is provable in (**ZF**+**HB**) :

Every normed space which has a Šmulian test of uniform smoothness is uniformly smooth.

Notice that this last converse statement is not provable in **ZF** (see Remark 13).

3. Uniform smoothness entails the effective Mazur property

This Section leads to our main result (see Theorem 1 and Corollary 2).

Lemma 1 (Uniqueness of the tangent form). *Let $(E, \|.\|)$ be a Gâteaux differentiable normed space which admits a Šmulyan test. Let C be a nonempty closed convex subset of E such that $0 \notin C$. If g is a tangent form to C seen from 0 then $g \in \cap_{\lambda>\rho}\overline{\{G(a,.)\ : a \in C, \|a\| \leq \lambda\}}$.*

Proof. Let η be a Šmulian test for E. Let $\rho := d(0, C)$. Let $\varepsilon > 0$ and let $\lambda > \rho$; by definition of ρ, let $a \in C$ such that $\|a\| < \rho(1 + \eta(\varepsilon))$ and $\|a\| \leq \lambda$; then $g(a) \geq \rho$ hence $g(\frac{a}{\|a\|}) \geq \frac{\rho}{\rho(1+\eta(\varepsilon))} > 1-\eta(\varepsilon)$, while $G(a, a) = 1 \geq 1-\eta(\varepsilon)$, hence $\|g - G(a, .)\| < \varepsilon$. It follows that $g \in \cap_{\lambda>\rho}\overline{\{G(a,.)\ : a \in C, \|a\| \leq \lambda\}}$. □

Definition 3. Given a nonempty subset C of a normed space E, and a point $a \in E$, a point $m \in C$ is said to be a *best approximation* of a in C when $\|m - a\| = \inf_{z\in C} \|z - a\|$.

Remark 8. A point may have several best approximations in a given closed convex set : for example, consider the space $E = \mathbb{R}^2$ endowed with the *"sup norm"*, then the set of best approximations of $(2, 0)$ in Γ_E is the segment $[(1, -1), (1, 1)]$.

Remark 9. Given a nonempty closed convex subset C of a finite dimensional normed space E, it is provable in **ZF** that every point of E has a best approximation in C. More generally, the same conclusion holds in **ZFC** for every *reflexive* space E, but this statement is not provable in **ZF** (see Subsection 4.4).

Lemma 2 (Ishihara, see [6])**.** *Let $(E, \|.\|)$ be a normed space. Let C be a nonempty convex subset of E such that $0 \notin C$. If 0 has a best approximation $a \in C$ and if the norm $\|.\|$ is Gâteaux differentiable at point a, then $G(a, .)$ is a tangent form to C seen from 0.*

Proof. Since a is a best approximation of 0 in C, $d(0, C) = \|a\|$. For every $z \in C$, $G(a, z) = G(a, a) + G(a, z - a) = \|a\| + G(a, z - a)$; moreover, for every $t \in [0, 1]$, $a + t(z - a) = (1 - t)a + tz \in C$, hence $\|a + t(z - a)\| \geq \|a\|$. It follows that for every $z \in C$, $G(a, z - a) = \lim_{t\to 0^+} \frac{\|a+t(z-a)\|-\|a\|}{t} \geq 0$ whence $G(a, z) \geq \|a\|$. □

Lemma 3. *Let $(E, \|.\|)$ be a Gâteaux differentiable normed space which admits a Šmulyan test η. Let ε, μ be positive real numbers, let D be the crown $\{x \in E : \mu \leq \|x\| \leq \mu(1+\eta(\varepsilon))\}$, and let C be a nonempty closed convex subset of this crown. Then the diameter of the subset $\{G(x, .) : x \in C\}$ of the dual E' is less than 2ε.*

