The continuous Hahn-Banach property and the Axiom of Choice

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ICBSOS, Chern Institute of Mathematics, Nankai University, Tianjin, China, 22 July 2007
The Continuous Hahn-Banach (CHB) property

**sub-linear functional**

Given a (real) vector space $E$, a mapping $p : E \to \mathbb{R}$ is *sublinear* if for every $x, y \in E$ $p(x + y) \leq p(x) + p(y)$, and for every $\lambda \in \mathbb{R}_+$, $p(\lambda x) = \lambda p(x)$.

**CHB property on a (real) topological vector space $E$**

For every *continuous* sub-linear functional $p : E \to \mathbb{R}$, every vector subspace $F$ of $E$, and every linear mapping $f : F \to \mathbb{R}$ such that $f \leq p|_F$, there exists some linear mapping $g : E \to \mathbb{R}$ extending $f$ such that $g \leq p$.

**The HB property on a real vector space**

Similar statement but $p$ is not assumed to be continuous.
The Axiom of Choice says:

**AC**: If \((A_i)_{i \in I}\) is a non-empty family of non-empty sets, there exists a mapping \(f : I \rightarrow \bigcup_{i \in I} A_i\) such that for every \(i \in I\), \(f(i) \in A_i\).

**ZFC** and **ZF**

- In **ZF + AC** (set-theory with the Axiom of Choice), “Every (real) normed space satisfies CHB”.
- In **ZF** (set-theory without the Axiom of Choice), \(\ell^\infty/c_0\) may fail to satisfy CHB ([Ho-Ru], [J], [P]).

We work in set-theory without the Axiom of Choice **ZF**.
Spaces satisfying \( CHB \) (in \( ZF \))

An equivalent of the \( CHB \) property ([Dodu-M])

The \( CHB \) property on a normed space \( E \) is equivalent (in \( ZF \)) to the following “geometric form”: given a closed subset \( C \) of \( E \) and a point \( a \in E \setminus C \), there exists \( f \) in the unit sphere of the continuous dual \( E' \) such that \( f[B(a, R)] \leq f[C] \), where \( R := d(a, C) \).

The following spaces satisfy (in \( ZF \)) the \( CHB \) property:

- Normed spaces with a dense well-orderable subset (e.g. separable normed spaces): see [Fossy-M].
- Uniformly convex Gâteaux differentiable Banach spaces (e.g. Hilbert spaces): see [Dodu-M].
- Uniformly smooth Banach spaces: see [Albius-M].
UG differentiability implies CHB (in ZF)

Axiom of Dependent Choices (DC)

Given a binary relation $R$ on a non-empty set $X$ such that $\forall x \in X \ \exists y \in X \ xRy$, there exists a sequence $(x_n)_{n \in \mathbb{N}}$ in $E$ such that $\forall n \in \mathbb{N} \ x_n R x_{n+1}$.

Recall that $AC \Rightarrow DC \Rightarrow AC(\mathbb{N})$ (converses are false).

Theorem ([Dodu-M])

In $ZF + DC$, Gâteaux differentiable normed spaces satisfy the CHB property.

I do not know if this statement holds in $ZF$ (or in $ZF + AC(\mathbb{N})$).

Theorem ([M], preprint 2007)

Every uniformly Gâteaux differentiable normed space satisfies (in $ZF$) the CHB property.
$G^+(a, .)$ and $G^-(a, .)$ in a normed space $(E, \| . \|)$

**Definition**

Given $a \in E\{0\}$ and $h \in E$, let

$$G^\pm(a, h) := \lim_{t \to 0^\pm} \frac{\|a + th\| - \|a\|}{t}$$

The mapping $G^+(a, .)$ (resp. $G^-(a, .)$) is sub-linear (resp. super-linear) and continuous, and $G^-(a, .) \leq G^+(a, .)$.

- The norm $\| . \|$ is Gâteaux differentiable (G-diff.) at point $a$ if $G^-(a, .) = G^+(a, .)$.

- The normed space $E$ is G-differentiable if its norm is G-diff. at each non-null point.

If $E$ is G-diff at a point $a \in E\{0\}$, then $G(a, .)$ is the unique norming form at point $a$. 
The *FE* property on a normed space *E*

**Definition**

*E* satisfies the *Finite Extension (FE)* property if for every finite dim. subspace *V* of *E*, and every linear form *f*: *V* → ℝ, there exists a linear form *g*: *E* → ℝ extending *f* with \( \|g\| = \|f\| \).

