SYNTHÈSE

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ABSTRACT. Ce travail présente une synthèse de quelques résultats obtenus, dans le cadre de la théorie des ensembles sans axiome du choix **ZF**, principalement en topologie générale et Analyse fonctionnelle. Il s'agit d'un "guide de lecture" donnant un aperçu de travaux que j'ai effectués après ma thèse, et je renvoie aux articles que j'ai publiés (voir liste à la fin de ce document) pour la preuve détaillée de ces résultats.

ABSTRACT. This synthesis presents an overview of some results I obtained after my Thesis, in set-theory **ZF** (without choice), mainly in general topology and Functional Analysis. Details and proofs for the results formulated in this "reading guide" can be found in my published papers, listed at the end of this document.

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Introduction

The following synthesis consists of four quite independent parts. The first part deals with some results in general topology, namely relative to the normality of linearly ordered sets, and to the Baire category property of Hausdorff compact spaces. The second part describes a theory of integration in Gelfand algebras. The third part exposes some points of functional analysis, concerning the Hahn-Banach property and various notions of reflexivity. The last part presents results in "discrete mathematics" obtained in collaboration.

Part 1. Topology and the Axiom of Choice

1. Normality of linearly ordered sets

In this Section, we introduce some results obtained in [P90] and [P91].

1.1. **van Douwen's question.** Given a topological space X, we denote by $\mathcal{O}(X)$ the set of its open subsets. Say that a topological space X is *normal* if it is Hausdorff and if for every couple (F,G) of disjoint closed subsets of X, there exists a couple (U,V) of disjoint open subsets of X satisfying $F \subseteq U$ and $G \subseteq V$. Denoting by $\mathcal{N}_X$ the set of couples (F,G) of disjoint closed subsets of X, say that a mapping $\Phi : \mathcal{N}_X \to \mathcal{O}(X) \times \mathcal{O}(X)$ is a *normality operator* on X if it associates to every $(F,G) \in \mathcal{N}_X$ a couple (U,V) of disjoint open subsets of X satisfying $F \subseteq U$ and $G \subseteq V$. Notice that the existence of such an operator is equivalent to the existence of a mapping $\Psi : \mathcal{N}_X \to \mathcal{O}(X)$ satisfying for every $(F,G) \in \mathcal{N}_X$ the constraint

$$F \subseteq \Psi(F,G) \subseteq \overline{\Psi(F,G)} \subseteq X \backslash G \tag{1}$$

Say that a normality operator $\Phi = (\Phi_1, \Phi_2)$ on the topological space X is *monotone* if for every $(F,G),(F',G') \in \mathcal{N}_X$ satisfying $F \subseteq F'$ and $G \subseteq G'$, $\Phi_1(F,G') \subseteq \Phi_1(F',G)$ and $\Phi_2(F',G) \subseteq \Phi_2(F,G')$: Φ_1 is increasing in the first argument and decreasing in the second argument, while Φ_2 is decreasing in the first argument and increasing in the second argument. Say that the topological space X is *monotonely normal* if X is Hausdorff and admits a monotone normality operator. In [17], van Douwen introduced the following statement :

(**LN**, **Linear Normal**). *Every linearly ordered set is normal (for the order topology).*

He proved that **LN** is equivalent to each of the following statements :

> *"Every linearly ordered set has a normality operator."*
>
> *"Every linearly ordered family of conditionally complete linearly ordered sets has a choice function".*

Here a poset $(X, \leq)$ is *conditionally complete* if every nonempty bounded from above subset A of X has an upper bound in X. van Douwen also proved that the following weak form of **LN** is not even provable in **ZF** :

(**LN$\mathbb{Z}$**). *Every linearly ordered family of $\mathbb{Z}$-ordered sets has a choice function.*

Here, a $\mathbb{Z}$-ordered set is a poset isomorphic with the poset $\mathbb{Z}$. Then van Douwen asked whether **LN** implies the following statement :

(**LMN**, **Linear Monotone Normal**). *Every linearly ordered set is monotonely normal (for the order topology).*

In my Thesis, see [P88] and [R88], I solved van Douwen's above question. More precisely, I proved that **LN** $\Rightarrow$ **TOR** $\Rightarrow$ **LMN** where **TOR** is the following statement :

> **TOR (Total Order Representation)** *For every linearly ordered set X, there exists an ordinal α and a strictly increasing mapping $f : X \to \{0,1\}^\alpha$, where $\{0,1\}^\alpha$ is endowed with the lexicographic order.*

Given a linearly ordered set $(X, \leq)$, an *extreme choice* on X is a mapping $* : X \times X \to X$ associating to every $(x, y) \in X^2$ satisfying $]x, y[\neq \varnothing$, an element $x * y \in]x, y[$ such that, for every $x, y, z \in X$ satisfying $x < y < z$:

(1) $x * z \in \{x * y, y, y * z\}$ if $]x, y[\neq \varnothing$ and $]y, z[\neq \varnothing$;
(2) $x * z \in \{y, y * z\}$ if $]x, y[= \varnothing$ and $]y, z[\neq \varnothing$;
(3) $x * z \in \{x * y, y\}$ if $]x, y[\neq \varnothing$ and $]y, z[= \varnothing$.

The following Theorem 1 solves a question left open in my Thesis :

Theorem 1. *The statement* **LN** *is equivalent to the following statement :*

"Every conditionally complete linearly ordered set has an extreme choice."

Proof. See [P91]. □

1.2. **Birkhoff's question.** The following Theorem 2 answers a question raised by Birkhoff (see [6]), which was left open in [17] :

Theorem 2. LN *does not imply* **AC**.

Proof. See [P90], where we show that the basic Cohen model satisfies **LN**. □

1.3. **Choice in $\mathbb{Z}$-ordered posets and $\mathbb{Z}$-chameleons.** A.R.D. Mathias introduced various sorts of *chameleons* (see [30]) as a way of building non determined subsets of $\mathbb{R}$. We now introduce $\mathbb{Z}$-chameleons on a given infinite set X : denoting by $[X]^\omega$ the set of its infinite subsets, say that a mapping $\chi : [X]^\omega \to \mathbb{Z}$ is a $\mathbb{Z}$-*chameleon* if for every infinite subset A of X and every element $a \in X \backslash A$, $\chi(A \sqcup \{a\}) = \chi(A) + 1$ where $+$ is the additive law of the group $\mathbb{Z}$. We now consider the following statement, which is a consequence of **LN$\mathbb{Z}$**+**OP**, where **OP** is the *ordering principle*, according to which "Every set has a linear order" :

(**C$\mathbb{Z}$**, **Choice in $\mathbb{Z}$-ordered posets**). *Every family of $\mathbb{Z}$-ordered posets has a choice function.*

Theorem 3. *The axiom* **C$\mathbb{Z}$** *is equivalent to the existence of $\mathbb{Z}$-chameleons on every infinite set.*

Proof. (unpublished) Given an infinite set X, we denote by $\mathcal{B}_X$ the boolean algebra $\mathcal{P}(X)/fin$ where $fin(X)$ is the ideal of finite subsets of X, and $\mathcal{P}_X$ the poset $\mathcal{B}_X\backslash\{0\}$. Notice that a mapping $\chi : [X]^\omega \to \mathbb{Z}$ is a $\mathbb{Z}$-chameleon if and only if for each class $x \in \mathcal{P}_X$, the restriction $\chi_{\restriction x}$ is a $\mathbb{Z}$-chameleon.
$\Rightarrow$. Let X be an infinite set. For every class $x \in \mathcal{P}_X$, let C_x be the set of $\mathbb{Z}$-chameleons on x. Then, for every $\chi, \chi' \in C_x$ and every $n \in \mathbb{Z}$, the two following conditions are equivalent :

(1) There exists $A \in x$ such that $\chi(A) - \chi'(A) = n$;
(2) For every $A \in x$, $\chi(A) - \chi'(A) = n$.

The binary relation R on C_x which is defined by "$\chi R \chi'$ if and only if there exists some $A \in x$ satisfying $\chi(A) \geq \chi'(A)$" is a total order isomorphic with $\mathbb{Z}$. Then, use **C**$\mathbb{Z}$ to choose a $\mathbb{Z}$-chameleon in each class $x \in \mathcal{P}_X$.
$\Leftarrow$. Given some family $(X_i, \leq_i)_{i\in I}$ of $\mathbb{Z}$-ordered sets, assume that there exists a $\mathbb{Z}$-chameleon χ on the disjointed sum $X := \sqcup_{i\in I}(\{i\} \times X_i)$. For every $i \in I$, there exists a unique element $x_i \in X_i$ satisfying $\chi(\{x \in X_i : x \leq_i x_i\}) = 0$. $\square$

In particular, in the basic Cohen model, there exist $\mathbb{Z}$-chameleons on every infinite set (because the basic Cohen model satisfies **LN**+**OP**).

2. Hausdorff compact spaces and the Baire category property

In this Section, we introduce some results obtained in [P94] and [P98].

2.1. **Brunner's question.** Recall that a topological space X is a *Baire space* if every sequence of dense open subsets of X has a dense intersection. It is known (see [7]) that the statement *"Every complete metric space is a Baire space."* is equivalent to the axiom of Dependent Choices :

(**DC**, **axiom of Dependent Choices**). *For every non-empty set X and every binary relation R on X satisfying $\forall x \in X\ \exists y \in X\ xRy$, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of X satisfying $\forall n \in \mathbb{N}\ x_n R x_{n+1}$.*

Now, consider the following statement :

(**BC**, **Baire for Compact Hausdorff spaces**). *Every compact Hausdorff space is a Baire space.*

and the statement (introduced by Blass, [9]) :

(**DMC**, **Dependent Multiple Choice**). *Every pruned tree has a pruned subtree whose levels are finite.*

Here, a *tree* is a poset $(T, \leq)$ having a smallest element (the *root*) such that for every $x \in T$, the set $\{y \in T : y \leq x\}$ is finite and linearly ordered by $\leq$; a *pruned* tree is a tree $(T, \leq)$ such that for every $x \in T$, there exists $y \in T\backslash\{x\}$ satisfying $x \leq y$; a *subtree* of the tree $(T, \leq)$ is a subset S of T containing the root of T, which is closed under predecessor, and which is endowed with the order induced by $\leq$ on S.

It is rather easy to prove that **DMC** implies **BC** ; our Theorem 4 solves a question raised by Brunner (cf. [12]).

Theorem 4. **BC** $\Rightarrow$ **DMC**. *More precisely, the statement* "Every scattered compact space is a Baire space." *implies* **DMC**.

Proof. See [P98, Corollary 3 p. 6]. □

Here, a *scattered* space (see [11, TGIX.62, "espaces éparpillés"]) is a Hausdorff topological space having a basis of clopen (closed and open) subsets.

Question 1. Does **BC** imply **DC** ? (Goldblatt, [19]). Or, equivalently, does **DMC** imply **DC** ? (Blass, [9], notices that **DMC** together with the axiom of choice for countably many finite sets implies **DC**).

2.2. **The power of DMC.**

2.2.1. *Various consequences of* **DMC**. Consider the two following statements :

(**AC**($\mathbb{N}$), **countable axiom of choice**). *If* $(A_n)_{n\in\mathbb{N}}$ *is a sequence of nonempty subsets, then* $\prod_{n\in\mathbb{N}} A_n$ *is non-empty.*

(**AC^fin**($\mathbb{N}$), **countable axiom of Choice for finite sets**). *If* $(A_n)_{n\in\mathbb{N}}$ *is a sequence of nonempty finite subsets, then* $\prod_{n\in\mathbb{N}} A_n$ *is non-empty.*

Given a set X, consider the following restricted forms of **AC** to some set X :

(**DC**(X)). *For every binary relation* R *on* X *satisfying* $\forall x \in X\ \exists y \in X\ xRy$*, there exists a sequence* $(x_n)_{n\in\mathbb{N}}$ *of elements of* X *satisfying* $\forall n \in \mathbb{N}\ x_n R x_{n+1}$.