Proof. We are to prove that

$$\forall a, b \in C\ \ \|G(a, .) - G(b, .)\| \leq 2\varepsilon$$

hence it is sufficient to prove the Lemma in the case E is finite-dimensional (and even three-dimensional) : let ρ be the distance between 0 and C (notice that $\rho \geq \mu$). Since E is finite-dimensional, let $u \in C$ be a best approximation of 0 in C ; let $f = G(u,.)$; then $f \in S_{E'}$ and, using Lemma 2, $\rho \leq f[C]$. Given $a \in C$, $f(a) \geq \rho$ so $f(\frac{a}{\|a\|}) \geq \frac{\rho}{\|a\|} \geq \frac{\rho}{\mu(1+\eta(\varepsilon))} \geq \frac{1}{1+\eta(\varepsilon)} \geq 1-\eta(\varepsilon)$, while $G(a, \frac{a}{\|a\|}) = 1 \geq 1-\eta(\epsilon)$, hence $\|G(a,.)-f\| \leq \varepsilon$. Thus, for every $a, b \in C$, $\|G(a,.)-G(b,.)\| \leq 2\varepsilon$. □

Definition 4 (complete metric space). We say that a metric space is *complete* if and only if every Cauchy filter has a limit point.

Remark 10. Note that the above notion of completeness implies the notion of *sequential completeness* : "Every Cauchy sequence is convergent.", but the converse does not hold in **ZF**, see Subsection 4.5.

It is easy to prove that $\mathbb{R}$ endowed with the standard metric is complete, and more generally, for every set I, the normed space $\ell^\infty(I)$ endowed with the uniform norm (see Notation 2) is complete. It follows that *the continuous dual E' of every normed space E is complete*, since it is a closed subset of $\ell^\infty(\Gamma_E)$.

Lemma 4 (Geometric interpretation of uniform smoothness). *Let $(E, \|.\|)$ be a Gâteaux differentiable normed space which admits a Šmulyan test and let C be a nonempty closed convex subset of E such that $0 \notin C$. Let ρ be the distance between 0 and C, and for every integer $n \geq 1$, denote by C_n the closure in E' of $\{G(x,.) \ : x \in C$ and $\|x\| \leq \rho + \frac{1}{n+1}\}$. The set $\bigcap_{n\in\omega} C_n$ is a singleton.*

Proof. The sequence of nonempty sets $(C_n)_{n\in\omega}$ is decreasing and, using Lemma 3, the sequence of the diameters of the sets C_n tends to 0. Since E' is complete, it follows that $\bigcap_{n\in\omega} C_n$ is a singleton. □

Notation 1. In the conditions of Lemma 4, the element of the singleton $\bigcap_{n\in\omega} C_n$ is denoted by f_C.

Lemma 5. *Let $(E, \|.\|)$ be a Gâteaux differentiable normed space which admits a Šmulyan test and let C be a nonempty closed convex subset of E such that $0 \notin C$. If 0 has a best approximation $a \in C$, then $f_C = G(a,.)$ and f_C is the unique tangent form to C seen from 0.*

Proof. Let $\rho := d(0, C)$. Since

$$G(a,.) \in \bigcap_{n\in\omega} \{G(x,.) \ : x \in C \text{ and } \|x\| \leq \rho + \frac{1}{n+1}\}$$

it follows that $G(a,.) = f_C$; moreover, using Lemma 2, $G(a,.)$ is a tangent form to C seen from 0. The uniqueness of the tangent form follows from Lemma 1. □

Remark 11 (**ZFC**). If E is uniformly smooth space, if C is a nonempty closed convex subset of E such that $0 \notin C$, one can prove in **ZFC** that f_C is the *unique tangent form to C seen from* 0 : in fact, since E is a uniformly smooth Banach space, every bounded closed convex subset of E is weakly compact because E is reflexive (here, we use the Axiom of Choice), so 0 has a best approximation in C, hence, using Lemma 5, f_C is the unique tangent form to C seen from 0.