**Lemma**

If *E* is G-differentiable, then *E* satisfies the *FE* property.

**Proof.**

Let *V* be a finite dimensional vector subspace of *E*. Let *f*: *V* → ℝ be some non null linear mapping. Since *V* is finite dimensional, the closed unit ball of *V* is compact, thus *f* attains its norm at a point *a* in the unit sphere of *V*. Since *G(a, .)* is the unique norming form at point *a*, it extends \( \frac{f}{\|f\|} \); thus \( \|f\|.G(a, .) \) extends *f* and has the same norm as *f*. \( \square \)
Convex compactness

Centered family

A non-empty family $\mathcal{C}$ of sets is centered if every finite intersection of elements of $\mathcal{C}$ is non-empty.

c-compactness

Given a (real) vector space $E$, a topology $\mathcal{T}$ on $E$, and a convex subset $C$ of $E$, say that $C$ is convex-compact if for every family of $\mathcal{T}$-closed convex subsets $\mathcal{C}$ of $E$, if $\{C \cap A : A \in \mathcal{C}\}$ is centered, then $C \cap \cap \mathcal{C}$ is non-empty.
Another equivalent of $CHB$ on a normed space $E$

We denote by $B_{E'}$, the closed unit ball of the continuous dual $E'$ of $E$.

**Theorem 2 ([Fossy-M])**

The following statements are equivalent:

- The normed space $E$ satisfies the $CHB$ property.
- $E$ satisfies the $FE$ property and $B_{E'}$ is c-compact in the weak* topology of $E'$. 
Uniformly Gâteaux differentiable normed spaces

**Definition**

A normed space \((E, \| \cdot \|)\) is uniformly Gâteaux differentiable (UG) if \(E\) is G-differentiable, and if, for every \(h \in E\),

\[
\lim_{t \to 0, t \neq 0} \left( \frac{\|a + th\| - \|a\|}{t} - G(a, h) \right) = 0 \quad \text{uniformly for } a \in S_E.
\]

**Remark**

\[ UG \quad \text{G-diff.} \quad \text{Frechet-diff.} \quad \text{Unif. Frechet-diff} \]
Normed space $E$ with a $w^*UR$ dual ball

**Proposition (classical)**

If a normed space is $UG$, then its dual ball is $w^*UR$.

**Notation**

Given a normed space $E$, and some $h \in E$, we denote by $d_h : E' \times E' \to \mathbb{R}_+$ the pseudo-metric associating to each $(f, g) \in E' \times E'$ the real number $|f(h) - g(h)|$.

**Weak* Uniformly Rotund dual ball**

Say that the dual ball $B_{E'}$ of $E$ is $w^*UR$ if for every $h \in E$, and every $\varepsilon > 0$, there exists some $\delta \in ]0, 1[$ such that for every $a \in S_E$, the $d_h$-diameter of the set $C_a := \{f \in B_{E'} : f(a) > \delta\}$ is $< \varepsilon$. 
Convex-compactness of a $w^*UR$ dual ball

Theorem 3 ([M], preprint 2007)
If the dual ball $B_{E'}$ of $E$ is $w^*UR$, then $B_{E'}$ is convex-compact in the weak* topology.

Proof.
The proof relies on a purely topological criterion of compactness in complete gauge spaces with a sub-basis of closed sets which are small in thin crowns.

Corollary
Every UG normed space satisfies the CHB property.
Smallness in thin crowns

Definition

Let $X$ be a space and let $d, d'$ be two pseudo-metrics on $X$. Let $a \in X$. A set $C$ of subsets of $X$ satisfies the property of $d'$-smallness in thin $d$-crowns centered at $a$ if for every $R \in \mathbb{R}_+^*$, for every $\varepsilon > 0$ there exists $\eta \in ]0, R[$ such that for every $C \in C$,

$$C \subseteq D_d(a, R - \eta, R + \eta) \Rightarrow \text{diam}_{d'}(C) < \varepsilon$$

If $(X, (d_i)_{i \in I})$ is a gauge space, $C$ satisfies the property of $(d_i)_{i \in I}$-smallness in thin $d$-crowns centered at $a$ if for every $i \in I$, $C$ satisfies the property of $d_i$-smallness in thin $d$-crowns centered at $a$. 


