The continuous Hahn-Banach property and the Axiom of Choice

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The Continuous Hahn-Banach (*CHB*) property

sub-linear functional

Given a (real) vector space E, a mapping $p : E \to \mathbb{R}$ is *sublinear* if for every $x, y \in E$ $p(x+y) \leq p(x) + p(y)$, and for every $\lambda \in \mathbb{R}_+$, $p(\lambda x) = \lambda p(x)$.

CHB property on a (real) topological vector space E

For every continuous sub-linear functional $p : E \to \mathbb{R}$, every vector subspace F of E, and every linear mapping $f : F \to \mathbb{R}$ such that $f \leq p_{\restriction F}$, there exists some linear mapping $g : E \to \mathbb{R}$ extending f such that $g \leq p$.

The *HB* property on a real vector space

Similar statement but p is not assumed to be continuous.

The Axiom of Choice

The Axiom of Choice says:

> **AC** : *If $(A_i)_{i\in I}$ is a non-empty family of non-empty sets, there exists a mapping $f : I \to \cup_{i\in I} A_i$ such that for every $i \in I$, $f(i) \in A_i$.*

ZFC and ZF

- In **ZF**+**AC** (set-theory with the Axiom of Choice), "Every (real) normed space satisfies *CHB*".
- In **ZF** (set-theory without the Axiom of Choice), ℓ^∞ / c_0 may fail to satisfy *CHB* ([Ho-Ru], [J], [P]).

We work in set-theory without the Axiom of Choice **ZF**.

Spaces satisfying *CHB* (in **ZF**)

An equivalent of the *CHB* property ([Dodu-M])

The *CHB* property on a normed space E is equivalent (in **ZF**) to the following "geometric form": given a closed subset C of E and a point $a \in E \setminus C$, there exists f in the unit sphere of the continuous dual E' such that $f[B(a, R)] \leq f[C]$, where $R := d(a, C)$.

The following spaces satisfy (in **ZF**) the *CHB* property:

- Normed spaces with a dense well-orderable subset (*e.g.* separable normed spaces): see [Fossy-M].
- Uniformly convex Gâteaux differentiable Banach spaces (*e.g.* Hilbert spaces): see [Dodu-M].
- Uniformly smooth Banach spaces: see [Albius-M].

UG differentiability implies **CHB** (in **ZF**)

Axiom of Dependent Choices (*DC*)

Given a binary relation R on a non-empty set X such that $\forall x \in X\ \exists y \in X\ xRy$, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in E such that $\forall n \in \mathbb{N}\ x_n R x_{n+1}$.

Recall that **AC** $\Rightarrow$ **DC** $\Rightarrow$ **AC**$(\mathbb{N})$ (converses are false).

Theorem ([Dodu-M])

In **ZF**+**DC**, Gâteaux differentiable normed spaces satisfy the *CHB* property.

I do not know if this statement holds in **ZF** (or in **ZF**+**AC**$(\mathbb{N})$).

Theorem ([M], preprint 2007)

Every uniformly Gâteaux differentiable normed space satisfies (in **ZF**) the *CHB* property.

$G^+(a,.)$ and $G^-(a,.)$ in a normed space $(E, \|.\|)$

Definition

Given $a \in E \backslash \{0\}$ and $h \in E$, let

$$G^{\pm}(a,h) := \lim_{t->0^{\pm}} \frac{\|a+th\| - \|a\|}{t}$$

The mapping $G^+(a,.)$ (*resp.* $G^-(a,.)$) is sub-linear (*resp.* super-linear) and continuous, and $G^-(a,.) \leq G^+(a,.)$.
-The norm $\|.\|$ is *Gâteaux differentiable (G-diff.) at point a* if $G^-(a,.) = G^+(a,.)$.
-The normed space E is *G-differentiable* if its norm is G-diff. at each non-null point.

If E is G-diff at a point $a \in E \backslash \{0\}$, then $G(a,.)$ is the unique norming form at point a.

The *FE* property on a normed space E

Definition

E satisfies the *Finite Extension* (*FE*) property if for every finite dim. subspace V of E, and every linear form $f : V \to \mathbb{R}$, there exists a linear form $g : E \to \mathbb{R}$ extending f with $\|g\| = \|f\|$.