In fact, Remark 11 holds in **ZF** :

Theorem 1. *Let $(E, \|.\|)$ be a Gâteaux differentiable normed space which admits a Šmulyan test. Let C be a nonempty closed convex subset of E such that $0 \notin C$. Then f_C is the unique tangent form to C seen from 0.*

Proof. The uniqueness of the tangent form follows from Lemma 1. We now show that the mapping f_C is a tangent form to C seen from 0. Let $\rho := d(0, C) > 0$. Since $f_C \in S_{E'}$, it is sufficient to prove that $\forall x \in C\ f_C(x) \geq \rho$. Denote by $\mathcal{F}$ the set of finite-dimensional subspaces F of E meeting C ; the partial ordered set $(\mathcal{F}, \subseteq)$ is directed. For every $F \in \mathcal{F}$, let C_F be the closed convex subset $C \cap F$ of F, let ρ_F be the distance between 0 and C_F, and let $f_{C_F} \in F'$ be the (unique) tangent form to C_F, seen from 0. Let $x \in E$. By definition of f_C :

$$f_C(x) = \lim_{a \in C, \|a\| \to \rho} G(a, x) = \lim_{F \in \mathcal{F}, a \in C_F, \|a\| = \rho_F} G(a, x) \tag{11}$$

For each $F \in \mathcal{F}$ such that $F \ni x$, for each $a \in C_F$ such that $\|a\| = \rho_F$, it follows from Lemma 5 that $G(a, x) = f_{C_F}(x)$, while $f_{C_F}(x) \geq \rho_F \geq \rho$, hence using equality (11), $f_C(x) \geq \rho$. □

Note that given a Gâteaux differentiable normed space $(E, \|.\|)$ which admits a Šmulian test, a nonempty closed convex subset C of E, and a point $a \in E \backslash C$, then the closed convex set $C_a := C - a$ does not contain 0, and the affine mapping $f_{C_a} - f_{C_a}(a)$ is the unique tangent form to C seen from a (see Remark 4).

Corollary 1. *Every Gâteaux differentiable normed space which admits a Šmulian test satisfies the effective Mazur property.*

Proof. Let E be Gâteaux differentiable normed space which admits a Šmulian test. Consider the mapping Φ which, to every ordered pair (C, a) where C is a nonempty closed convex subset of E and $a \in E \backslash C$, associates the (unique) tangent form to C seen from a. Then Φ witnesses the effective Mazur property on E. □

Using Remark 7, Corollary 1 can be reformulated as follows :

Corollary 2. *Every uniformly smooth normed space satisfies the effective Mazur property.*

Remark 12. Using Lemma 6 below, every *finite*-dimensional normed space E satisfies the two following properties :

> **Hahn-Banach property :** *For every sublinear mapping $p : E \to \mathbb{R}$, for every vector subspace $F \subseteq E$, and for every linear mapping $f : F \to \mathbb{R}$ such that $f \leq p_{\restriction F}$, there exists a linear mapping $g : E \to \mathbb{R}$ such that $g \leq p$ and g extends f.*
>
> **Mazur property :** *For every ordered pair (C, a) such that C is a nonempty closed convex subset of E and $a \in E \backslash C$, there exists a tangent form to C seen from a.*

Lemma 6. *(Hahn-Banach's "finite extension lemma") Let V be a vector space, let K be a subspace of V, let $p : V \to \mathbb{R}$ be a sublinear mapping, and let $f : K \to \mathbb{R}$ be a linear mapping satisfying $f \leq p$. If there exists a finite subset F of V such that $K \cup F$ spans the whole vector space V, then there exists a linear mapping $g : V \to \mathbb{R}$ that extends f and such that $g \leq p$.*

Proof. Let F be a minimal finite subset F such that $K \cup F$ spans V.
First case : F is a singleton $\{e\}$. Let $m = \sup_{y \in V} \{f(y) - p(y - e)\}$ and $M = \inf_{y \in V} \{p(y + e) - f(y)\}$. Since p is sublinear, $m \leq M$. For every real number $\lambda \in [m, M]$, the linear mapping g which extends f and such that $g(e) = \lambda$ satisfies $g \leq p$.
General case. By induction on the cardinal of F, using the previous case. □

