# COUNTABLE CHOICE AND COMPACTNESS

MARIANNE MORILLON

Abstract. We work in set-theory without choice **ZF**. Denoting by **AC**($\mathbb{N}$) the countable axiom of choice, we show in **ZF**+**AC**($\mathbb{N}$) that the closed unit ball of a uniformly convex Banach space is compact in the convex topology (an alternative to the weak topology in **ZF**). We prove that this ball is (closely) convex-compact in the convex topology. Given a set $I$, a real number $p \geq 1$ (*resp.* $p = 0$), and some closed subset $F$ of $[0,1]^I$ which is a bounded subset of $\ell^p(I)$, we show that **AC**($\mathbb{N}$) (*resp.* **DC**, the axiom of Dependent Choices) implies the compactness of $F$.

EQUIPE RÉUNIONNAISE DE MATHÉMATIQUES ET INFORMATIQUE THÉORIQUE (ERMIT)

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## 1. Introduction

1.1. **Presentation of the results.** We work in **ZF**, Zermelo-Fraenkel set-theory without the Axiom of Choice (for short **AC**). Consider the *countable Axiom of Choice*, which is not provable in **ZF**, and which does not imply **AC**:

> **AC**($\mathbb{N}$): *If $(A_n)_{n\in\mathbb{N}}$ is a family of non-empty sets, then there exists a mapping $f : \mathbb{N} \to \cup_{n\in\mathbb{N}} A_n$ associating to every $k \in \mathbb{N}$ an element $f(k) \in A_k$.*

In this paper, we first provide in **ZF**+**AC**($\mathbb{N}$) a criterion of compactness for topological spaces coarser than some complete metric space, having a sub-basis of closed sets satisfying "good" properties with respect to the distance (see Theorem 1 in Section 2.3.4). We then consider an alternative topology for the weak topology on a normed space, namely the *convex topology* (which is the weak topology in **ZF**+**HB**), and we provide some properties of this convex topology: in particular, using the (choiceless) Lusternik-Schnirelmann theorem, we show (see Theorem 2 in Section 3.3) that the closure of the unit sphere of $E$ for the convex topology is the closed unit ball of $E$. Applying our criterion of compactness to the convex topology, we obtain some new results. First, in **ZF**+**AC**($\mathbb{N}$), "The closed unit ball of a uniformly convex Banach space is compact in the convex topology." (see Theorem 3 in Section 4.2): this extends a result obtained by Fremlin for Hilbert spaces (see [6, Chapter 56, Section 566P]) and this solves a question raised in [3, Question 2]. We then prove in **ZF** that "The closed unit ball of a uniformly convex Banach space is (closely) *convex-compact* in the convex topology." (see Theorem 4 in Section 4.3). Given a set $I$, we apply our results to closed subsets of $[0,1]^I$. In Section 5, we show that $\mathbf{AC}(\mathbb{N})$ implies the compactness of closed subsets of $[0,1]^I$ which are bounded subsets of some $\ell^p(I)$, $1 \leq p < +\infty$. In Section 6, we prove that the Axiom of *Dependent Choices* **DC** implies that every closed subset of $[0,1]^I$ which is contained in $\ell^0(I)$ is compact.

1.2. **Some weak forms of AC.** We now review some weak forms of the Axiom of Choice which will be used in this paper and the known links between them. For detailed references and much information on this subject, see [7].

1.2.1. **DC** *and* $\mathbf{AC}(\mathbb{N}, \mathbf{fin})$. The axiom of *Dependent Choices* asserts that:

> **DC**: Given a non-empty set $X$ and a binary relation $R$ on $X$ such that $\forall x \in X \exists y \in X \; xRy$, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of $X$ such that for every $n \in \mathbb{N}$, $x_n R x_{n+1}$.

The *countable Axiom of Choice for finite sets* says that:

> $\mathbf{AC}(\mathbb{N}, \mathbf{fin})$: *If $(A_n)_{n\in\mathbb{N}}$ is a family of finite non-empty sets, then there exists a mapping $f : \mathbb{N} \to \cup_{n\in\mathbb{N}} A_n$ associating to every $n \in \mathbb{N}$ an element $f(n) \in A_n$.*

Of course, $\mathbf{AC} \Rightarrow \mathbf{DC} \Rightarrow \mathbf{AC}(\mathbb{N}) \Rightarrow \mathbf{AC}(\mathbb{N}, \mathbf{fin})$. However, the converse statements are not provable in **ZF**, and $\mathbf{AC}(\mathbb{N}, \mathbf{fin})$ is not provable in **ZF** (see references in [7]).