Lemma

If E is G-differentiable, then E satisfies the *FE* property.

Proof.

Let V be a finite dimensional vector subspace of E. Let $f : V \to \mathbb{R}$ be some non null linear mapping. Since V is finite dimensional, the closed unit ball of V is compact, thus f attains its norm at a point a in the unit sphere of V. Since $G(a, .)$ is the unique norming form at point a, it extends $\frac{f}{\|f\|}$; thus $\|f\|.G(a, .)$ extends f and has the same norm as f. $\square$

Convex compactness

Centered family

A non-empty family $\mathcal{C}$ of sets is *centered* if every finite intersection of elements of $\mathcal{C}$ is non-empty.

c-compactness

Given a (real) vector space E, a topology $\mathcal{T}$ on E, and a convex subset C of E, say that C is *convex-compact* if for every family of $\mathcal{T}$-closed convex subsets $\mathcal{C}$ of E, if $\{C \cap A : A \in \mathcal{C}\}$ is centered, then $C \cap \cap\mathcal{C}$ is non-empty.

Another equivalent of *CHB* on a normed space E

We denote by $B_{E'}$ the closed unit ball of the continuous dual E' of E.

Theorem 2 ([Fossy-M])

The following statements are equivalent:

- The normed space E satisfies the *CHB* property.
- E satisfies the *FE* property and $B_{E'}$ is c-compact in the weak* topology of E'.

Uniformly Gâteaux differentiable normed spaces

Definition

A normed space $(E, \|.\|)$ is *uniformly Gâteaux differentiable* (UG) if E is G-differentiable, and if, for every $h \in E$, $\lim_{t->0, t\neq 0} \left(\frac{\|a+th\|-\|a\|}{t} - G(a,h)\right) = 0$ uniformly for $a \in S_E$.

Remark

$$\begin{tikzcd}
 & \text{G-diff.} & \\
UG \arrow[ur] & & \text{Frechet-diff.} \arrow[ul] \\
 & \text{Unif. Frechet-diff} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Normed space E with a *w*UR* dual ball

Proposition (classical)

If a normed space is *UG*, then its dual ball is *w*UR*.

Notation

Given a normed space E, and some $h \in E$, we denote by $d_h : E' \times E' \to \mathbb{R}_+$ the pseudo-metric associating to each $(f, g) \in E' \times E'$ the real number $|f(h) - g(h)|$.

Weak* Uniformly Rotund dual ball

Say that the dual ball $B_{E'}$ of E is *w*UR* if for every $h \in E$, and every $\varepsilon > 0$, there exists some $\delta \in]0, 1[$ such that for every $a \in S_E$, the d_h-diameter of the set $C_a := \{f \in B_{E'} : f(a) > \delta\}$ is $< \varepsilon$.

Convex-compactness of a *w*UR* dual ball

Theorem 3 ([M], preprint 2007)

If the dual ball $B_{E'}$ of E is w^*UR, then $B_{E'}$ is convex-compact in the weak* topology.

Proof.

The proof relies on a purely topological criterion of compactness in complete gauge spaces with a sub-basis of closed sets which are small in thin crowns. □

Corollary

Every UG *normed space satisfies the* CHB *property.*

Smallness in thin crowns

Definition

Let X be a space and let d, d' be two pseudo-metrics on X. Let $a \in X$. A set $\mathcal{C}$ of subsets of X satisfies the of d'-smallness in thin d-crowns centered at a if for every $R \in \mathbb{R}_+^*$, for every $\varepsilon > 0$ there exists $\eta \in]0, R[$ such that for every $C \in \mathcal{C}$,

$$C \subseteq D_d(a, R - \eta, R + \eta) \Rightarrow \mathrm{diam}_{d'}(C) < \varepsilon$$

If $(X, (d_i)_{i \in I})$ is a gauge space, $\mathcal{C}$ satisfies the property of $(d_i)_{i \in I}$-smallness in thin d-crowns centered at a if for every $i \in I$, $\mathcal{C}$ satisfies the property of d_i-smallness in thin d-crowns centered at a.