4. Horrors of functional analysis without choice

In this Section, we give some examples of classical theorems of Functional Analysis which hold in **ZFC** but which are not provable in **ZF**. Given a normed space E, we denote by $j_E : E \to E''$ the *canonical mapping* which is defined as follows : for all $x \in E$, for all $f \in E'$, $j_E(x)(f) = f(x)$. Clearly, j_E is linear, continuous and $\|j_E\| \leq 1$. Using **HB**, one can prove that j_E is *isometric*, *i.e.* for every $x \in E$, $\|j_E(x)\| = \|x\|$, but this result does not hold in **ZF** (see Subsection 4.2 below). In **ZFC**, there are many equivalent definitions of reflexivity for a normed space ; since we work in **ZF**, we now provide a definition for reflexivity :

Definition 5. A normed space E is said to be *reflexive* when j_E is isometric and onto.

Notation 2. For every set I, denote by $\ell^\infty(I)$ the space of bounded mappings $f : I \to \mathbb{R}$ endowed with the *uniform* norm $\|.\|_\infty$ which is defined as follows : for every $f \in l^\infty(I)$, $\|f\|_\infty := \sup_{i \in I} |f(i)|$. Denote by $\ell^0(I)$ the subspace of $\ell^\infty(I)$ which is defined as follows :

$$\ell^0(I) := \{f \in \ell^\infty(I) : \forall \varepsilon > 0 \; \exists F_0 \in \mathcal{P}_f(I) \; \forall i \in I \backslash F_0 \;\; |f(i)| \leq \varepsilon\}$$

Denote by $\ell^1(I)$ the space of elements $f \in \mathbb{R}^I$ such that $\sum_{i \in I} |f(i)| < +\infty$, endowed with the norm $\|.\|_1$ which is defined as follows : for every $f \in \ell^1(I)$, $\|f\|_1 := \sum_{i \in I} |f(i)|$.

It is well known that the continuous dual of $\ell^0(I)$ is $\ell^1(I)$, and that the continuous dual of $\ell^1(I)$ is $\ell^\infty(I)$. If I is finite, then $\ell^\infty(I) = l^0(I)$, so the continuous dual of $\ell^\infty(I)$ is $\ell^1(I)$, and the following question is natural :

If I is infinite, what is the continuous dual of $\ell^\infty(I)$?

Note that in **ZF**, the canonical mapping $j_{\ell^1(I)} : \ell^1(I) \to (\ell^1(I))''$ is isometric, and in (**ZF**+**HB**), $j_{\ell^1(I)}$ is not onto, but this is not provable in **ZF**, see Subsection 4.1.

4.1. A model of ZF in which $\ell^1(\omega)$ and $\ell^\infty(\omega)$ are reflexive.

Pincus and Solovay (see [8]) have built a model $\mathcal{M}$ of set theory **ZF** in which, for every set I, every finitely additive measure m on I is *discrete*, *i.e.* there is an element $(\lambda_i)_{i \in I} \in \ell^1(I)$ such that for every subset A of I, $m(A) = \sum_{i \in A} \lambda_i$. In this model, the continuous dual of the normed space $\ell^\infty(I)$ is $\ell^1(I)$, hence for every set I, each of the normed spaces $\ell^1(I)$ and $\ell^\infty(I)$ is reflexive. Also observe that the reflexive space $l^\infty(\omega)$ has a closed subspace which is not reflexive : for example, consider the subspace $\ell^0(\omega)$: $(\ell^0(\omega))'' = (\ell^1(\omega))' = \ell^\infty(\omega)$ hence $\ell^0(\omega)$ is not reflexive. Also remark that, though $\ell^1(\omega)$ is a reflexive separable Banach space, its continuous dual $\ell^\infty(\omega)$ is not separable. Also note that, although $\ell^0(\omega)$ is not reflexive, its continuous dual $\ell^1(\omega)$ is reflexive.