1.2.2. **BPI** *and* **HB**. The *Boolean Prime Ideal* axiom says that:

> **BPI**: *Every non trivial boolean algebra has a prime ideal.*

It is known that **BPI** is not provable in **ZF** and that **BPI** does not imply **AC**. The following well-known statements of functional analysis are equivalent to the axiom **BPI** (see for example [13]): the *Tychonov theorem* for product of compact Hausdorff spaces, the *Alaoglu theorem*, the fact that for every set $I$ the product space $[0,1]^I$ is compact.

*Remark* 1. If a set $I$ is well-orderable, then the product topological space $[0,1]^I$ is compact in **ZF**.

1.2.3. *Hahn-Banach.* Given a (real) vector space $E$, a mapping $p : E \to \mathbb{R}$ is *sub-linear* if for every $x, y \in E$, and every $\lambda \in \mathbb{R}_+$, $p(x+y) \leq p(x) + p(y)$ (sub-additivity), and $p(\lambda.x) = \lambda p(x)$ (positive homogeneity). Consider the *"Hahn-Banach"* axiom, a well known consequence of **AC** which is not provable in **ZF**:

> **HB**: *Let $E$ be a (real) vector space. If $p : E \to \mathbb{R}$ is a sub-linear mapping, if $F$ is a vector subspace of $E$, if $f : F \to \mathbb{R}$ is a linear mapping such that $f \leq p_{\restriction F}$, then there exists a linear mapping $g : E \to \mathbb{R}$ extending $f$ such that $g \leq p$.*

Given a (real) topological vector space $E$ (*i.e.* $E$ is a vector space such that the "sum" $+ : E \times E \to E$ and the external multiplicative law $. : \mathbb{R} \times E \to E$ are continuous for the product topology), say that $E$ satisfies the *Continuous Hahn-Banach property* (for short *CHB* property) if *"For every* continuous *sub-linear mapping $p : E \to \mathbb{R}$, for every vector subspace $F$ of $E$, if $f : F \to \mathbb{R}$ is a linear mapping such that $f \leq p_{\restriction F}$, then there exists a linear mapping $g : E \to \mathbb{R}$ extending $f$ such that $g \leq p$."* The statement **HB** is not provable in **ZF**, however, some real normed spaces satisfy (in **ZF**) the *CHB* property: for example normed spaces with a well-orderable dense subset (in particular separable normed spaces), but also Hilbert spaces, spaces $\ell^0(I)$ (see [5]), uniformly convex Banach spaces with a Gâteaux-differentiable norm ([4]), uniformly smooth Banach spaces (see [1]).

It is rather easy to prove that **BPI** implies **HB** and that **HB** is equivalent to most of its classical geometrical forms (see [4, Section 6]). It is also easy to see that **AC** $\Rightarrow$ (**BPI**+**DC**) $\Rightarrow$ **BPI** $\Rightarrow$ **HB**. The converse statements are not provable in **ZF** and **HB** is not provable in **ZF**+**DC** (see [7]).

*Remark* 2. There exist various (**ZFC**-equivalent) definitions of reflexivity for Banach spaces: most of them are equivalent in **ZF**+**HB**+**DC** (see [11]), including James' sup theorem (see [12]).

## 2. A criterion of compactness

### 2.1. Filters.

2.1.1. *Filters in lattices of sets.* Given a set $X$, a *lattice* of subsets of $X$ is a subset $\mathcal{L}$ of $\mathcal{P}(X)$ containing $\varnothing$ and $X$, which is closed by finite intersections and finite unions. A *filter* of the lattice $\mathcal{L}$ is a non-empty proper subset $\mathcal{F}$ of $\mathcal{L}$ such that for every $A, B \in \mathcal{L}$:

$$(A, B \in \mathcal{F}) \Rightarrow A \cap B \in \mathcal{F} \tag{1}$$

$$(A \in \mathcal{F} \text{ and } A \subseteq B) \Rightarrow B \in \mathcal{F} \tag{2}$$

A subset $\mathcal{A}$ of $\mathcal{L}$ is contained in a filter of $\mathcal{L}$ if and only if $\mathcal{A}$ is *centered* *i.e.* every finite subset of $\mathcal{A}$ has a non-empty intersection; in this case, the intersection of all filters of $\mathcal{L}$ containing $\mathcal{A}$ is called the *filter generated by* $\mathcal{A}$ and we denote it by $fil(\mathcal{A})$.