A criterion of compactness

Theorem

Let \((X, (d_i)_{i \in I})\) be a Hausdorff complete gauge space. Let \(\mathcal{T}\) be a topology on \(X\) which is included in the associated gauge topology, and let \(\mathcal{C}\) be a sub-basis of closed sets of \((X, \mathcal{T})\) which is closed by finite intersections. Let \(a \in X\). Let \(d\) be a metric on \(X\) such that \(d\)-closed balls centered at \(a\) belong to \(\mathcal{C}\), and such that \(\mathcal{T}\) is included in the topology \(\mathcal{T}_d\) associated to \(d\). If \(\mathcal{C}\) satisfies the property of \((d_i)_{i \in I}\)-smallness in thin \(d\)-crowns centered at \(a\), then every every large \(d\)-ball with center \(a\) (and thus, every \(d\)-bounded element of \(\mathcal{C}\)) is closely \(\mathcal{C}\)-compact.

Proof.

See [6] or a sketch of proof in next frame.
Proof of the criterion of compactness

Sketch of the Proof

Let \( \mathcal{L} \) be the lattice generated by \( \mathcal{C} \). Let \( \rho > 0 \) and let \( B \) be the large \( d \)-ball \( B_d(a, \rho) \). Let \( \mathcal{A} \) be subset of non-empty elements \( \mathcal{C} \) which is closed by finite intersection and such that \( B \in \mathcal{A} \). Let us show (in \( \text{ZF} \)) that \( \cap \mathcal{A} \) is non-empty. Let \( \mathcal{F} \) be the filter of \( \mathcal{L} \) generated by \( \mathcal{A} \). Let \( R := \inf\{r \in \mathbb{R}^+ : B_d(a, r) \in S(\mathcal{F})\} \). Denote by \( \mathcal{A}' \) the set \( \{A \cap B_d(a, r) : A \in \mathcal{A} \text{ and } r > R\} \). If \( R = 0 \) then \( a \in \cap \mathcal{A} \). If \( R > 0 \), then for every \( \varepsilon > 0 \), there exists some element of \( \mathcal{A}' \) which is included in the crown \( D_d(a, R - \varepsilon, R + \varepsilon) \). Since \( \mathcal{C} \) satisfies the property of \((d_i)_{i \in I}\)-smallness in thin \( d \)-crowns centered at \( a \), the centered family \( \mathcal{A}' \) is Cauchy in the gauge space \((X, (d_i)_{i \in I})\); since this gauge space is complete and Hausdorff, \( \cap \mathcal{A}' \) is a singleton \( \{a\} \). This singleton is \( \text{ZF} \)-definable from \((X, (d_i)_{i \in I}), d, \mathcal{C} \) and \( \mathcal{A} \) whence the close c-compactness.
The weak* topology on $E'$

Let $E$ be a normed space.

**Evaluation functionals**

Given some $a \in E$, we denote by $\tilde{a} : E' \to \mathbb{R}$ the linear form $f \mapsto f(a)$.

**Topology $\sigma(E', E)$**

We denote by $\sigma(E', E)$ the weakest topology on $E'$ for which all evaluation functionals $\tilde{a}$, $a \in E$ are continuous.

**Proposition**

The topology $\sigma(E', E)$ on $E$ is the topology associated to the gauge space $(E, (d_a)_{a \in E})$. 
*-polyhedras of the continuous dual \( E' \)

polyhedras of a (real) tvs \( E \)

A strict (resp. large) hemi-space of \( E \) is a subset of \( E \) of the form \((f < \lambda)\) (resp. \((f \leq \lambda)\)) where \( f \in E' \). A strict (resp. large) polyhedra of \( E \) is a finite intersection of strict (resp. large) hemi-spaces.

*polyhedras of the continuous dual of a normed space \( E \)

A strict (resp. large) *-polyhedra of \( E' \) is a polyhedra of the tvs \( E' \) endowed with the weak* topology.

\( \mathcal{P} \)

We denote by \( \mathcal{P} \) the class of large *-polyhedras of \( E' \): \( \mathcal{P} \) is a sub-basis of closed subsets of \( \sigma(E', E) \), which is closed by finite intersection.
Theorem (Dodu-M, Lemma 1)

Let $E$ be a Hausdorff tvs. If $C$ is a convex subset of $E$, and if $P$ is a strict polyhedra of $E$ disjoint from $C$, then there exists $f \in E'$ such that $f[P] < f[C]$.

Beginning of the proof.