4.2. **A Banach space E for which j_E is not isometric.** Given any model $\mathcal{M}$ of (**ZF**+¬**HB**), there exists in this model, (see [4], Lemma 5 p.12), an infinite dimensional normed space $(E, \|.\|)$ such that $E' = \{0\}$ (for example, in the above model of Pincus and Solovay, the normed space $\ell^\infty(\omega)/\ell^0(\omega)$ is infinite dimensional, but its continuous dual is $\{0\}$). It follows that the canonical mapping j_E is equal to 0 and j_E is not isometric. Moreover, the *weak topology* $\sigma(E, E')$ on E (*i.e.* the topology generated by the sets $\{x \in E : f(x) < \lambda\}$ where $f \in E'$ and $\lambda \in \mathbb{R}$) has only two open sets which are $\emptyset$ and E. It follows that the closed unit ball of E is not weakly closed, and that singletons of E are not weakly closed either, hence the weak topology on E is not Hausdorff.

4.3. **A Banach space which is smooth at a point but not Gâteaux differentiable at this point.** Let $\mathcal{M}$ be a model of (**ZF**+¬**HB**) : in this model, there exists an infinite dimensional Banach space $(E, \|.\|)$ such that $E' = \{0\}$. Let $F = E \oplus \mathbb{R}$ the vector space endowed with the norm $N : F \to \mathbb{R}$ which is defined as follows : for all $x \in E$, for all $\lambda \in \mathbb{R}$, let $N(x, \lambda) = \|x\| + \mid \lambda \mid$. Clearly, every $f \in F'$ is of the following type : $(x, \lambda) \mapsto m\lambda$ where $m \in \mathbb{R}$. Let $a = (0_E, 1)$. Since $(x, \lambda) \mapsto \lambda$ is the only norming mapping at point a, the normed space (F, N) is smooth at point a ; however, (F, N) is not Gâteaux differentiable at point a because, given any point $u \in S_E$, observe that for $h := (u, 0) \in S_F$, $G^+(a, h) = 1$ and $G^-(a, h) = -1$.

Remark 13. Given a model (**ZF** + ¬**HB**), and an infinite dimensional normed space E of this model such that $E' = \{0\}$, observe that E is not Gâteaux differentiable though any mapping $\eta : \mathbb{R}_+^* \to \mathbb{R}_+^*$ is a Šmulian test of uniform smoothness for E.

4.4. **A separable reflexive space which does not satisfy the projection property.** We say that a normed space E has the *projection property* if and only if for every nonempty closed convex subset of E, every point of E has a best approximation in C. In **ZFC**, it is well known (see [9] Theorem 28.41. p.177) that, for a Banach space E, the three following properties are equivalent:
i) The space E is reflexive ;
ii) The closed unit ball of E is Hausdorff compact in the weak topology ;
iii) The space E satisfies the projection property.

It is easy to show that statement "ii) ⇒ iii)" holds in **ZF**, but statement "i) ⇒ iii)" is not provable in **ZF** : in fact, the space $\ell^1(\omega)$ does not satisfy the projection property, because if we consider the continuous linear mapping $f : \ell^1(\omega) \to \mathbb{R}$ such that for every $x = (x_i)_{i\in\omega} \in \ell^1(\omega)$, $f(x) = \sum_{i\in\omega} \frac{i}{i+1} x_i$, then f does not attain its norm on the closed unit ball hence 0 does not have a best approximation on the closed hyperplane $H := \{x \in l^1(\omega); f(x) = 1\}$; however, in some models of **ZF**, the space $\ell^1(\omega)$ is reflexive (see Subsection 4.1).

4.5. **A metric space which is sequentially complete but not complete.** In Cohen's first model (see [7]), there are dense subsets of $\mathbb{R}$ without any infinite countable subset : such a subset A of $\mathbb{R}$ is not complete (because it is not closed), but it is sequential complete because every Cauchy sequence of A converges (such a sequence is eventually constant).

5. Gâteaux differentiability and smoothness

In this Section, we compare in **ZF** the two notions of Gâteaux differentiability and smoothness for a norm.