2.1.2. *Stationary sets.* Given a filter $\mathcal{F}$ of a lattice $\mathcal{L}$ of subsets of a set $X$, an element $S \in \mathcal{L}$ is $\mathcal{F}$*-stationary* if for every $A \in \mathcal{F}$, $A \cap S \neq \varnothing$. The set $\mathcal{S}_{\mathcal{L}}(\mathcal{F})$ (also denoted by $\mathcal{S}(\mathcal{F})$) of $\mathcal{F}$-stationary elements of $\mathcal{L}$ satisfies the following properties:

(i) If $\mathcal{A}$ is a chain of $\mathcal{L}$ and if $\mathcal{A} \subseteq \mathcal{S}(\mathcal{F})$, then $\mathcal{A} \cup \mathcal{F}$ is centered.
(ii) Let $F_1, \dots, F_m \in \mathcal{L}$. If $F_1 \cup \cdots \cup F_m \in \mathcal{S}(\mathcal{F})$, then there exists some $i_0 \in \{1..m\}$ such that $F_{i_0}$ is $\mathcal{F}$-stationary.

2.2. **Complete metric spaces.** Given a metric space $(X, d)$, some point $a \in X$ and real numbers $R, R'$ satisfying $R \leq R'$, we define *large* $d$*-balls* and *large* $d$*-crowns* as follows:

$$B_d(a, R) := \{x \in X : \ d(a, x) \leq R\}$$

$$D_d(a, R, R') := \{x \in X : \ R \leq d(a, x) \leq R'\}$$

Moreover, if $A$ is a subset of $X$, we define the $d$*-diameter* of $A$:

$$\operatorname{diam}_d(A) := \sup\{d(x, y) : x, y \in A\} \in [0, +\infty]$$

In particular, $\operatorname{diam}_d(\varnothing) = 0$. A metric space $(X, d)$ is said to be *complete* if every Cauchy filter of the lattice of closed subsets of $X$ has a non-empty intersection. Here, a set $\mathcal{A}$ of subsets of $X$ is *Cauchy* if for every $\varepsilon > 0$, there exists $A \in \mathcal{A}$ satisfying $\operatorname{diam}_d(A) < \varepsilon$.

## 2.3. **A criterion of compactness in ZF+AC($\mathbb{N}$).**

2.3.1. *Compactness.*

**Definition 1** ($\mathcal{C}$-compactness, closed $\mathcal{C}$-compactness)**.** Given a class $\mathcal{C}$ of subsets of a set $X$, say that a subset $A$ of $X$ is $\mathcal{C}$*-compact* if for every family $(C_i)_{i \in I}$ of $\mathcal{C}$ such that $(C_i \cap A)_{i \in I}$ is centered, $A \cap \cap_{i \in I} C_i$ is non-empty; say that $\mathcal{A}$ is *closely* $\mathcal{C}$-compact if there is a mapping associating to every family $(C_i)_{i \in I}$ of $\mathcal{C}$ such that $(C_i \cap A)_{i \in I}$ is centered, an element of $A \cap \cap_{i \in I} C_i$.

Recall that a topological space $X$ is *compact* if $X$ is $\mathcal{C}$-compact, where $\mathcal{C}$ is the set of closed subsets of $X$. Equivalently, every filter of the lattice of closed sets of $X$ has a non-empty intersection.

2.3.2. *Sub-basis of closed sets.*

**Definition 2** (basis, sub-basis of closed subsets)**.** A set $\mathcal{B}$ of closed subsets of a topological space $X$ is a *basis of closed sets* if every closed set of $X$ is an intersection of elements of $\mathcal{B}$. A set $\mathcal{S}$ of closed subsets of $X$ is a *sub-basis of closed sets* if the set $\mathcal{B}$ of finite unions of elements of $\mathcal{S}$ is a basis of closed sets of $X$.

The following result is easy.