$P$ is of the form $P = \bigcap_{i=1}^{m} \{ x \in E : f_i(x) < \alpha_i \}$. where $f_1, ..., f_m \in E' \setminus \{0\}$ and $\alpha_1, ..., \alpha_m \in \mathbb{R}$. Let $V := \bigcap_{i=1}^{m} \text{Ker}(f_i)$, let $F$ be a finite dimensional subspace of $E$ such that $V \oplus F = E$ and let $p : E \to F$ be the projection on $F$ with kernel $V$. Since the $f_i$ are continuous, $p : E \to F$ is continuous. Moreover, since the tvs $E$ is Hausdorff, its finite dimensional subspace $F$ is isomorphic with the usual space $\mathbb{R}^d$ where $d$ is the dimension of $F$. 
End of the proof.

The finite dimensional tvs $F$ satisfies the various classical geometrical Hahn-Banach properties. Let $K := p[C]$ and $U := p[P]$; the convex subsets $K$ and $U$ are disjoint in $F$, and $U$ is open in $F$ hence, since $F$ is finite dimensional, there is $g \in F'$ such that $g[U] < g[K]$. Let $f := g \circ p$. Then $f[P] < f[C]$. 

\[ f[P] < f[C]. \]
Separating convex sets and *polyhedras

weak* continuous forms (classical)

Given a Banach space $E$, weak* continuous linear mappings $\phi : E' \to \mathbb{R}$ are evaluation mappings $\tilde{a} : E' \to \mathbb{R}$ for $a \in E$.

Theorem

Let $E$ be a normed space, let $C$ be a convex subset of $E'$. If $P$ is a strict *-polyhedra of $E'$ which is disjoint from $C$, then there exists $a \in E$ such that $\tilde{a}[C] < \tilde{a}[P]$. 
smallness of *polyhedras in a w*UR dual ball

**Theorem**

Let $E$ be a normed space. Let $d$ be the distance given by the dual norm on $E'$. Let $(d_h)_{h \in E}$ be the canonical gauge space of the weak* topology on $E'$. If $E$ has a w*UR dual norm, then the class $\mathcal{P}_b := \{P \cap B(0, r) : P \in \mathcal{P} \text{ and } r \in \mathbb{R}_+\}$ has the property of $(d_h)_{h \in E}$-smallness in thin $d$-crowns centered at $0_{E'}$.

**Proof.**

Let $h \in E$. Let $\varepsilon > 0$. Since the dual norm of $E'$ is w*UR, let $\eta > 0$ such that for every $a \in S_E$, and for every $f, g \in C_{a, \eta}$, $|f(h) - g(h)| < \varepsilon$. Let $P \in \mathcal{P}$ such that $P \cap B(0, 1 - \eta) = \emptyset$. Using the previous ZF-provable consequence of HB, there exists $a \in S_E$ such that $\tilde{a}[B(0, 1 - \eta)] < \tilde{a}[P]$, thus $(P \cap B_E) \subseteq C_{a, \eta}$, thus for every $f, g \in (P \cap B_E)$, $|f(h) - g(h)| < \varepsilon$. 

□
Some open questions

Question 1
Does a Gateaux (or Frechet) differentiable normed (or Banach) space satisfy the CHB property in \( \text{ZF} \)? in \( \text{ZF} + \text{AC}(\mathbb{N}) \) (where \( \text{AC}(\mathbb{N}) \) is the “countable axiom of choice”)?

Equivalently, is the dual ball of a G-diff (or F-diff) normed space weak* c-compact?

Question 2 (Bell and Fremlin, 1972)
In \( \text{ZF} + \text{AC} \), given a real locally convex topological vector space \( E \), “Every non-empty c-compact convex subset \( C \) of \( E \) has an extremal point”. Does this statement imply \( \text{AC} \) in \( \text{ZF} \)?
Some open questions (continued)

**Question 3**
Which closed subsets of $[0, 1]^I$ are c-compact (resp. compact) in $\mathbf{ZF}$? For example, if $F$ is Corson (i.e. every element of $F$ has a countable support), is $F$ compact in $\mathbf{ZF} + \mathbf{DC}$? in $\mathbf{ZF} + \mathbf{AC}(\mathbb{N})$?

**Question 4 (van Rooij, [R])**
In $p$-adic functional analysis, Ingleton’s theorem is a statement analogous to the Hahn-Banach theorem, which holds for Banach spaces over a spherically complete field. Does Ingleton’s theorem imply $\mathbf{AC}$?


References (continued)