(**AC**($\mathbb{N}, X$)). *If* $(A_n)_{n\in\mathbb{N}}$ *is a sequence of nonempty subsets of* X*, then* $\prod_{n\in\mathbb{N}} A_n$ *is non-empty.*

($\mathbf{AC}^{fin}(\mathbb{N}, X)$). *If* $(A_n)_{n\in\mathbb{N}}$ *is a sequence of nonempty finite subsets of* X*, then* $\prod_{n\in\mathbb{N}} A_n$ *is non-empty.*

($\mathbf{WO}^{fin}(\mathbb{N}, X)$). *If* $(A_n)_{n\in\mathbb{N}}$ *is a sequence of finite subsets of* X*, then* $\cup_{n\in\mathbb{N}} A_n$ *is countable.*

Notice that **DMC** implies **DC**(X) for any set X satisfying $\mathbf{WO}^{fin}(\mathbb{N}, X)$ (for example a linearly orderable set X). For every set X, $\mathbf{DC}(X) \Rightarrow \mathbf{AC}(\mathbb{N}, X) \Rightarrow \mathbf{AC}^{fin}(\mathbb{N}, X)$. Although $\mathbf{AC^{fin}}(\mathbb{N})$ is equivalent to the statement *"Every sequence of finite sets has a countable union."*, I do not know if for a given set X, $\mathbf{AC}^{fin}(\mathbb{N}, X)$ and $\mathbf{WO}^{fin}(\mathbb{N}, X)$ are equivalent ; I do not know whether **DC**(X) implies $\mathbf{WO}^{fin}(\mathbb{N}, X)$.

Now consider the following statement :

(**PDC**, **Power Dependent Choice**). *For every set* X*, if* $\mathbf{DC}(X)+\mathbf{WO}^{fin}(\mathbb{N}, X)$ *then* $\mathbf{DC}(\mathcal{P}(X))+\mathbf{WO}^{fin}(\mathbb{N}, \mathcal{P}(X))$.

Proposition 1. **DMC** $\Rightarrow$ **PDC**.

Proof. (Unpublished) Assume that X is a set such that $\mathbf{DC}(X)+\mathbf{WO}^{fin}(\mathbb{N},X)$ holds. We are to prove that **DMC** implies $\mathbf{WO}^{fin}(\mathbb{N},\mathcal{P}(X))$. Let us introduce the following notion : given two sets A and B, say that B *subsumes* A if the mapping $A \to \mathcal{P}(B)$ associating to each $x \in A$ the set $x \cap B$ is one-to-one. Of course, for every set A, $\cup A$ subsumes A. If A is a finite set, then there is (at least) one finite subset of $\cup A$ subsuming A. Also notice that if a set B subsumes A, any set containing B also subsumes A. We now show that, given a sequence $(A_n)_{n\in\mathbb{N}}$ of non-empty finite subsets of $\mathcal{P}(X)$, $\cup_{n\in\mathbb{N}}A_n$ is countable. Since each A_n is subsumed by some finite subset of X, it is possible, using **DMC**, to consider a sequence $(B_n)_{n\in\mathbb{N}}$ of finite subsets of X such that each B_n subsumes A_n ; for each $n \in \mathbb{N}$, denote by $\phi_n : A_n \to \mathcal{P}(B_n)$ the one-to-one mapping $x \mapsto x \cap B_n$. Now $\mathbf{WO}^{fin}(\mathbb{N},X)$ implies that $\cup_{n\in\mathbb{N}}B_n$ is countable, whence $\cup_{n\in\mathbb{N}}\mathcal{P}(B_n)$ is countable. It follows that $\cup_{n\in\mathbb{N}}A_n$ is also countable. □

It follows that **DMC** implies for every ordinal α the statements $\mathbf{DC}\big(\mathcal{P}(\alpha)\big)$, $\mathbf{DC}\big(\mathcal{P}(\mathcal{P}(\alpha))\big)$, ... So **DMC** implies $\mathbf{DC}(\mathbb{R})$, $\mathbf{DC}(\mathcal{P}(\mathbb{R}))$, ..., $\mathbf{DC}(\mathcal{P}(\mathcal{P}(\mathbb{R})))$. In particular, **DMC** implies the statement $\mathbf{AC}(\mathbb{N},\mathbb{R})$ which is useful in measure theory : in fact, one can build in **ZF** the (finitely additive) Lebesgue measure on the boolean algebra $\mathcal{L}_n$ of Lebesgue-measurable subsets of $\mathbb{R}^n$; now in the setting $\mathbf{ZF}+\mathbf{AC}(\mathbb{N},\mathbb{R})$, one can prove that the boolean algebra $\mathcal{L}_n$ is σ-complete, and also the σ-additivity of the Lebesgue measure on $\mathcal{L}_n$ and other consequences (the dominated convergence theorem, the Fubini theorem).

Question 2. Does **PDC** imply **DMC** ?

2.2.2. *Ekeland's variational principle.* It is known, see [13] or [P99a, Lemma **5(i)** and Theorem **4**], that **DC** is equivalent to the following Ekeland's principle :

(**Ek**, **Ekeland's principle**). *If (E,d) is a nonempty sequentially complete metric space, if $f : E \to \mathbb{R}$ is lower semi-continuous and bounded below, if $\epsilon \in \mathbb{R}_+^*$, then there exists $a \in E$ such that for all $x \in E$, $f(a) \le f(x) + \epsilon d(x,a)$.*

Consider the following weak form of Ekeland's principle :

(**WEk**, **Weak Ek**). *If C is a nonempty sequentially complete convex subset of a normed space $(E, \|.\|)$, if $f : C \to \mathbb{R}$ is* convex*, lower semi-continuous and bounded below, if $\epsilon \in \mathbb{R}_+^*$, then there exists $a \in C$ such that for all $x \in C$, $f(a) \le f(x) + \epsilon\,\|x-a\|$.*

Theorem 5. **DMC** $\Rightarrow$ **WEk**.

Proof. See [P99a, Lemma **5(ii)**]. □

Question 3. Does **WEk** imply **DMC** ?

2.3. **Extremally disconnected compact Hausdorff spaces.** Say that a topological space X is *extremally disconnected* if for every open subset U of X, the closure $\overline{U}$ of U is open. Finite discrete spaces are extremally disconnected (and Hausdorff compact). Given a set X, we denote by βX the set of ultrafilters of X endowed with the Zariski topology[1]. If every filter on X is contained in a ultrafilter of X, then the space βX is Hausdorff compact and extremally disconnected (it is the Stone-Čech compactification of the discrete space X). If a set X is infinite and *amorphous* (every subset of X is finite or cofinite), then there exists a unique non-trivial ultrafilter on X : the set $\mathcal{U}_X$ of infinite (here cofinite) subsets of X ; moreover, given a filter $\mathcal{F}$ on X, either there is a finite subset F of X such that $F \in \mathcal{F}$, and $\mathcal{F}$ is contained in a trivial ultrafilter of X, or $\mathcal{F}$ is contained in $\mathcal{U}_X$: it follows that the space $\beta X := X \cup \{\mathcal{U}_X\}$ is compact Hausdorff and extremally disconnected (and it is also the Alexandrov compactification of the discrete space X). Thus, infinite amorphous sets yield infinite (amorphous) extremally disconnected Hausdorff compact spaces.

A set I is *Dedekind-finite* (“D-finite” for short) if there is no one-to-one mapping from $\mathbb{N}$ into I. In the opposite case, the set I is Dedekind-infinite (D-infinite). Consider the following axiom :

(**D**, **Dedekind-infinite**). *"Every infinite set is Dedekind-infinite".*

The class of *almost well-ordered* sets is (see [8]) the smallest class containing all singletons and closed under well-ordered union. This class is definable by a formula of **ZF** (see [26, p. 140-141]).

Theorem 6. *Assume that $\mathcal{M}$ is a model of* **ZF** *where there is a filter on ω which is not contained in any ultrafilter of ω.*

(1) *If $\mathcal{M}$ satisfies* **D***,*
(2) *or if every set of $\mathcal{M}$ is almost well-ordered,*

then every extremally disconnected Hausdorff compact space of $\mathcal{M}$ is finite.

Proof. 1. See [P94, Corollary 6 p. 10].
2. See See [P94, Corollary 7 p. 12]. □

For example, Blass' model without any non-trivial ultrafilter on $\mathbb{N}$ (see [8]) satisfies condition 2. of Theorem 6. Notice that Blass' model does not satisfy $\mathbf{AC}^{fin}(\mathbb{N}, \mathcal{P}(\mathbb{R}))$ since in this model, there is a sequence $(P_n)_{n\in\mathbb{N}}$ of two-element subsets of $\mathcal{P}(\mathbb{R})$ such that $\prod_{n\in\mathbb{N}} P_n = \varnothing$. It follows that the following statement **BCED** does not imply $\mathbf{AC}^{fin}(\mathbb{N}, \mathcal{P}(\mathbb{R}))$:

(**BCED**, **BC for Extremally Disconnected spaces**). *Extremally disconnected Hausdorff compact spaces are Baire.*

In particular, **BCED** does not imply **DMC**.

[1] See Appendix A.3

$$\begin{tikzcd}
 & \mathbf{DC} \arrow[d] & \\
 & \mathbf{DMC} \arrow[dl] \arrow[d] \arrow[dr] & \\
\mathbf{PDC} \arrow[d] & \mathbf{WEk} & \mathbf{BCED} \\
\mathbf{DC}(\mathcal{P}(\mathbb{R})) \arrow[d] & & \\
\mathbf{DC}(\mathbb{R}) \arrow[d] & & \\
\mathbf{AC}(\mathbb{N}, \mathbb{R}) \arrow[d] & & \\
\mathbf{AC}_{fin}(\mathbb{N}, \mathcal{P}(\mathbb{R})) & &
\end{tikzcd}$$

FIGURE 1. Around **DMC**.

2.4. **Countable compact Hausdorff spaces.** Consider the following statements :

(**BCC**, **BC for Countable spaces**). *Every countable compact Hausdorff is Baire.*

(**WBCC**, **Weak BCC**). *Every non-empty countable compact Hausdorff space has an isolated point.*

(**BCCC**, **BC for Countable Connected spaces**). *Every non-empty countable compact connected Hausdorff space is Baire (whence it is a singleton).*

On september 96, I asked Miller the following question : "Does **ZF** prove **BCC** ?" and Miller inserted my question into his list of open question ([32]). On 15 December 1996, I asked the more precise question (see my postage to the forum *sci.math.research*, entitled *"Countable Continua")* : does **ZF** prove **BCCC** ? Such a question has a "philosophical flavour" since it is a way of asking if there may exist an infinite continuum (*i.e.* a connected compact Hausdorff space) which is somewhat "discrete" (equipotent with $\mathbb{N}$). In his Thesis ([15], [16]), Omar de la Cruz answered both questions : he first built a model of **ZF** where there is an infinite countable compact Hausdorff space without any isolated point ; he then built another model of **ZF** with an infinite countable compact connected Hausdorff space.

Part 2. Boolean algebras and Gelfand algebras

3. STONE AND GELFAND REPRESENTATION THEOREMS

In this Section, I present some equivalents of the "Boolean prime ideal axiom".

3.1. **The Boolean Prime Ideal axiom.** Consider the following axiom :

(**BPI, Boolean Prime Ideal**). *Every non-trivial boolean algebra has a prime ideal.*

The statement **BPI** has numerous equivalents (see [22]). For example, it is known that **BPI** is equivalent to the following *Stone representation Theorem* : *"Every boolean algebra* $\mathbb{B}$ *is isomorphic with the algebra of clopen subsets of its spectrum* $\Sigma_{\mathbb{B}}$ via *the morphism* $a \mapsto \Omega_a$*."* In my Thesis ([P87], [R88]), I found the following equivalent of **BPI** :

(**PIC**, **Prime ideal Choice**). *For every bounded distributive lattice*[2] L*, there is a mapping associating to every proper ideal of* L *a prime ideal*[3] *of* L *containing* I*.*

My proof of **BPI** $\Rightarrow$ **PIC** (see [P87]) is based on the two following consequences of **BPI** :

(1) Given a bounded distributive lattice and some proper ideal I of L, the subspace of prime ideals of L containing I endowed with the pro-constructible[4] topology is non-empty and Hausdorff compact ;
(2) Every family of non-empty Hausdorff compact spaces has a non-empty product.