**Proposition 1.** *Let $X$ be a topological space, and $\mathcal{L}$ be a lattice of closed subsets of $X$. If $\mathcal{L}$ is a basis of closed subsets of $X$, and if every filter of $\mathcal{L}$ has a non-empty intersection, then $X$ is compact.*

2.3.3. *Property of smallness.* Given real numbers $a, b$, we denote by $]a, b[$ the open interval $\{x \in \mathbb{R} : \ a < x < b\}$.

**Definition 3** (smallness in thin crowns)**.** Let $(X, d)$ be a metric space and let $a \in X$. Say that a set $\mathcal{C}$ of subsets of $X$ satisfies the *property of $d$-smallness in thin crowns centered at $a$* if for every $R \in \mathbb{R}_+^*$, for every $\varepsilon > 0$ there exists $\eta \in ]0, R[$ such that for every $C \in \mathcal{C}$,

$$C \subseteq D_d(a, R-\eta, R+\eta) \Rightarrow \mathrm{diam}_d(C) < \varepsilon$$

2.3.4. *Criterion of compactness.*

**Theorem 1.** *Let $(X, d)$ be a complete metric space. Let $\mathcal{T}$ be a topology on $X$ which is included in the topology $\mathcal{T}_d$ of $(X, d)$. Let $\mathcal{C}$ be a sub-basis of closed sets of $(X, \mathcal{T})$, which is closed by finite intersection. If $a \in X$, if $\mathcal{C}$ contains all large $d$-balls centered at $a$, and if $\mathcal{C}$ satisfies the property of $d$-smallness in thin crowns centered at $a$, then:*

*(i) In* **ZF**+**AC**$(\mathbb{N})$*, every large $d$-ball with center $a$ is $\mathcal{T}$-compact (and thus, every $d$-bounded $\mathcal{T}$-closed subset of $X$ is $\mathcal{T}$-compact).*
*(ii) In* **ZF***, every large $d$-ball with center $a$ is $\mathcal{C}$-compact (and thus, every $d$-bounded element of $\mathcal{C}$ is closely $\mathcal{C}$-compact).*

*Proof.* Let $\mathcal{L}$ be the sub-lattice of $\mathcal{P}(X)$ generated by $\mathcal{C}$. Let $\rho > 0$ and let $B$ be the large $d$-ball $B_d(a, \rho)$.
(i) Let $\mathcal{F}$ be a filter of $\mathcal{L}$ containing $B$. Let us prove in **ZF**+**AC**$(\mathbb{N})$ that $\cap\mathcal{F}$ is non-empty (using Proposition 1, this will imply that $B$ is $\mathcal{T}$-compact). Let $R := \inf\{r \in \mathbb{R}_+ : \ B_d(a, r) \in \mathcal{S}(\mathcal{F})\}$. The set of balls $\{B_d(a, r) : \ r > R\}$ is a chain of $\mathcal{F}$-stationary sets of $\mathcal{L}$, thus $\mathcal{F} \cup \{B_d(a, r) : \ r > R\}$ generates a filter $\mathcal{G}$ of $\mathcal{L}$ (see Section 2.1.2-(i)). If $R = 0$ then $\cap\mathcal{G} = \{a\}$ (because elements of $\mathcal{F}$ are $\mathcal{T}_d$-closed) thus $a \in \cap\mathcal{G} \subseteq \cap\mathcal{F}$. If $R > 0$, for every $\varepsilon > 0$, there exists some element of $\mathcal{G}$ which is included in the crown $D_d(a, R-\varepsilon, R+\varepsilon)$; with **AC**$(\mathbb{N})$, choose