5.1. **Gâteaux differentiability at a point implies smoothness at this point.** Given a normed space E, given a point $a \in E\backslash\{0\}$, Gâteaux differentiability of the norm at point a provides a linear mapping $f \in S'_E$ which is norming at point a ; the following easy Lemma shows that Gâteaux differentiability at point a also implies *uniqueness* of a norming linear mapping at point a :

Lemma 7. *Let $(E, \|.\|)$ be a normed space, let $a \in E\backslash\{0\}$. For every linear mapping $f : E \to \mathbb{R}$, the three following conditions are equivalent :*
i) The mapping f is norming at point a ;
ii) For every $h \in E$, $G^-(a,h) \leq f(h) \leq G^+(a,h)$;
iii) For every $h \in E$, $f(h) \leq G^+(a,h)$.

Proof. *i)* $\Rightarrow$ *ii)*. If f is norming at point a, then, given any $h \in E$:

$$\forall t > 0, \ f(h) = \frac{f(th)}{t} = \frac{f(a+th) - f(a)}{t} \leq \frac{\|a+th\| - \|a\|}{t}$$

thus $f(h) \leq G^+(a,h)$. In particular, for every $h \in E$, $f(-h) \leq G^+(a,-h)$, hence $-f(h) \leq -G^-(a,h)$ thus $G^-(a,h) \leq f(h)$.
ii) $\Rightarrow$ *iii)* is trivial.
iii) $\Rightarrow$ *i)*. Since $G^+(a,.) \leq \|.\|$, it follows that f is continuous and $\|f\| \leq 1$; moreover, $f(-a) \leq G^+(a,-a) = -\|a\|$, hence $f(a) \geq \|a\|$ so $f(a) = \|a\|$ and f is norming at point a. □

5.2. **From smoothness at a point to Gâteaux differentiability at this point.** Consider the following statement :

> **S2G** ("Smoothness to Gâteaux differentiability") : *Every normed space which is smooth at a point $a \in E\backslash\{0\}$ is Gâteaux differentiable at this point a.*

Proposition 2. *Axiom* **HB** *is equivalent to* **S2G**.

Proof. **HB** $\Rightarrow$ **S2G** : classical, see [1] page 181. Suppose that some normed space $(E, \|.\|)$ is not Gâteaux differentiable at a given point $a \in S_E$; then there exists $h \in E$ such that $G^-(a,h) < G^+(a,h)$. Using **HB**, consider linear mappings $f, g \in E'$ such that $f \leq G^+(a,.)$, $g \leq G^+(a,.)$, $f(h) = G^-(a,h)$ and $g(h) = G^+(a,h)$. Then, using Lemma 7, f and g are both norming at point a, while $f \neq g$. It follows that E is not smooth at point a.
S2G $\Rightarrow$ **HB** : it follows from Subsection 4.3 that $\neg$**HB** implies $\neg$**S2G**. □

We end with some questions.

Question 1. Is the Mazur property provable in **ZF** for Gâteaux differentiable Banach spaces ? (it is provable in (**ZF**+**DC**), see [3]).

Recall that uniform smoothness means that $\lim_{t\to 0,\, t\neq 0} \frac{\|a+th\|-\|a\|}{t}$ exists for each $a, h \in S_E$, and is uniform in $(a,h) \in S_E \times S_E$.

In the published version of this paper,
I exchanged "Fréchet differentiability" and "uniform Gâteaux differentiability".

Since uniform smoothness yields the Mazur property, it is natural to ask whether the existence for every $a \in E\backslash\{0\}$ of $\lim_{t\to 0,\, t\neq 0} \frac{\|a+th\|-\|a\|}{t}$ uniformly in $h \in S_E$ (*i.e.* Fréchet differentiability), or the existence for every $h \in E\backslash\{0\}$ uniformly in $a \in S_E$ (*i.e. uniform Gâteaux differentiability*, see [2], Definition 6.5 p.63) is sufficient to imply the Mazur property in **ZF** :

Question 2. Is the Mazur property provable in **ZF** for Fréchet differentiable Banach spaces ?

Question 3. Is the Mazur property provable in **ZF** for uniformly Gâteaux differentiable Banach spaces ?

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