Now consider the following statement, introduced by Johnstone (see [25]) :

AMIT, **(Almost Maximal Ideal Theorem)** *Every bounded distributive lattice has an almost maximal ideal.*

Here, an *almost maximal* ideal of a bounded distributive lattice L is a prime ideal which is fixed by the "Jacobson radical" j, associating to every ideal I of L the ideal

$$j(I) := \{a \in L : \forall b \in L(b \vee a = 1 \Rightarrow \big(\exists i \in I \; b \vee i = 1)\big)\}$$

The equivalence **BPI** $\Rightarrow$ **PIC** allowed me to answer affirmatively a question raised by Johnstone (see [25]) :

Theorem 7. BPI $\Rightarrow$ **AMIT**.

Proof. See [P87]. □

Notice that Blass ([10]) and Banaschewski ([2]) also solved Johnstone's question.

3.2. **Gelfand algebras.** Say that a $\mathbb{R}$-commutative unitary normed algebra A is a *pre-Gelfand algebra* if A satisfies the two following properties :

$$\forall f \in A \quad \left\|f^2\right\| = \|f\|^2 \tag{2}$$

$$\forall f \in A \;\; 1 + f^2 \text{ is invertible in } A \tag{3}$$

Moreover, if the pre-Gelfand algebra A is complete, say that A is a Gelfand algebra. Every Gelfand algebra is a Riesz space[5] for the order $\leq$ defined by the formula $x \leq y$ iff $Sp(y-x) \subseteq \mathbb{R}_+$.

Example 1. Given a topological space X, the $\mathbb{R}$-algebra $A := C^*_{\mathbb{R}}(X)$ of continuous bounded mappings $f : X \to \mathbb{R}$ endowed with the uniform norm is a Gelfand algebra.

[2] see Appendix A.2
[3] see Appendix A.2
[4] see Appendix A.1
[5] See Appendix B.3

Given a Gelfand algebra A, and a maximal ideal $\mathfrak{M}$ of A, the $\mathbb{R}$-algebra $A/\mathfrak{M}$ is isomorphic with $\mathbb{R}$ through a (unique) isomorphism that we denote by $\chi_{\mathfrak{M}} : A/\mathfrak{M} \to \mathbb{R}$. For every $a \in A$, we consider the mapping $\tilde{a} : Max(A) \to \mathbb{R}$ associating to every $\mathfrak{M} \in Max(A)$ the real number $\chi_{\mathfrak{M}}(a + \mathfrak{M})$.

Theorem (Banaschewki, [1]**). BPI** *is equivalent to the following representation theorem :* For every Gelfand algebra A, the subspace $Max(A)$ of the spectrum Σ_A is compact, Hausdorff and completely regular, and the Banach algebra A is isomorphic with $C^*(Max(A))$ through the mapping $a \mapsto \tilde{a}$.

4. Measured boolean algebras and measured Gelfand algebras

In this Section, we present results obtained in [P87], [P90] and [P91].

4.1. **A geometric form of the Boolean Prime Ideal.** In [5], Bell and Fremlin showed that **AC** is equivalent to the following statement :

> *For every commutative Banach algebra A, the closed unit ball of the continuous dual A' has an extreme point.*

Here, recall that given a convex subset of a real vector space E, a point $e \in C$ is an *extreme point* of C if for every $x, y \in C$, and for every real number $\lambda \in [0, 1]$, $\lambda x + (1 - \lambda)y = e \Rightarrow e \in \{x, y\}$. In their proof, Bell and Fremlin consider some family $(A_i)_{i \in I}$ of non-empty sets and they define the (commutative) Banach algebra $A := \prod_{\ell^1(I)} \ell^0(A_i)$. The continuous dual A' is $\prod_{\ell^\infty(I)} \ell^1(A_i)$, and the closed unit ball of A' is $C := \prod_{i \in I} \Gamma_i$ where for each $i \in I$, Γ_i is the closed unit ball of $\ell^1(A_i)$. A point $e = (e_i)_{i \in I}$ of C is extreme if and only if for each $i \in I$, e_i is extreme in $\ell^1(A_i)$, which means that e_i is the indicator of some singleton $\{a_i\}$ where $a_i \in A_i$; so a choice function for the family $(A_i)_{i \in I}$ is precisely an extreme point of C. Using the *unitization* of a Banach algebra (see [35] p. 2-3), **AC** is also equivalent to the following statement :

> *For every commutative unitary Banach algebra A, the closed unit ball of the continuous dual A' has an extreme point.*

In [P87], [R88], we showed that **BPI** is equivalent to the following statement :

> **GE** *: For every non-null Gelfand algebra A, the closed unit ball of the continuous dual A' has a non-null extreme point.*

Our proof relied on Theorem 28[6]. We then introduced the following weak form of **GE** :

(**WGE**, **Weak GE**). *For every Gelfand algebra A, the closed unit ball of the continuous dual A' has an extreme point.*

and we asked the following question, which is still open :

Question 4. Does **WGE** imply **GE** ? or, equivalently, does **WGE** imply **BPI** ?

[6] See Appendix B.5

4.2. **Measured Gelfand algebras.** Given a commutative unitary normed algebra A, a *state* on A is a positive linear functional $f : A \to \mathbb{R}$ satisfying $f(1) = 1$. Every state is continuous, with norm $\|1\|$. A *measured normed algebra* is a commutative unitary normed algebra A endowed with a state f.

Theorem 8. *The statement* **WGE** *is equivalent to each of the following statements :*
WGE1 *: Every measured Gelfand algebra has a maximal ideal.*
WGE2 *: Every measured Gelfand algebra has a prime ideal.*

Proof. **WGE** $\Rightarrow$ **WGE1** : Use Proposition 5. **WGE1** $\Rightarrow$ **WGE2** is straightforward. **WGE2** $\Rightarrow$ **WGE** : Let A be a Gelfand algebra. If $A' = \{0\}$ then 0 is an extreme point of the closed unit ball of A' ; else, there exists some non-null element $f \in A'$; but $f = f^+ - f^-$ where f^+ and f^- are positive linear functionals satisfying $\|f\| = \|f^+\| + \|f^-\|$. Thus, there exists some state g on A. Using **WGE2**, this implies a prime ideal I on A. Now, the adherence $\overline{I}$ is a maximal ideal of the Gelfand algebra A (see [1]). □

4.3. **Measured Boolean algebras.** Given a bounded lattice $(L, \vee, \wedge, 0, 1)$, say that a mapping $m : L \to \mathbb{R}_+$ is a *measure* if for every $x, y \in L$, $x \wedge y = 0 \Rightarrow m(x \vee y) = m(x) + m(y)$. Moreover, if $m(1) = 1$, say that m is a *probability.* A *measured bounded lattice* is a bounded lattice endowed with a probability. Consider the following statements :

MBPI : *Every measured Boolean algebra has a prime ideal.*

MLPI : *Every measured bounded distributive lattice has a prime ideal.*

(**WMBPI**, **Weak MBPI**). *For every set X, for every sub-algebra $\mathcal{B}$ of the boolean algebra $\mathcal{P}(X)$, for every probability $m : \mathcal{B} \to [0, 1]$, there exists a maximal filter $\mathcal{U}$ of $\mathcal{B}$ containing the set $\{x \in \mathcal{B} : m(x) = 1\}$.*

(**EMBPI**). *For every set X, for every probability $m : \mathcal{P}(X) \to [0, 1]$, there exists a maximal filter $\mathcal{U}$ of $\mathcal{B}$ containing the set $\{A \in \mathcal{P}(X) : m(A) = 1\}$.*

($\mathbf{U}_\omega$). *There exists a non-trivial ultrafilter on* $\mathbb{N}$.

Clearly, **MBPI** $\Rightarrow$ **WMBPI** $\Rightarrow$ **EMBPI**.

Theorem 9. (1) **WGE** $\Rightarrow$ **MBPI** $\Rightarrow$ **MLPI**.
(2) **WMBPI** $\Rightarrow$ $\mathbf{U}_\omega$.
(3) **EMBPI** $\not\Rightarrow$ $\mathbf{U}_\omega$.

Proof. 1. **WGE** $\Rightarrow$ **MBPI** : see [P90, Théorème p.15.04].
MBPI $\Rightarrow$ **MLPI** : see [P90, Théorème p.15.12].
2. See [P90, Théorème 2 p.15.07].
3. See [P90, Théorème 3 p.15.07]. □

Theorem 10. (1) **WMBPI** *is equivalent to its "multiple form" :*
Given a family $(X_i)_{i \in I}$ of sets, a family $(\mathcal{B})_{i \in I}$ such that each $\mathcal{B}_i$ is a sub-algebra of the boolean algebra $\mathcal{P}(X_i)$, given a family $(m_i)_{i \in I}$ such that each m_i is a probability on $\mathcal{B}_i$, there exists a family $(\mathcal{U}_i)_{i \in I}$ such that each $\mathcal{U}_i$ is a maximal filter of $\mathcal{B}_i$ containing the set $\{x \in \mathcal{B}_i : m_i(x) = 1\}$.

(2) **WMBPI** $\Rightarrow$ $\mathbf{AC^{fin}}(\mathbb{N})$.

Proof. See [P90, Théorème 1 p.15.08 and Théorème 2 p.15.09]. $\square$

Theorem 11. **MBPI** *is equivalent to its "multiple form".*

Proof. See [P90, Théorème p.15.11]. In this proof, we build in **ZF** the "coproduct" of an infinite family of Boolean algebras. $\square$

4.4. **The Gelfand algebra** $L^\infty(A, f)$**.** Say that a measured Gelfand algebra (A, f) is *reduced* if the state f satisfies $\forall x \in A_+ \big(f(x) = 0 \Rightarrow x = 0\big)$. Say that a measured bounded lattice (L, m) is *reduced* if $\forall x \in L \big(m(x) = 0 \Rightarrow x = 0\big)$. Say that a norm N on a Riesz space[7] E is a *lattice norm* if for every $x, y \in E$, $\big(|x| \le |y| \Rightarrow \|x\| \le \|y\|\big)$. A Riesz space endowed with a lattice norm is called a *normed Riesz space.* A normed Riesz space which is complete (every Cauchy filter converges) is called a *Banach lattice.*

Notation 1 ($L_1(A, f)$)**.** Given a reduced Gelfand algebra (A, f), the mapping $N_1 : A \to \mathbb{R}_+$ associating to every $x \in A$ the real number $f(|x|)$ is a lattice norm. We denote by $L_1(A, f)$ the Banach completion of the normed space (A, N_1) : then, the lattice laws $\vee$ and $\wedge$ have unique extensions to $L_1(A, f)$, and $L_1(A, f)$ is a Banach lattice.

Notation 2 ($L^\infty(A, f)$)**.** Given a reduced Gelfand algebra (A, f), we denote by $L^\infty(A, f)$ the vector subspace $\{x \in L_1(A, f) : \ \exists\lambda \in \mathbb{R}\ |x| \le \lambda\}$ endowed with the norm $\|.\| : L^\infty(A, f) \to \mathbb{R}_+$ associating to every $x \in L^\infty(A, f)$ the real number $\inf\{\lambda \in \mathbb{R} : |x| \le \lambda\}$.

Given two posets P, Q, say that a mapping $f : P \to Q$ is *order continuous* if for every upward directed non empty subset A of P with a least upper bound in P, $f(\bigvee A) = \bigvee f[A]$. Say that f is *σ-order continuous* if for every upward directed sequence $(a_i)_{i\in\mathbb{N}}$ of P with a least upper bound in P, $f(\vee_{i\in\mathbb{N}} a_i) = \vee_{i\in\mathbb{N}} f(a_i)$.