A criterion of compactness

Theorem

Let $(X, (d_i)_{i \in I})$ be a Hausdorff complete gauge space. Let $\mathcal{T}$ be a topology on X which is included in the associated gauge topology, and let $\mathcal{C}$ be a sub-basis of closed sets of $(X, \mathcal{T})$ which is closed by finite intersections. Let $a \in X$. Let d be a metric on X such that d-closed balls centered at a belong to $\mathcal{C}$, and such that $\mathcal{T}$ is included in the topology $\mathcal{T}_d$ associated to d. If $\mathcal{C}$ satisfies the property of $(d_i)_{i \in I}$-smallness in thin d-crowns centered at a, then every every large d-ball with center a (and thus, every d-bounded element of $\mathcal{C}$) is closely $\mathcal{C}$-compact.

Proof.

See [6] or a sketch of proof in next frame. $\square$

Proof of the criterion of compactness

Sketch of the Proof

Let $\mathcal{L}$ be the lattice generated by $\mathcal{C}$. Let $\rho > 0$ and let B be the large d-ball $B_d(a, \rho)$. Let $\mathcal{A}$ be subset of non-empty elements $\mathcal{C}$ which is closed by finite intersection and such that $B \in \mathcal{A}$. Let us show (in **ZF**) that $\cap\mathcal{A}$ is non-empty. Let $\mathcal{F}$ be the filter of $\mathcal{L}$ generated by $\mathcal{A}$. Let $R := \inf\{r \in \mathbb{R}_+ : \ B_d(a, r) \in \mathcal{S}(\mathcal{F})\}$. Denote by $\mathcal{A}'$ the set $\{A \cap B_d(a, r) : A \in \mathcal{A} \text{ and } r > R\}$. If $R = 0$ then $a \in \cap\mathcal{A}$. If $R > 0$, then for every $\varepsilon > 0$, there exists some element of $\mathcal{A}'$ which is included in the crown $D_d(a, R - \varepsilon, R + \varepsilon)$. Since $\mathcal{C}$ satisfies the property of $(d_i)_{i\in I}$-smallness in thin d-crowns centered at a, the centered family $\mathcal{A}'$ is Cauchy in the gauge space $(X, (d_i)_{i\in I})$; since this gauge space is complete and Hausdorff, $\cap\mathcal{A}'$ is a singleton $\{a\}$. This singleton is **ZF**-definable from $(X, (d_i)_{i\in I})$, d, $\mathcal{C}$ and $\mathcal{A}$ whence the close c-compactness.

The weak* topology on E'

Let E be a normed space.

Evaluation functionals

Given some $a \in E$, we denote by $\tilde{a} : E' \to \mathbb{R}$ the linear form $f \mapsto f(a)$.

Topology $\sigma(E', E)$

We denote by $\sigma(E', E)$ the weakest topology on E' for which all evaluation functionals $\tilde{a}$, $a \in E$ are continuous.

Proposition

The topology $\sigma(E', E)$ on E is the topology associated to the gauge space $(E, (d_a)_{a \in E})$.

*-polyhedras of the continuous dual E'

polyhedras of a (real) *tvs* E

A strict (resp. large) hemi-space of E is a subset of E of the form $(f < \lambda)$ (resp. $(f \leq \lambda)$) where $f \in E'$. A strict (*resp.* large) *polyhedra* of E is a finite intersection of strict (*resp.* large) hemi-spaces.

*polyhedras of the continuous dual of a normed space E

A strict (resp. large) **-polyhedra* of E' is a polyhedra of the *tvs* E' endowed with the weak* topology.

$\mathcal{P}$

We denote by $\mathcal{P}$ the class of large *-polyhedras of E': $\mathcal{P}$ is a sub-basis of closed subsets of $\sigma(E', E)$, which is closed by finite intersection.

A **ZF**-provable form of Hahn-Banach

Theorem (Dodu-M, Lemma 1)

Let E be a Hausdorff tvs*. If C is a convex subset of E, and if P is a strict polyhedra of E disjoint from C, then there exists $f \in E'$ such that $f[P] < f[C]$.*

Beginning of the proof.