for every $n \in \mathbb{N}$, a finite subset $\mathcal{Z}_n$ of $\mathcal{C}$ such that $\cup\mathcal{Z}_n \in \mathcal{G}$ and $\cup\mathcal{Z}_n \subseteq D_d(a, R-\frac{1}{n+1}, R+\frac{1}{n+1})$. With $\mathbf{AC}(\mathbb{N}, \mathbf{fin})$, the set $\cup_{n\in\mathbb{N}}\mathcal{Z}_n$ is countable. We define by induction a sequence $(C_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathcal{Z}_n$ such that for every $n \in \mathbb{N}$, $\mathcal{G} \cup \{C_i : \ i < n\}$ generates a filter $\mathcal{G}_n$ and $C_n \in \mathcal{S}(\mathcal{G}_n)$: given some $n \in \mathbb{N}$, $\cup\mathcal{Z}_n \in \mathcal{G} \subseteq fil(\mathcal{G}, (C_i)_{i<n}) \subseteq \mathcal{S}(fil(\mathcal{G}, (C_i)_{i<n}))$; using Section 2.1.2-(ii), it follows that there exists $C_n \in \mathcal{Z}_n$ satisfying $C_n \in \mathcal{S}(fil(\mathcal{G}, (C_i)_{i<n}))$. Since $\mathcal{C}$ satisfies the property of $d$-smallness in thin $d$-crowns centered at $a$, the filter $\mathcal{H} := \cup_{n\in\mathbb{N}}\mathcal{G}_n$ is Cauchy in the metric space $(X, d)$. Since $\mathcal{T} \subseteq \mathcal{T}_d$ and $(X, d)$ is complete, $\cap\mathcal{H}$ is a singleton $\{b\}$. Thus $b \in \cap\mathcal{H} \subseteq \cap\mathcal{G} \subseteq \cap\mathcal{F}$.

(ii) Let $\mathcal{A}$ be subset of non-empty elements $\mathcal{C}$ which is closed by finite intersection and such that the ball $B$ belongs to $\mathcal{A}$. Let us show (in **ZF**) that $\cap\mathcal{A}$ is non-empty. Let $\mathcal{F}$ be the filter of $\mathcal{L}$ generated by $\mathcal{A}$. Let $R := \inf\{r \in \mathbb{R}_+ : \ B_d(a, r) \in \mathcal{S}(\mathcal{F})\}$. Denote by $\mathcal{A}'$ the set $\{A \cap B_d(a, r) : A \in \mathcal{A} \text{ and } r > R\}$. If $R = 0$ then $\{a\} = \cap\mathcal{A}' \subseteq \cap\mathcal{A}$. If $R > 0$, then for every $\varepsilon > 0$, there exists some element of $\mathcal{A}'$ which is included in the crown $D_d(a, R-\varepsilon, R+\varepsilon)$. Since $\mathcal{C}$ satisfies the property of $d$-smallness in thin $d$-crowns centered at $a$, the centered family $\mathcal{A}'$ is Cauchy in the metric space $(X, d)$; since this metric space $(X, d)$ is complete, $\cap\mathcal{A}'$ is a singleton $\{b\}$, and $\{b\} = \cap\mathcal{A}' \subseteq \cap\mathcal{A}$; moreover, $b$ is **ZF**-definable from $(X, d)$ and $\mathcal{A}$. $\square$

## 3. The convex topology on a normed space

*In this paper, all vector spaces that we consider are defined over the field* $\mathbb{R}$ *of* real *numbers.*

### 3.1. **Banach spaces.**

Given a normed space $E$ endowed with a norm $\|.\|$, we denote by $B_E$ the closed unit ball $\{x \in E : \|x\| \leq 1\}$, and by $S_E$ the unit sphere of $E$. The topology on $E$ associated to the norm is called the *strong topology*. A *Banach* space is a normed space which is (Cauchy)-complete for the metric associated to the norm (*i.e.* every Cauchy filter of closed sets has a non-empty intersection).

### 3.2. **Weak topologies on normed spaces.**

3.2.1. *The continuous dual $E'$ of a normed space $E$.* We endow the vector space $E'$ of continuous linear mappings $f : E \to \mathbb{R}$ with the *dual norm* $\|.\|^*$, and we call this space the *continuous dual* of the normed space $E$. We also denote by $can : E \to E''$ the *canonical mapping* associating to every $x \in E$ the "evaluating mapping" $\tilde{x} : E' \to \mathbb{R}$, satisfying for every $f \in E'$ the equality $\tilde{x}(f) = f(x)$.

3.2.2. *The weak topology $\sigma(E, E')$ on $E$.* It is the weakest topology $\mathcal{T}$ on $E$ such that elements $f \in E'$ are $\mathcal{T}$-continuous. The vector space $E$ endowed the weak topology is a locally convex topological vector space.