Theorem 12. *Let (A, f) be a reduced Gelfand algebra.*

(1) *The lattice order on the Banach lattice $L_1(A, f)$ is conditionally complete ;*
(2) *The (unique) continuous extension $\overline{f} : L_1(A, f) \to \mathbb{R}$ of f is order continuous ;*
(3) *The multiplicative law of A has a unique continuous extension to $L^\infty(A)$;*
(4) $(L^\infty(A), +, \lambda., ., \|.\|)$ *is a Gelfand algebra extending A ; moreover, the lattice order on $L^\infty(A)$ is conditionally complete.*

Proof. See [P91, Théorème p. 6]. $\square$

Given a commutative unitary Banach algebra A, endow A with the spectrum order ($a \in A_+ \Leftrightarrow Sp(a) \subseteq \mathbb{R}_+$) and consider the "square root" mapping $\sqrt{}:[[0, 1]] \to [[0, 1]]$, satisfying for every element $a \in [[0, 1]]$, the equality

$$\sqrt{a} = 1 - \frac{1}{2}u - \frac{1}{8}u^2 - \cdots - \frac{1.3.\ldots.(2n-3)}{2^n n!}u^n - \ldots$$

where $u = 1 - a \in [[0, 1]]$. Denoting by Z the set $\{2^k : k \in \mathbb{N}^*\}$, one can iterate the function $\sqrt{}$ and define for every $n \in Z$, a mapping $\sqrt[n]{}:[[0, 1]] \to [[0, 1]]$.

[7]See Appendix B.3

Given a commutative unitary ring $(A, +, ., 0, 1)$, we denote by $\mathbb{A}$ the set $\{x \in A : x^2 = x\}$. Then, consider the law $\oplus : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$ associating to every $x, y \in A$ the element $x \oplus y := x + y - 2x.y$. It is known that $(\mathbb{A}, \oplus, ., 0, 1)$ is a Boolean algebra.

Theorem 13. *Assume that A is a Gelfand algebra which is countably conditionally complete (*i.e. every non-empty countable bounded subset of A has an upper bound in A*). Consider the mapping $r : A \to \mathbb{A}$, associating to every $x \in A$ the idempotent* $\sup_{n \in Z}\{\sqrt[n]{\frac{|x|}{\|x\|}}\}$. *Then,*

1. *If $a \in [[-1, 1]]$ then $r(a) = \sup_{n \in Z}\{\sqrt[n]{|a|}\}$;*
2. *if $a, b \in A_+$, then $r(a) \vee r(b) = r(a \vee b)$; in particular, r is increasing on A_+ ;*
3. *For every $a, b \in A$, $r(a).r(b) = r(a.b)$;*
4. *If I is an ideal of the (bounded distributive) lattice $[[0, 1]]$, then $J := \{x \in A : r(x) \in I\}$ is an ideal of the ring A ; moreover, if the ideal I of $[[0, 1]]$ is prime, then J is a prime ideal of the ring A.*

Proof. See [P91, Théorème p. 4]. □

Corollary 1. MLPI ⇒ WGE.

Proof. See [P91, Théorème p.8]. □

Question 4 can be reformulated as follows :

Question 5 ([P90], [P91]). Does **MBPI** imply **BPI** ?

Part 3. Functional Analysis and the Axiom of Choice

All vector spaces that we consider in this Part are vector spaces over $\mathbb{R}$, the field of reals.

5. The Hahn-Banach property

In this Section, we present results obtained in [P92], [P98], [P99a] and [P01]. Given a normed space $(E, \|.\|)$, we denote by Γ_E its closed unit ball $\{x \in E : \|x\| \leq 1\}$, and we denote by E' its continuous dual.

5.1. **Bell and Fremlin's question.** Consider the following strong form of Krein-Milman's principle :

(**VKM**, **"Version" of Krein-Milman**). *If E is a locally convex Hausdorff topological vector space, if C is a non-empty convex subset of E which is convex-compact, then C has an extreme point.*

Here, given a topological vector space E, a convex subset C of E is said to be *convex-compact* if for every family $\mathcal{F}$ of closed convex subsets of C satisfying the finite intersection property, $\cap\mathcal{F}$ is non-empty. The following question is open :

Question 6 (Bell and Fremlin, [5]). Does **VKM** imply **AC** ?

Following Luxemburg's idea (see [28]), we showed (see Theorem 18, Section 5.6) that, given a normed space E, convex-compactness of $\Gamma_{E'}$ in the *weak topology follows from some Hahn-Banach property on E. So our search for convex-compact sets leads us to searching Banach spaces satisfying this Hahn-Banach property in **ZF**.

5.2. **Various Hahn-Banach properties.** Given a vector space E, a *sublinear functional* on E is a mapping $p : E \to \mathbb{R}$ such that for every $x, y \in E$ and $\lambda \in \mathbb{R}_+$, $p(x+y) \leq p(x) + p(y)$ and $p(\lambda.x) = \lambda.p(x)$. Say that the vector space E satisfies the *Hahn-Banach property (*resp. *continuous Hahn-Banach property)* if, for every sublinear functional (*resp.* continuous sublinear functional) $p : E \to \mathbb{R}$, for every subspace F of E and every linear functional $f : F \to \mathbb{R}$ such that $f \leq p_{|F}$, there exists a linear functional $g : E \to \mathbb{R}$ extending f and satisfying $g \leq p$. For short, in the first case say that E satisfies the **HB** *property*, and in the second case say that E satisfies the **CHB** *property*. Let S (*resp.* S_c) be the set of pairs (p, f) where $p : E \to \mathbb{R}$ is a *sublinear functional (*resp. *continuous sublinear functional)* on E and $f : F \to \mathbb{R}$ is a linear functional defined on a subspace F of E satisfying $f \leq p_{|F}$. Say that E satisfies the *multiple Hahn-Banach property (*resp. *multiple continuous Hahn-Banach property)* if there is a function ϕ which is defined on the set S, and which associates to every $(p, f) \in S$ (*resp.* $\in S_c$) a linear functional $g = \phi(p, f)$ which is defined on E, such that g extends f and $g \leq p$; here ϕ is called a *witness* of the multiple (*resp.* multiple continuous) Hahn-Banach property on E. For short, in the first case say that E satisfies the **MHB** *property*, and in the second case say that E satisfies the **MCHB** *property*.

5.3. **Various geometric Hahn-Banach properties.**

5.3.1. *Large separation.* The following Theorem presents equivalent "large separation" properties in topological vector spaces :

Theorem 14. *Given a topological vector space E, the following properties are equivalent :*

1. *The space E satisfies the* **CHB** *property ;*
2. *For every affine subspace C of E and every non-empty open convex subset O of E such that $C \cap O = \varnothing$, there exists a linear functional $f : E \to \mathbb{R}$ satisfying for every $x \in O$ the inequality $f(x) < \inf_{z \in C} f(z)$.*
3. *If C is a nonempty convex subset of E and if O is a nonempty open convex subset in E such that $C \cap O = \varnothing$ then there exists a linear functional f on E such that $\forall x \in O\ f(x) < \inf_C f$.*
4. *If $a \in E$ and if O is a nonempty open convex subset in E such that $a \notin O$ then there exists a linear functional f on E such that $\forall x \in O\ f(x) < f(a)$.*
5. *If C and O are two nonempty disjoint convex subsets of E, and if O is open then there exist a linear functional f on E such that $f[O] < f[C]$.*
6. *If C and O are two nonempty disjoint open convex subsets of E, then there exist a linear functional f on E such that $f[O] < f[C]$.*

Proof. See [P99a, Theorem 2]. □

5.3.2. *Strict separation.*

Lemma 1. *Let E be a topological vector space E, C a non-empty convex subset of E and $a \in E \backslash C$. If there are $f_1, \dots, f_m \in E'$ and $\alpha_1, \dots, \alpha_m \in \mathbb{R}$ such that the weak open set $O := \cap_{1 \leq i \leq m} \{x \in E : f_i(x) < \alpha_i\}$ satisfies $O \cap C = \varnothing$ and $a \in O$, then there exists $f \in E'$ satisfying $f(a) < \inf_C f$.*

Proof. See [P99a, Lemma 1] . □

Theorem 15. *Given a Hausdorff locally convex topological vector space E, each of the following properties is equivalent to the* **CHB** *property on E :*

1. *For every nonempty closed convex subset C of E, and every point $a \in E \backslash C$, there exists a continuous linear functional $f : E \to \mathbb{R}$ such that $f(a) < \inf_C f$;*
2. *Every closed convex subset of E is weakly closed ;*
3. *If C is a nonempty closed convex subset of E and if K is a nonempty compact convex subset of E then there exists a linear functional f on E such that $\sup_K f < \inf_C f$.*

Proof. See [P99a, Lemma 1 and Corollary 2]. □

Effective versions of Theorems 14 and 15 can be proved : for example, a topological vector space E satisfies the **MCHB** property if and only if there is a mapping associating to each ordered pair (C, O) of disjoint open convex subsets of E, a linear functional $f : E \to \mathbb{R}$ satisfying $f[C] < f[O]$.

5.3.3. *Various Mazur properties.* Say that a topological vector space E satisfies the *Mazur property* if every closed convex subset of E is weakly closed. Say that a closed convex subset C of a normed space $(E, \|.\|)$ satisfies the *strong Mazur property* if for every $x \in E \backslash C$, there exists some continuous linear functional $f : E \to \mathbb{R}$ such that $\|f\| = 1$ and $f(x) + \rho \leq f[C]$ where $\rho := \inf\{\|z - x\| : \ z \in C\}$ is the distance between x and C ; say that the closed convex subset C satisfies the *almost strong Mazur property* if for every $x \in E \backslash C$, for every $\delta \in]0, 1[$, there exists some continuous linear functional $f : E \to \mathbb{R}$ such that $\|f\| = 1$ and $f(x) + \delta\rho \leq f[C]$. Say that a normed space satisfies the *strong Mazur property (*resp. *the almost strong Mazur property)* if every closed convex subset satisfies the strong Mazur property (*resp.* the almost strong Mazur property).

Remark 1. Given a normed space E, the following properties are equivalent :

1. The **CHB** property on E ;
2. The strong Mazur property on E ;
3. The almost strong Mazur property on E ;
4. The Mazur property on E.

Proof. (1) $\Rightarrow$ (2) If the normed space E satisfies the **CHB** property, then, given a non-empty closed convex subset C of E and a point $a \in E \backslash C$, and denoting by ρ the distance between a and C, Theorem 14 ("large separation") applied to the open ball $B(a, \rho)$ and to the closed convex set C yields a linear functional $f : E \to \mathbb{R}$ satisfying $f[B(a, \rho)] < f[C]$. Now $f \neq 0$, so, dividing f by its norm, we may assume that $\|f\| = 1$. This implies that $f(a) + \rho \leq f[C]$.
(2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are trivial.
(4) $\Rightarrow$ (1) If every non-empty closed convex subset of E is weakly closed, then, applying Theorem 15 ("strict separation") yields the **CHB** property on E. □

Say that a normed space satisfies the *bounded* Mazur property if every bounded closed convex subset is weakly closed. Say that a normed space satisfies the *bounded* strong Mazur property if every bounded closed convex subset satisfies the strong Mazur property.

Question 7 ([P99a, Question 2])**.** Given a normed space E, does the bounded Mazur property on E imply the Mazur property on E ?

Question 8. Given a normed space E, does the bounded strong Mazur property on E imply the Mazur property on E ?

5.4. **Geometric proofs of various Hahn-Banach properties.** Say that a normed space satisfies the *"projection" property* if for every closed convex subset C of E and every $x \in E \backslash C$, there exists at least one point $a \in C$ satisfying $\|a - x\| = \inf\{\|z - x\| : z \in C\}$. It is known that in **ZFC**, the "projection" property for a Banach space is equivalent to "reflexivity" of this space : this is a consequence of James' sup theorem, according to which, in **ZFC**, a Banach space is "reflexive" if and only if every continuous linear functional on E attains its upper bound on the closed unit ball of E.