P is of the form $P = \bigcap_{i=1}^{m} \{x \in E : \; f_i(x) < \alpha_i\}$. where $f_1, ..., f_m \in E' \backslash \{0\}$ and $\alpha_1, ..., \alpha_m \in \mathbb{R}$. Let $V := \bigcap_{i=1}^{m} Ker(f_i)$, let F be a finite dimensional subspace of E such that $V \oplus F = E$ and let $p : E \to F$ be the projection on F with kernel V. Since the f_i are continuous, $p : E \to F$ is continuous. Moreover, since the *tvs* E is Hausdorff, its finite dimensional subspace F is isomorphic with the usual space $\mathbb{R}^d$ where d is the dimension of F.

A **ZF**-provable form of Hahn-Banach (continued)

End of the proof.

The finite dimensional *tvs* F satisfies the various classical geometrical Hahn-Banach properties. Let $K := p[C]$ and $U := p[P]$; the convex subsets K and U are disjoint in F, and U is open in F hence, since F is finite dimensional, there is $g \in F'$ such that $g[U] < g[K]$. Let $f := g \circ p$. Then $f[P] < f[C]$.

Separating convex sets and *polyhedras

weak* continuous forms (classical)

Given a Banach space E, weak* continuous linear mappings $\phi : E' \to \mathbb{R}$ are evaluation mappings $\tilde{a} : E' \to \mathbb{R}$ for $a \in E$.

Theorem

*Let E be a normed space, let C be a convex subset of E'. If P is a strict *-polyhedra of E' which is disjoint from C, then there exists $a \in E$ such that $\tilde{a}[C] < \tilde{a}[P]$.*

smallness of *polyhedras in a w*UR dual ball

Theorem

Let E be a normed space. Let d be the distance given by the dual norm on E'. Let $(d_h)_{h \in E}$ be the canonical gauge space of the weak topology on E'. If E has a w*UR dual norm, then the class $\mathcal{P}_b := \{P \cap B(0,r) : P \in \mathcal{P} \text{ and } r \in \mathbb{R}_+\}$ has the property of $(d_h)_{h \in E}$-smallness in thin d-crowns centered at $0_{E'}$.*

Proof.

Let $h \in E$. Let $\varepsilon > 0$. Since the dual norm of E' is w*UR, let $\eta > 0$ such that for every $a \in S_E$, and for every $f, g \in C_{a,\eta}$, $|f(h) - g(h)| < \varepsilon$. Let $P \in \mathcal{P}$ such that $P \cap B(0, 1-\eta) = \varnothing$. Using the previous **ZF**-provable consequence of **HB**, there exists $a \in S_E$ such that $\tilde{a}[B(0, 1-\eta)] < \tilde{a}[P]$, thus $(P \cap B_E) \subseteq C_{a,\eta}$, thus for every $f, g \in (P \cap B_E)$, $|f(h) - g(h)| < \varepsilon$. $\square$

Some open questions

Question 1

Does a Gateaux (or Frechet) differentiable normed (or Banach) space satisfy the *CHB* property in **ZF**? in **ZF**+**AC**($\mathbb{N}$) (where **AC**($\mathbb{N}$) is the "countable axiom of choice")?

Equivalently, is the dual ball of a G-diff (or F-diff) normed space weak* c-compact?

Question 2 (Bell and Fremlin, 1972)

In **ZF**+**AC**, given a real locally convex topological vector space E, *"Every non-empty c-compact convex subset C of E has an extremal point"*. Does this statement imply **AC** in **ZF**?

Some open questions (continued)

Question 3

Which closed subsets of $[0,1]^I$ are c-compact (*resp.* compact) in **ZF**? For example, if F is Corson (*i.e.* every element of F has a countable support), is F compact in **ZF**+**DC**? in **ZF**+**AC**$(\mathbb{N})$?

Question 4 (van Rooij, [R])

In p-adic functional analysis, Ingleton's theorem is a statement analogous to the Hahn-Banach theorem, which holds for Banach spaces over a spherically complete field. Does Ingleton's theorem imply **AC**?

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