Theorem 16. (1) *Normed spaces with a dense well-orderable subset satisfy the* **MCHB**.
(2) *Hilbert spaces satisfy the* **MCHB** *property.*
(3) *For every set I, the normed space $\ell^0(I)$*[8] *satisfies the* **MCHB** *property.*
(4) *Gâteaux differentiable*[9] *Banach spaces satisfying the projection property also satisfy the* **CHB** *property.*
(5) *Uniformly convex*[10] *Gâteaux differentiable Banach spaces satisfy the* **MCHB** *property.*
(6) **DC** *implies that Gâteaux differentiable Banach spaces satisfy the* **CHB** *property.*
(7) **WEk** *implies that Gâteaux differentiable Banach spaces satisfy the bounded almost strong Mazur property.*
(8) *Uniformly smooth*[11] *Banach spaces satisfy the* **MCHB** *property.*

Proof. (1), (2) and (3). See [P98].
(4), (5), (6) and (7). See [P99a].
(8) See [P01]. □

Question 9 ([P99a, Question 1])**.** Does a Gâteaux-differentiable Banach space satisfy the **CHB** property or the bounded Mazur property in **ZF** ?

There are (at least) two well-known notions of differentiability for a norm, which are intermediate between Gâteaux-differentiability and uniformly smooth differentiability : Fréchet differentiability[12] and uniform Gâteaux differentiability[13].

Question 10. Does a Fréchet-differentiable Banach space satisfy the **CHB** property or the bounded Mazur property in **ZF** ?

Question 11. Does a uniformly Gâteaux-differentiable Banach space satisfy the **CHB** property or the bounded Mazur property in **ZF** ?

[8]See Appendix D
[9]See Appendix E.2.2
[10]See Appendix E.1
[11]See Appendix E.2.4
[12]See Appendix E.2.3
[13]See Appendix E.2.6

5.5. **The Hahn-Banach property for separable normed spaces.** Clearly, every separable normed space satisfies the **CHB** property. More generally, every normed space having a dense well-orderable subset satisfies the **CHB** property. Given a set I and an element $i \in I$, the *Dirac mapping* m which associates to every subset A of I the real number 0 if $i \notin A$ and 1 if $i \in A$ is a probability on the boolean algebra $\mathcal{P}(I)$. More generally, given some family $(\lambda_i)_{i \in I} \in [0,1]^I$ satisfying $\sum_{i \in I} \lambda_i = 1$, the mapping $m := \sum_{i \in I} \lambda_i \delta_i$ is a measure on I : such a measure is said to be *trivial*. Pincus and Solovay ([34]) built a model of **ZF+DC** where every measure on the algebra $\mathcal{P}(\mathbb{N})$ is trivial. It follows that the following statement is not provable in **ZF** :

(**M Measure**). *There exists a non-trivial measure on* $\mathbb{N}$.

Now consider the following "countable" Hahn-Banach axiom :

(**HB**($\mathbb{N}$)). *Every separable normed space satisfies the Hahn-Banach property.*

and its weak form :

(**WHB**($\mathbb{N}$)). *Every separable Banach space satisfies the Hahn-Banach property.*

Theorem 17. **HB**($\mathbb{N}$) $\Rightarrow$ **M**.

Proof. See [P99a, Theorem 6]. □

It follows that **HB**($\mathbb{N}$) is not provable in **ZF**.

Question 12. ([P99a, Question 3]) Is **WHB**($\mathbb{N}$) provable in **ZF** ?

5.6. **Convex compactness.** Say that C is *effectively convex-compact* if there is a mapping ψ associating to every family $\mathcal{F}$ of closed convex subsets of C satisfying the finite intersection property, an element $\phi(\mathcal{F}) \in \cap\mathcal{F}$. In these conditions, ϕ is called a *witness* of the effective convex-compactness of C.

Luxemburg, see [28], showed that the two following statements are equivalent :

(**HB**, **Hahn-Banach**). *Every vector space satisfies the Hahn-Banach property.*

(**WA**, **Weak Alaoglu**). *For every normed space E, the closed unit ball of the normed space E' is convex-compact in the *weak topology.*

Our next Theorem specifies Luxemburg's result :

Theorem 18. *Let E be a normed space.*

(1) *If E satisfies the* **CHB** *property, then the closed unit ball of E' is convex-compact in the *weak topology.*
(2) *If E satisfies the* **MCHB** *property, then the closed unit ball $\Gamma_{E'}$ of E' is effectively convex-compact in the *weak topology. Moreover, a witness of the effective c-compactness of $\Gamma_{E'}$ is definable from E and a witness of the* **CHB** *property on E.*

Proof. See [P98, Theorem 4]. □

Corollary 2. *Given a set I, the closed unit ball of $\ell^1(I)$ is effectively convex-compact in the *weak topology.*

Proof. See [P98, Corollary 6]. ☐

Theorem 19. *Given a set I, the following properties are equivalent :*

(1) *The normed space $\ell^1(I)$ satisfies the* **CHB** *property ;*
(2) *The convex subset $[0,1]^I$ is convex-compact in the locally convex Hausdorff space $\mathbb{R}^I$.*

Proof. See [P98, Lemma 6]. ☐

Say that a normed space E satisfies the *finite extension property* (for short *FEP*) if for every finite-dimensional subspace F of E, and every (continuous) linear functional $f : E \to \mathbb{R}$, there exists some continuous functional $g : E \to \mathbb{R}$ extending f such that $\|g\| = \|f\|$. For example, every Gâteaux-differentiable normed space satisfies the *FEP*.

Theorem 20. *Given a normed space E, the following properties are equivalent :*

(1) *The space E satisfies the* **CHB** *property ;*
(2) *The space E satisfies the* FEP *and the closed unit ball of E' is convex-compact in the $*$weak topology.*

Proof. See [P98, Theorem 6]. ☐

5.7. **A partial answer to Bell and Fremlin's question.** Our next Theorem is a partial anwser to Bell and Fremlin's question. Consider the following weak form of the **VKM** axiom :

(**WVKM**, **Weak VKM**). *If E is a locally convex Hausdorff topological vector space, if C is a non-empty convex subset of E which is* effectively *convex-compact, then C has an extreme point.*

Also consider the following axiom of choice for well-ordered families of sets :

($\mathbf{AC_{wo}}$). *For every ordinal α, for every family $(A_\lambda)_{\lambda\in\alpha}$ of non-empty sets, $\prod_{\lambda\in\alpha} A_\lambda$ is non-empty.*

Theorem 21. WVKM $\Rightarrow$ $\mathbf{AC_{wo}}$.

Proof. See [P98, Corollary 7]. ☐

Notice that it is known that $\mathbf{AC_{wo}}$ implies **DC** (see [24]).

5.8. **DC and the Baire Category Theorem for convex-compact sets.** We have obtained the following equivalent of **DC**, in terms of Baire property for somewhat compact spaces :

Theorem 22. DC *is equivalent to the following statement :* "In a Hausdorff locally convex topological vector space, convex-compact convex subsets are Baire spaces."

Proof. See [P98, Theorem 10]. ☐

6. Reflexive Banach spaces

In this Section, we review some results obtained in [P99b] and [P04b].

6.1. **Weak compactness in uniformly convex Banach spaces.** Consider the following statements :

(**A**, **Alaoglu**). *For every normed space E, the closed unit ball of E' is *weakly compact.*

(**AH**, **Alaoglu Hilbert**). *The closed unit ball of a Hilbert space is weakly compact.*

(**RCuc**, **Reflexive Compactness for uniformly convex Banach spaces**). *The closed unit ball of a uniformly convex Banach space is weakly compact.*

Theorem 23. **DC** $\Rightarrow$ **RCuc** $\Rightarrow$ **AH** $\Rightarrow$ $\mathbf{AC^{fin}}(\mathbb{N})$.

Proof. **DC** $\Rightarrow$ **RCuc** : see [P99b]. **RCuc** $\Rightarrow$ **AH** is trivial. The proof of **AH** $\Rightarrow$ $\mathbf{AC^{fin}}(\mathbb{N})$ is in [P98, Theorem 9]. □

6.2. **James sequences and J-reflexivity.** Given a normed space $(E, \|.\|)$ and some real number $\vartheta > 0$, a sequence $(a_k)_{k\in\mathbb{N}}$ of E is a *ϑ-sequence* if for every integer $i \in \mathbb{N}$, the distance between $span\{a_k : k < i\}$ and $conv\{a_k : i \le k\}$ is $> \vartheta$. Say that a Banach space E is *J-reflexive* if for every $\vartheta \in]0, 1[$, the closed unit ball of E does not contain any James sequence. In **ZFC**, James proved that every closed bounded convex subset of a J-reflexive Banach space is weakly compact.

Given a normed space E, we denote by L_E the lattice generated by the closed convex subsets of E : thus, L_E is the set of finite unions of closed convex subsets of E. We then define the *convex topology* on the normed space E as the topology for which a subset of E is closed if and only if it is the intersection of a subset of the lattice L_E. Say that a Banach space is *convex-reflexive* if its closed unit ball is compact in the convex topology. We now consider the following statement :

(**J2C**). *Every J-reflexive Banach space is convex-reflexive.*

Theorem 24. **DC** $\Rightarrow$ **J2C**.

Proof. See [P04b]. □

Say that a normed space is *J-convex* (see [3]) if there exists some integer $N \ge 2$ and some real number $\vartheta \in]0, 1[$ satisfying :

$$(4) \quad \forall a_1, \dots, a_N \in S_E \ \exists i_0 \in \{1..N\} \quad \left\| \frac{(a_1 + \dots + a_{i_0}) - (a_{i_0+1} + \dots + a_N)}{N} \right\| \ge \vartheta$$

This notion of J-convexity is is also due to James. We now consider the following statement statements :

(**A3**, **convex-reflexivity of J-convex Banach spaces**). *The closed unit ball (and every closed bounded convex subset) of a J-convex Banach space is compact in the convex topology.*

(**A2**, **convex-reflexivity of uniformly convex Banach spaces**). *The closed unit ball (and every closed bounded convex subset) of a uniformly convex Banach space is compact in the convex topology.*

In a Hilbert space, every closed convex subset is weakly closed so the convex topology and the weak topology are equal in a Hilbert space. In conclusion,

Corollary 3. **DC** $\Rightarrow$ **J2C** $\Rightarrow$ **A3** $\Rightarrow$ **A2** $\Rightarrow$ **RCuc** $\Rightarrow$ **AH** $\Rightarrow$ $\mathbf{AC^{fin}}(\mathbb{N})$.

Proof. Here, the implication **J2C** $\Rightarrow$ **A**3 is due to James, who proved (in **ZF**) that J-convex spaces are J-reflexive (see [23] or [3]). Notice that a direct proof of **DC** $\Rightarrow$ **A2** was previously obtained in [P99b]. ☐

6.3. **A consequence of J2C.** Recall the following axiom **D** : *"Every infinite set is Dedekind-infinite".*

Theorem 25. **J2C** $\Rightarrow$ **D**.

Proof. See [P04b]. ☐

6.4. **Various distinct notions of reflexivity.** Say that a Banach space E is :

- *convex-reflexive* if the closed unit ball Γ_E of E is compact in the convex-topology ;
- *ω-reflexive* if Γ_E is ω-compact ;
- *simply-reflexive* if the canonical mapping $j_E : E \to E''$ is isometric and onto ;
- *onto-reflexive* if the canonical mapping j_E is onto.

In the following diagram, see Figure 2, we sum up some results obtained in [P04b] about reflexivity in **ZF**.

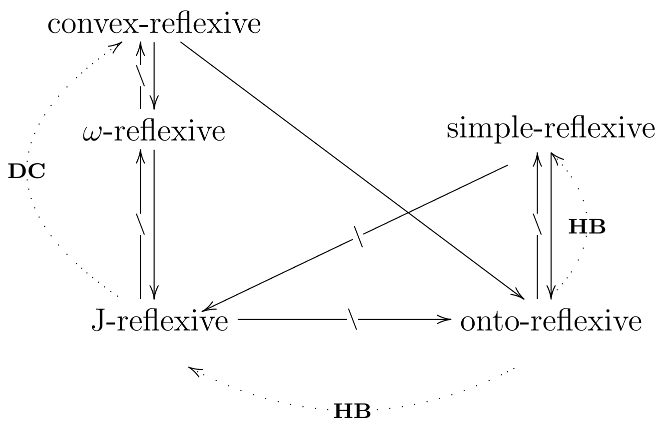

FIGURE 2. Reflexivity in **ZF**.

6.5. **Answer to some question about BPI.** It is well known (see [24], [22]) that **BPI**, being equivalent to the compactness of $\{0,1\}^I$ for every set I, implies numerous compactness results. In particular, the classical proof of Alaoglu's theorem shows that **BPI** implies the following result of functional analysis :

(**RC**, **Reflexive compactness**). *The closed unit ball of every simply-reflexive Banach space is weakly compact.*

We now state that **RC** does not imply **BPI**, solving Question **2.11** of [P99b].

Theorem 26.
(1) *The axiom* **HB** *implies that every onto-reflexive Banach space is J-reflexive.*
(2) (**HB**+**DC**) *implies that every onto-reflexive Banach space is convex-reflexive (whence* (**HB**+**DC**) *implies* **RC***).*
(3) **RC** *does not imply* **BPI***.*

Proof. See [P04b]. □

Question 13 (see [P04b]**).** It follows from our study that, in **ZF**+**DC**+**HB**, convex-reflexivity, ω-reflexivity, simple-reflexivity, J-reflexivity and onto-reflexivity are equivalent. Is there some "classical" notion of reflexivity which is not equivalent to these notions in **ZF**+**DC**+**HB** ?

Part 4. Other works

7. Spanning Graphs

We consider simple undirected and loop-free graphs. A *forest* is a graph with no cycles, a *tree* is a connected forest. A graph G' is a *subgraph* of a graph G if every vertex of G' is a vertex of G and every edge of G' is an edge of G ; such a subgraph is *spanning* if every vertex of G incident to an edge of G is also incident to an edge of G'.

AC implies that every connected graph has a spanning tree, and the converse is easily seen to hold. Indeed, the axiom of choice follows from the fact that every connected graph has a connected spanning sub-graph without cycles of all even lengths (whereas, it is a theorem of **ZF** that every connected graph has connected spanning subgraphs with no odd cycles at all). We show that relaxing the connectedness assumption on the spanning graph still implies the axiom of choice, and in particular **AC** follows from every connected graph admitting a spanning forest. Indeed consider the statement

> **SC**, **Spanning Coppice** : *Every bipartite connected graph has a spanning subgraph omitting some finite complete bipartite graph* $K_{n,n}$.

Theorem 27. **AC** $\Leftrightarrow$ **SC**.

Proof. See [S04a]. □

8. Algorithmics over boolean functions

This Section is not (or does not seem to be) related to **ZF** *or the Axiom of Choice.*

Let X be a finite set. Let n, k be two strictly positive integers. Given some mapping $f : X^n \to X^n$, a finite sequence $(h_j)_{j\in\{0..nk-1\}}$ of nk mappings from X^n to X is a *closed iterative calculus* of f if, for any $a = (a_0, \ldots, a_{n-1})$ in X^n the sequence of assignments

$$\texttt{for j:=-n to -1 do } x_{j[n]} := a_{j[n]} \texttt{ od}$$
$$\texttt{for j:=0 to n k-1 do } x_{j[n]} := h_j(x_{j[n]}, x_{(j+1)[n]}, \ldots, x_{(j+n-1)[n]}) \texttt{ od}$$

ends with $(x_0, \ldots, x_{n-1}) = f(a)$. The integer nk is said to be the *length* of this closed iterative calculus. The previous definition was given by Serge Burckel in "Closed Iterative Calculus", Theoretical Computer Science 158 (1996) 371-378, where he proved that that every "boolean" mapping $f : \{0,1\}^n \to \{0,1\}^n$ has a closed iterative calculus. In [P00], we proved that boolean mappings can be computed using only three sorts of assignments (cycles, transpositions and collapsings). In [P03], we proved that for every integer $n \geq 1$, every mapping $f : \{0,1\}^n \to \{0,1\}^n$ has a closed iterative calculus in $\frac{n(n+1)}{2}$ steps. In [P04a], we proved that for every finite field $\mathbb{K}$, every *linear* mapping $f : \mathbb{K}^n \to \mathbb{K}^n$ has a closed iterative calculus in $2n - 1$ *linear* assignments.

APPENDIX

Appendix A. Spectra

A.1. **The spectrum of a commutative unitary ring.** Given a commutative unitary ring $(A,+,.,0,1)$, we denote by Σ_A the *spectrum of* A, *i.e.* the set of its prime ideals endowed with the *Zariski* topology. Recall that the Zariski topology on Σ_A is the topology generated by subsets of the following form :

$$\Omega_a := \{I \in \Sigma_A : \ a \notin I\}, \quad a \in A$$

Since for every $a, b \in A$, $\Omega_a \cap \Omega_b = \Omega_{a.b}$, open subsets of Σ_A are of the form $\{I \in \Sigma_A : J \not\subseteq I\}$ where J is some ideal of A. Given an element $I \in \Sigma_A$, the adherence $\overline{\{I\}}$ is the set $\{J \in \Sigma_A : I \subseteq J\}$; so closed singletons of Σ_A correspond to maximal ideals of A. We denote by $Max(A)$ the subspace of maximal ideals of A.

The topological space Σ_A is *sober* :

(1) Σ_A is T_0 *i.e.* for every distinct points $\in \Sigma_A$, there exists some open subset of Σ_A containing one of the point and not the other one ;
(2) Every *irreducible* closed subset of Σ_A is of the form $\overline{\{I\}}$ where I is some prime ideal of A.

Using **BPI**, one can prove that the topological space Σ_A is *spectral* (see [27] p.279-281), which means that :

(1) Σ_A is sober ;
(2) Σ_A is quasicompact ;
(3) The set of open quasicompact subsets of Σ_A is closed by finite intersection ;
(4) The set of open quasicompact subsets of Σ_A generate the topology of Σ_A.

Using **BPI**, one can also prove the following statement, due to Höchster (see [27] p. 429-444) :

(**Ho**, **Höchster**). *Every spectral space is homeomorphic with the spectrum of some commutative unitary ring.*

Question 14. Does Höchster's statement **Ho** imply **BPI** ?

Given a spectral topological space X, denote by $\mathcal{C}$ the set of its closed subsets, and by Ω the set of its quasicompact open subsets. Then, the *pro-constructible* topology on the spectral space X is the topology having $\mathcal{C} \cup \Omega$ as a sub-basis of closed subsets : thus, a subset of X is closed for the pro-constructible topology if it is an intersection of subsets of the form $F \cup O$ where $F \in \mathcal{C}$ and $O \in \Omega$. The proof of **BPI** $\Rightarrow$ **Ho** relies on the fact that **BPI** implies the following statement :

(**SHo**). *The pro-constructible topology on a spectral space is (Hausdorff) compact.*

Question 15. Does **SHo** imply **BPI** ?

A.2. **The spectrum of a bounded distributive lattice.** An (abstract) *lattice* is a structure $(L, \vee, \wedge)$ where $\vee, \wedge : L \times L \to L$ are two associative, commutative and idempotent binary laws satisfying the following for every $x, y \in L$ (see [21]) :

$$(x \wedge y) \vee y = y \tag{5}$$

$$(x \vee y) \wedge y = y \tag{6}$$

Given an ordered set $(P, \leq)$ which is *reticulated*, *i.e.* such that every pair $\{x, y\} \subseteq P$ has an upper bound $x \vee y$ and a lower bound $x \wedge y$, then the structure $(P, \vee, \wedge)$ is a lattice. Given a lattice $(L, \vee, \wedge)$, there exists a unique reticulated order $\leq$ on L such that $\vee$ (*resp.* $\wedge$) is the "supremum" (*resp.* "infimum") associated law : namely, for every $x, y \in L$, $(x \leq y) \Leftrightarrow (x \vee y = y)$. We will refer to this order. The lattice $(L, \vee, \wedge)$ is *distributive* if the following properties are satisfied for every $x, y, z \in L$:

$$(x \wedge y) \vee z = (x \wedge z) \vee (y \wedge z) \tag{7}$$

$$(x \vee y) \wedge z = (x \vee z) \wedge (y \vee z) \tag{8}$$

A *bounded lattice* is a structure $(L, \vee, \wedge, 0, 1)$ which is a lattice endowed with two elements 0 and 1 satisfying the extra properties for every $x \in L$:

$$x \vee 0 = x \tag{9}$$

$$x \wedge 1 = x \tag{10}$$

Clearly, a bounded lattice $(L, \vee, \wedge, 0, 1)$ also satisfies $x \wedge 0 = 0$ and $x \vee 1 = 1$ for every $x \in L$. Given a lattice $(L, \vee, \wedge)$, say that a non-empty subset I of L is an *ideal* of $(L, \vee, \wedge)$ if I satisfies the following properties for every $x, y \in L$:

$$((y \in I) \text{ and } (x \leq y)) \Rightarrow (x \in I) \tag{11}$$

$$((x \in I) \text{ and } (y \in I)) \Rightarrow (x \vee y \in I) \tag{12}$$

Say that an ideal I of the lattice L is *prime* if I satisfies the following extra property for every $x, y \in L$:

$$((x \wedge y) \in I) \Rightarrow ((x \in I) \text{ or } (y \in I)) \tag{13}$$

Say that a non-empty subset F of L is a *filter (*resp. *prime filter)* of the lattice $(L, \vee, \wedge)$ if F is an ideal (*resp.* prime ideal) of the lattice $(L, \wedge, \vee)$. Clearly, a subset I of a lattice $(L, \vee, \wedge)$ is a prime ideal of L if and only if $L \backslash I$ is a prime filter of $(L, \vee, \wedge)$.

Given a bounded distributive lattice $(L, \vee, \wedge, 0, 1)$, we denote by Σ_L the set of prime ideals of L endowed with the topology generated by subsets of the following form :

$$\Omega_a := \{I \in \Sigma_L : \ a \notin I\} \qquad a \in L$$

Since for every $a, b \in L$, $\Omega_a \cap \Omega_b = \Omega_{a \wedge b}$, open subsets of Σ_L are of the form $\{I \in \Sigma_L : J \not\subseteq I\}$ where J is some ideal of L. Given an element $I \in \Sigma_L$, the adherence $\overline{\{I\}}$ is the set $\{J \in \Sigma_L : I \subseteq J\}$; so closed singletons of Σ_L correspond to maximal prime ideals of L. If L is a distributive bounded lattice, every maximal proper ideal of L is prime ; moreover, the topological space Σ_L is sober, and, using **BPI**, the topological space Σ_L is spectral.

A.3. **Boolean algebras.** Recall that a *boolean algebra* is a (commutative) unitary ring $(\mathbb{B}, +, \times, 0, 1)$ satisfying $x^2 = 1$ for every $x \in \mathbb{B}$. Equivalently, a boolean algebra is a bounded distributive lattice $(\mathbb{B}, \vee, \wedge, 0, 1, *)$ endowed with a 1-ary law $* : \mathbb{B} \to \mathbb{B}$ satisfying the following properties for every $x \in \mathbb{B}$:

$$x \vee x^* = 1 \tag{14}$$

$$x \wedge x^* = 0 \tag{15}$$

Given a set X, the structure $(\mathcal{P}(X), \Delta, \cap, \varnothing, X)$ is a boolean algebra (corresponding to the lattice $(\mathcal{P}(X), \cap, \cup, \varnothing, X)$). More generally, given a topological space X, the set of its clopen subsets (*i.e.* subsets which are both open and closed) is a boolean sub-algebra of $\mathcal{P}(X)$. Given a boolean algebra $\mathbb{B}$, the spectrum of the commutative unitary ring $\mathbb{B}$ and the spectrum of the lattice $\mathbb{B}$ are the same space. This spectrum is Hausdorff and scattered (clopen subsets form a basis of open sets). Moreover, if every ideal of $\mathbb{B}$ is contained in a maximal ideal, then this spectrum $\Sigma_{\mathbb{B}}$ is compact. In particular, in **ZF+BPI**, the spectrum of a Boolean algebra is compact. In **ZF+BPI**, the spectrum of a complete Boolean algebra is Hausdorff compact and extremally disconnected (the closure of every open subset is open).

A.4. **Maximal ideals and AC.** Hodges (see [20]) showed that **AC** is equivalent to the following statement :

> **MITR**, **Maximal Ideal Theorem for Rings** *Every non-null commutative unitary ring has a maximal ideal.*

Wel now give another proof of **MITR** $\Rightarrow$ **AC** relying on Höchster's statement and the fact that **AC** is equivalent to the following statement (see [4]) :

> **MITL**, **Maximal Ideal Theorem for bounded distributive Lattices** *Every non-null bounded distributive lattice has a maximal ideal.*

.

Proposition 2. MITR $\Rightarrow$ **MITL**.

Proof. First notice that **MITL** implies **BPI**, since boolean algebras are commutative rings. Given a non-null distributive lattice $(L, \vee, \wedge, 0, 1)$, consider the spectrum Σ_L of L. Then, using **BPI**, Σ_L is spectral ; thus, using Höchster's statement, which follow from **BPI**, Σ_L is homeomorphic with the spectrum of a commutative unitary ring A. Now **MITR** implies a closed singleton in the topological space $\Sigma_R = \Sigma_L$; this implies a maximal ideal in the lattice L. $\square$

Appendix B. Ordered vector spaces

B.1. **The positive cone of an ordered vector space.** An *ordered vector space* is a real vector space E endowed with a partial order $\leq$ satisfying the following property for every $x, y \in E$ and every $\lambda \in \mathbb{R}$:

$$\big((x \geq 0) \text{ and } (y \geq 0)\big) \Rightarrow (x + y \geq 0) \tag{16}$$

$$\big((\lambda \geq 0) \text{ and } (x \geq 0)\big) \Rightarrow (\lambda.x \geq 0) \tag{17}$$

The *positive part* of an ordered vector space $(E, \leq)$ is the set $E_+ := \{x \in E : x \geq 0\}$. The positive part E_+ is a *cone* (if $x \in E_+$ and $\lambda \in \mathbb{R}_+^*$ then $\lambda x \in E_+$), E_+ is convex, E_+ is "saillant" ($E_+ \cap (-E_+) = \{0\}$), E_+ is "pointé" ($0 \in E_+$). Conversely, given a real vector space E and some convex cone P of E which is "saillant" and "pointé", the binary relation $\leq$ on E defined by the property $x \leq y \Leftrightarrow (y - x \in P)$ is a partial order on E satisfying (16) and (B.1). Given an ordered vector space $(E, \leq)$, for every $x, y \in E$ we denote by $[[x, y]]$ the interval $\{z \in E : \; x \leq z \text{ and } z \leq y\}$.

B.2. **Bounded forms on an ordered vector space.** Given a vector space E, we denote by E^* its algebraic dual. If E is a *real* vector space, then E^* endowed with the order induced by the product order on $\mathbb{R}^E$ is an ordered vector space. Given an ordered vector space $(E, \leq)$, a linear mapping $f : E \to \mathbb{R}$ is said to be *bounded* if for every $x, y \in E$, the set $f([[x, y]])$ is bounded in $\mathbb{R}$. The set of bounded real linear mappings on E is a vector subspace of E^* that we denote by $B(E)$. We endow $B(E)$ with order induced by the order of E^* : then, $E^*_+ \subseteq B(E)$.

B.3. **Riesz spaces.** A *Riesz space* is an ordered vector space $(E, \leq)$ such that the order $\leq$ is reticulated. Given a Riesz space E, for any $x \in E$, we denote by x^+ the *positive part* $x \vee 0$ of x, by x^- the *negative part* $(-x) \vee 0$ of x, and by $|x|$ the *absolute part* $x \vee (-x)$ of x. Say that x and y are *disjoint* if $|x| \wedge |y| = 0$, and denote it by $x \perp y$.

Proposition ([31])**.** *A Riesz space E satisfies the following first-order properties for every $x, y, z \in E$ and every $\lambda \in \mathbb{R}_+$:*

(1) $x + y = x \vee y + x \wedge y$
(2) $x \vee y = -\big((-x) \wedge (-y)\big)$
(3) $x \vee y + z = (x + z) \vee (y + z)$
(4) $x \wedge y + z = (x + z) \wedge (y + z)$
(5) $x = x^+ - x^-$
(6) $|x| = x^+ + x^-$
(7) $|\lambda.x| = |\lambda||x|$
(8) $|x + y| \leq |x| + |y|$
(9) $x^+ \perp x^-$ *and the decomposition of x into the difference of two disjoint positive elements is unique.*
(10) $(x \leq y) \Leftrightarrow \big((x^+ \leq y^+)$ *and* $(y^- \leq x^-)\big)$
(11) $x \perp y \Leftrightarrow |x| \vee |y| = |x| + |y|$, *and in this case,* $|x + y| = |x| + |y|$
(12) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ *and* $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$
(13) *If $x, y, z \in E_+$, then $(x + y) \wedge z \leq x \wedge z + y \wedge z$ (whence $[[0, x + y]] = [[0, x]] + [[0, y]]$)*
(14) $|x - y| = |x \vee z - y \vee z| + |x \wedge z - y \wedge z|$

B.4. **Weakly Riesz spaces.** An ordered vector space $(E, \leq)$ is *weakly Riesz* if

(1) $E = E_+ - E_+$
(2) $\forall x \in E \; \forall y \in E \;\; [[0, x]] + [[0, y]] = [[0, x + y]]$

So every Riesz space is weakly Riesz.

Theorem ([14]). *Given a weakly Riesz space E, the ordered vector space $B(E)$ is conditionally complete. Moreover, for every $u \in B(E)$, for every $x \in E_+$,*

$$u^+(x) = \sup\{u(t) : t \in [[0, x]]\}$$

$$u^-(x) = \sup\{u(t) : t \in [[-x, 0]]\}$$

B.5. **Order units.** Given an ordered vector space E, say that an element $e \in E_+$ is a *order unit* if for every $x \in E$, there exist real numbers r, s satisfying $r.e \leq x \leq s.e$.

Proposition 3. *Assume that E is an ordered vector space with a unit order e.*

1. *The mapping $\|.\| : B(E) \to \mathbb{R}$ associating to every $u \in B(E)$ the real number $\sup\{|u(x)| \ : x \in [[-e, e]]\}$ is a norm on $B(E)$.*
2. *For every $u \in B(E)$, $u \geq 0 \Leftrightarrow \|u\| = u(e)$*

Moreover, if E is weakly Riesz, for every $u \in B(E)$, (u^+, u^-) is the unique element of $B_+(E) \times B_+(E)$ satisfying $u = u^+ - u^-$ and $\|u\| = \|u^+\| + \|u^-\|$; in particular, $\|u\| = \||u|\|$.

Proof. Easy. See [P86, Théorèmes 2 and 3]. □

Theorem 28. *Assume that E is a weakly Riesz space with a unit order e. Denote by Γ the closed unit ball of the normed space $B(E)$, by K the convex set $B(E) \cap E_+$, and by U the convex set*

$$\{(u, v) \in B_+(E) \times B_+(E) : \ u(e) + v(e) = 1\}$$

If $B(E)$ is non-null, then for every extreme point $u \in \Gamma$,

1. *(u^+, u^-) is an extreme point of U ;*
2. *u^+, u^- are extreme points of K ;*
3. *$u^+ = 0$ or $u^- = 0$, whence $u(e) = \pm 1$.*

Proof. See [P86, Théorème 4]. □

Appendix C. Banach algebras

C.1. **Normed algebras.** In this Section, $\mathbb{K}$ is the field $\mathbb{R}$ or $\mathbb{C}$. A *normed algebra* is a $\mathbb{K}$-algebra $(A, +, \times, \lambda., 0, 1)$ endowed with a norm $\|.\|$ satisfying the following property for every $x, y \in A$:

$$\|x \times y\| \leq \|x\| \, \|y\| \tag{18}$$

A *Banach algebra* is a normed algebra which is complete as a normed space. Given a Banach space E, the $\mathbb{K}$-algebra $\big(LC(E), +, \circ, \lambda., 0, Id_E\big)$ of continuous linear mappings $u : E \to E$ endowed with the "sup" norm is a (non commutative) unitary Banach algebra.

C.2. **Spectrum of an element.** Given a unitary complex Banach algebra A, for every $a \in A$, the *spectrum* of A is the set $Sp(a) := \{\lambda \in \mathbb{C} : \lambda - a \text{ is not invertible } inA\}$. Given a unitary *real* Banach algebra A, for every $a \in A$, the *spectrum* of A is the set $Sp(a) := \{\lambda \in \mathbb{C} : \lambda - a \text{ is not invertible in } A_{\mathbb{C}}\}$, where $A_{\mathbb{C}}$ is the complexified algebra of A.

Theorem ([35], [33])**.** *Given a unitary Banach algebra A, for every $a \in A$, the spectrum of a is a non-empty closed bounded subset of $\mathbb{C}$. Moreover,* $\sup\{|\lambda| : \lambda \in Sp(a)\} = sup\{\sqrt[n]{\|a^n\|}\} \leq \|a\|$.

Proposition 4. *Given a real, commutative, unitary Banach algebra A, the following sentences are equivalent :*

(1) *for every $a \in A$, $1 + a^2$ is invertible ;*
(2) *for every $a \in A$, $Sp(a) \subseteq \mathbb{R}$.*

If A satisfies these two sentences, then $A_+ := \{a \in A : Sp(a) \subseteq \mathbb{R}_+\} = \{a^2 : a \in A\}$ is a convex cone which is "saillant" and "pointé", and the associated ordered vector space is a Riesz space.

C.3. **Gelfand algebras.**

Proposition 5. *Let A be a Gelfand algebra A such that A' is non-null. Denote by K the convex set $\Gamma_{A'} \cap A'_+$, where $\Gamma_{A'}$ is the closed unit ball of A'. For every $\phi \in A'$, the following properties are equivalent :*

(1) *ϕ is an extreme point of $\Gamma_{A'}$ and $\phi(1) \geq 0$;*
(2) *ϕ is an extreme point of $\Gamma_{A'}$ and $\phi(1) = 1$;*
(3) *$\phi : A \to \mathbb{R}$ is a morphism of algebras ;*
(4) *ϕ is an extreme point of K and $\phi(1) = 1$.*

Proof. $1 \Rightarrow 2$: see [P87, Théorème 4]. $2 \Rightarrow 3$: see [P87, Théorème 7].
$3 \Rightarrow 4$ and $4 \Rightarrow 1$ are staightforward, using the fact that for $u \in A'_+$, $\|u\| = u(1)$. ☐

Appendix D. Some classical Banach spaces

Recall that given a set I, $\ell^0(I)$ is the following normed vector space endowed by the "sup" norm :

$$\ell^0(I) := \{(x_i)_{i\in I} : \forall \varepsilon > 0\ \exists F \in \mathcal{P}_f(I)\ \forall i \in I\backslash F\ |x_i| \leq \varepsilon\}$$

The continuous dual of $\ell^0(I)$ is (isometrically isomorphic with) the following normed vector space endowed with the "sum" norm :

$$\ell^1(I) := \{(x_i)_{i\in I} : \sum_{i\in I} |x_i| < +\infty\}$$

Moreover, the continuous dual of $\ell^1(I)$ is the following space endowed with the "sup" norm :

$$\ell^\infty(I) := \{(x_i)_{i\in I} : \sup_{i\in I} |x_i| < +\infty\}$$

For every $p \in]1, +\infty[$, the vector space

$$\ell^p(I) := \{(x_i)_{i\in I} : \sum_{i\in I} |x_i|^p < +\infty\}$$

is endowed with the N_p norm : $N_p((x_i)_{i\in I}) = \left(\sum_{i\in I}|x_i|^p\right)^{1/p}$.

Remark 2. Denoting by $\mathbb{R}^{(I)}$ the vector space of mapping $f \in \mathbb{R}^I$ such that the set $\{i \in I : f(i) \neq 0)\}$ is finite, the Banach space $\ell^0(I)$ is the completion of the normed space $\mathbb{R}^{(I)}$ endowed with the "sup" norm and, for every $p \in [1, +\infty[$, the Banach space $\ell^p(I)$ is the completion of the normed space $\mathbb{R}^{(I)}$ endowed with the N_p norm.

Appendix E. Some geometric properties of Banach spaces

E.1. **Uniform convexity.** Given a normed space $(E, \|.\|)$, the mapping

$$\delta_E : \varepsilon \mapsto \inf\{1 - \left\|\frac{x+y}{2}\right\| : x \in \Gamma_E,\ y \in \Gamma_E,\ \|x-y\| \geq \varepsilon\}$$

is called the *modulus of convexity* of E. Notice that $\delta_E :]0,2] \to \mathbb{R}_+$ is continuous and order-preserving. The space E is *uniformly convex* if $\forall \varepsilon > 0,\ \delta_E(\varepsilon) > 0$. In other words, E is uniformly convex if there exists a mapping $\delta : \mathbb{R}_+^* \to \mathbb{R}_+^*$ such that, for every real number $\varepsilon > 0$, for every $x, y \in \Gamma_E$,

$$\|x-y\| \geq \varepsilon \Rightarrow \left\|\frac{x+y}{2}\right\| \leq 1 - \delta(\varepsilon) \tag{19}$$

Any mapping $\delta :]0,2] \to \mathbb{R}_+^*$ satisfying (19) is called a *witness of uniform convexity for* E, and then, δ_E is the best witness of uniform convexity.

Remark 3. Every Hilbert space is uniformly convex, with modulus of convexity $\varepsilon \mapsto 1 - \sqrt{1-\frac{\varepsilon^2}{4}}$.

Proof. See [3, p. 189-190]. □

Remark 4. Uniformly convex Banach spaces satisfy the "projection" property, and, for these spaces, the projection is unique.

Proof. See [3, p. 194-195]. □

E.2. **Various notions of differentiability of a norm.** In this Section, we recall the following notions of differentiability for a norm $\|.\|$ on a vector space E : smoothness, Gâteaux differentiability, Fréchet differentiability and uniform smoothness (these notions are stated from the weakest to the strongest).

E.2.1. *Smoothness of the norm.* A continuous linear mapping $f \in S_{E'}$ is said to be *norming* at a point $a \in E\backslash\{0\}$ if and only if $f(a) = \|a\|$. The normed space $(E, \|.\|)$ is said to be *smooth at point* $a \in E\backslash\{0\}$ if there *exists* a *unique* norming mapping at this point a. The space $(E, \|.\|)$ is said to be *smooth* (see [3, p. 177]) if it is smooth at every point $a \in E\backslash\{0\}$.

Notice that, for every $a \in E\backslash\{0\}$, the existence of a norming linear mapping at point a is provable in **ZF+HB** ; so in **ZF+HB**, smoothness at a point is equivalent to the *uniqueness* of a norming linear mapping at this point.

E.2.2. *Gâteaux differentiability of the norm.* Here are some classic facts about *Gâteaux differentiability* for a norm (see [3, p. 178-179]). For every $a \in E\backslash\{0\}$ and every $h \in E$, the convexity of the norm implies that the function $\tau_a^h : t \mapsto \frac{\|a+th\|-\|a\|}{t}$ is non-decreasing, thus it has a limit when $t \to 0^-$ (*resp.* when $t \to 0^+$) and the following inequality holds :

$$\lim_{t \to 0^-} \frac{\|a+th\|-\|a\|}{t} \leq \lim_{t \to 0^+} \frac{\|a+th\|-\|a\|}{t} \tag{20}$$

Let $G^-(a,h) := \lim_{t\to 0^-} \frac{\|a+th\|-\|a\|}{t}$ and $G^+(a,h) := \lim_{t\to 0^+} \frac{\|a+th\|-\|a\|}{t}$. The sublinearity of the norm implies that the mapping $G^+(a,.)$ is sublinear and satisfies

$$\forall h \in E,\ G^+(a,h) \leq \|h\| \tag{21}$$

Moreover, since $\|x\| = \|-x\|$ holds for every $x \in E$,

$$\forall h \in E,\ \ G^+(a,-h) = -G^-(a,h) \tag{22}$$

Statement (22) implies that the mapping $G^-(a,.)$ is *superlinear*, *i.e.* the two following conditions are satisfied :

$$\forall h_1, h_2 \in E,\ \ G^-(a, h_1+h_2) \geq G^-(a,h_1) + G^-(a,h_2)$$

$$\forall \lambda \in \mathbb{R}_+\ \forall x \in E,\ \ G^-(a,\lambda x) = \lambda G^-(a,x)$$

Now, the normed space $(E, \|.\|)$ is *Gâteaux differentiable* at a point $a \in E\backslash\{0\}$ if, for every $h \in E$, $\lim_{t\to 0, t\neq 0} \frac{\|a+th\|-\|a\|}{t}$ exists in $\mathbb{R}$, *i.e.* $G^+(a,h) = G^-(a,h)$; in this case, for every $h \in E$, we denote by $G(a,h)$ the real number $\lim_{t\to 0, t\neq 0} \frac{\|a+th\|-\|a\|}{t}$; then the mapping $G(a,.)$ is linear (because $G^+(a,.)$ is sublinear and $G^-(a,.)$ is superlinear), it is continuous with norm ≤ 1 (because of (21)), and in fact $\|G(a,.)\| = 1$ (because $G(a,a) = \|a\|$). The normed space $(E, \|.\|)$ is *Gâteaux differentiable* if its norm is Gâteaux differentiable at every point $a \in E\backslash\{0\}$.

Remark 5. In **ZF**, Gâteaux differentiability at a given point implies smoothness of the norm at this point. But the converse statement *"Every normed space which is smooth at a point is Gâteaux differentiable at this point."* is equivalent to **HB**.

Proof. See [P01, Proposition 2 p. 438]. □

E.2.3. *Fréchet differentiability.* Given a normed space $(E, \|.\|)$, the norm $\|.\|$ is said to be *Fréchet differentiable* at a point $a \in E\backslash\{0\}$ if it is Gâteaux differentiable and

$$\lim_{t\to 0,\ t\neq 0} \frac{\|a+th\|-\|a\|}{t} \text{ is uniform in } h \in S_E \tag{23}$$

Since the function τ_a^h is non-decreasing on $\mathbb{R}_+$, the norm $\|.\|$ is Fréchet differentiable at point a if and only if

$$\lim_{t\to 0^+, t\neq 0} \left(\frac{\|a+th\|-\|a\|}{t} - \frac{\|a-th\|-\|a\|}{-t}\right) = 0 \text{ , uniformly in } h \in S_E \tag{24}$$

The normed space $(E, \|.\|)$ is said to be *Fréchet differentiable* if its norm is Fréchet differentiable at every point $a \in E\backslash\{0\}$.

E.2.4. *Uniform smoothness.* Let $(E, \|.\|)$ be a normed space such that $E \neq \{0\}$. The normed space $(E, \|.\|)$ is said to be *uniformly Fréchet differentiable* (see [18] Definition 1.9 p.8) if (25)

$$\lim_{t \to 0,\, t \neq 0} \frac{\|a + th\| - \|a\|}{t} \text{ exists for each } a \in S_E \text{ and each } h \in S_E, \text{ and is uniform in } (a, h) \in S_E \times S_E$$

Notice that $(E, \|.\|)$ is uniformly Fréchet differentiable if and only if

$$\lim_{t \to 0^+, t \neq 0} \left(\frac{\|a + th\| - \|a\|}{t} - \frac{\|a - th\| - \|a\|}{-t} \right) = 0 \text{ , uniformly in } a, h \in S_E \tag{26}$$

Now, consider the following function ρ_E, which is called the *modulus of smoothness of* the normed space E (see [3] page 204) :

$$\rho_E \ : t \to \sup_{\|a\| = \|b\| = 1} \left\{ \frac{\|a + tb\| + \|a - tb\|}{2} - 1 \right\}$$

The space E is said to be *uniformly smooth* if

$$\lim_{t \to 0^+} \frac{\rho_E(t)}{t} = 0 \tag{27}$$

This is equivalent to the following condition :

$$\lim_{h \to 0,\, h \neq 0} \frac{\|a + h\| + \|a - h\| - 2\,\|a\|}{\|h\|} = 0 \text{ , uniformly in } a \in S_E \tag{28}$$

Since (26) and (28) are equivalent, uniform Fréchet differentiability and uniform smoothness are equivalent.

It is easy to prove that every finite-dimensional normed space E which is Gâteaux differentiable is uniformly smooth (because Γ_E is compact), but in general, Gâteaux differentiability, Fréchet differentiability and uniform smoothness are three distinct notions.

E.2.5. *Šmulian tests.* A mapping η from $\mathbb{R}_+$ to $\mathbb{R}_+$ is said to be a *Šmulian test* of uniform smoothness for a normed space $(E, \|.\|)$ if and only if, for every $\varepsilon \in \mathbb{R}_+$, and for every $f, g \in S_{E'}$:

$$[\exists a \in S_E,\ \big(f(a) > 1 - \eta(\varepsilon) \text{ and } g(a) > 1 - \eta(\varepsilon)\big)] \Rightarrow \|f - g\| < \varepsilon$$

Proposition 6. *Let $(E, \|.\|)$ be a uniformly smooth normed space with modulus of smoothness ρ, and let $\delta : \mathbb{R}_+ \to \mathbb{R}_+$ be the mapping $\varepsilon \mapsto \sup\{t \in]0, 1] \ : \frac{\rho(t)}{t} \leq \frac{\varepsilon}{4}\}$. Then the mapping $\eta : \varepsilon \mapsto \frac{\varepsilon\delta(\varepsilon)}{4}$ is a Šmulian test of uniform smoothness for E.*

Proof. Follow the idea in the proof of [18, Theorem 1.4 page 3], but, in order to work in **ZF**, avoid the use of sequences (see [P01, Proposition 1 p. 430]). □

Remark 6. The following statement is provable in **ZF** :

> *Let E be a normed space which has a Šmulian test of uniform smoothness. If for every $a \in S_E$, for every real number $\delta \in]0, 1[$, there exists $f \in S_{E'}$ satisfying $f(a) \geq \delta$, then E is uniformly smooth.*

Proof. Adapt the proof of [18, Theorem 1.4 p.3-4], avoiding the use of sequences (see [P01, Remark 6 p. 431]). □

Remark 7. It follows from Remark 6 that in **ZF**, every Gâteaux differentiable normed space which has a Šmulian test is uniformly smooth. Since every uniformly smooth normed space has a Šmulian witness (see Proposition 6) and since uniform smoothness implies Gâteaux differentiability, the two following properties are equivalent : "E is uniformly smooth.", "E is Gâteaux differentiable and E admits a Šmulyan witness."

Using Remark 6, the following statement is provable in **ZF+HB** :

> *Every normed space which has a Šmulian test of uniform smoothness is uniformly smooth.*

Notice that this last converse statement is not provable in **ZF** (see Remark 8).

Remark 8. Given a model **ZF+¬HB**, there exists in this model an infinite dimensional normed space E such that $E' = \{0\}$ (see [P98, Lemma 5 p. 12] or [29, Theorem 2]). Such a space E is not Gâteaux differentiable though any mapping $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ is a Šmulian test of uniform smoothness for E.

E.2.6. *Uniform Gâteaux differentiability.* Recall that uniform smoothness of a normed space $(E, \|.\|)$ means that the following limit exists for each $a, h \in S_E$, and is uniform in $(a, h) \in S_E \times S_E$:

$$\lim_{t \to 0,\, t \neq 0} \frac{\|a + th\| - \|a\|}{t}$$

Now, *uniform Gâteaux differentiability* of E (see [18, Definition 6.5 p.63]), is the existence for every $h \in E \backslash \{0\}$, of $\lim_{t \to 0,\, t \neq 0} \frac{\|a+th\|-\|a\|}{t}$ uniformly in $a \in S_E$.

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