3.2.3. *The weak* topology $\sigma(E', E)$ on $E'$.* It is the weakest topology $\mathcal{T}$ on $E$ such that evaluating mappings $\tilde{x} : E' \to \mathbb{R}$, $x \in E$ are $\mathcal{T}$-continuous. The vector space $E'$ endowed the weak* topology is a Hausdorff locally convex topological vector space.

*Remark* 3. In a model of **ZF** where **HB** fails, there exists a non null (infinite dimensional) normed space $E$ such that $E' = \{0\}$ (see [5, Lemma 5] or [9]). In such a model of **ZF**, the weak topology on $E$ is trivial with only two open sets.

### 3.3. The convex topology on a normed space.

3.3.1. *Definition of the convex topology.* Since the weak topology on an infinite dimensional normed space $E$ may be trivial (in **ZF**), we define an alternative topology on $E$, the *convex topology* (which we introduced in [11]): it is the weakest topology $\mathcal{T}_c$ for which strongly closed convex subsets of $E$ are $\mathcal{T}_c$-closed. The lattice generated by strongly closed convex subsets of $E$ is called the *convex lattice* of $E$. Elements of this lattice are finite unions of strongly closed convex sets, so this lattice is a basis of closed subsets of the convex topology. Thus the set $\mathcal{C}$ of strongly closed convex subsets of $E$ is a sub-basis of closed sets of the convex topology, which is closed by finite intersection.

**Proposition 2.** *Let $E$ be a normed space.*

*(i) "weak topology on $E$" $\subseteq$ "convex topology on $E$" $\subseteq$ "strong topology on $E$".*
*(ii) If $E$ satisfies the continuous Hahn-Banach property, then the weak topology and the convex topology on $E$ are equal.*

*Proof.* (i) is trivial. (ii) follows from the fact that if a normed space $E$ satisfies the *CHB* property, then it satisfies several classical geometric forms of the geometric Hahn-Banach property (see [4]) and in particular, every closed convex set is weakly closed. □

In **ZF+HB**, the weak topology and the convex topology on a normed space are equal.

3.3.2. *Convex topology* vs *strong topology.*

**Theorem (Lusternik-Schnirelmann).** *Let $n \in \mathbb{N}^*$, let $N : \mathbb{R}^n \to \mathbb{R}$ be a norm, and let $S$ be the unit sphere of $N$. Let $a \in \mathbb{R}^n$ such that*

*$N(a) < 1$. Denote by $s_a : S \to S$ the "antipodal mapping" associating to every $x \in S$ the point $y \in S$ such that $(xa) \cap S = \{x, y\}$, where $(xa)$ is the line generated by $x$ and $a$. If $C_1, \ldots, C_n$ are $n$ closed subsets of $\mathbb{R}^n$ such that $S \subseteq \cup_{1 \leq i \leq n} C_i$, then there exists $i \in \{1..n\}$ and $x \in S$ such that $\{x, s_a(x)\} \subseteq C_i$.*

*Proof.* The proof of this famous result is choiceless (see for example [10] for $a = 0$). □

**Theorem 2.** *Let $E$ be a normed space which is not finite-dimensional. The closure of the unit sphere $S_E$ for the convex topology is the closed unit ball $B_E$. In particular, the convex topology on $E$ is strictly contained in the strong topology.*

*Proof.* Since $B_E$ is closed in the convex topology, the closure $C$ of $S_E$ in the convex topology is contained in $B_E$. We now prove that every point $a \in E$ such that $\|a\| < 1$ belongs to $C$. Consider some finite set $\{C_i : 1 \leq i \leq n\}$ of closed convex subsets of $E$ such that $F := \cup_{1 \leq i \leq n} C_i$ contains $S$. We have to show that $a \in F$. Let $V$ be a vector subspace of $E$, containing $a$, with dimension $\geq n$. The Lusternik-Schnirelmann theorem implies that for some $i_0 \in \{1..n\}$, $C_{i_0} \cap V$ contains two $a$-antipodal points of $S_E$; by convexity of $C_{i_0}$, $a \in C_{i_0}$. □

**Question 1.** The convex topology $\mathcal{T}_c$ on a normed space $E$ is $T_1$ (*i.e.* every singleton is closed). Is it Hausdorff? Is the space $(E, \mathcal{T}_c)$ a topological vector space? Is it locally convex? Is $\mathcal{T}_c$ the topology associated to some family of pseudo-metrics on $E$? Is $\mathcal{T}_c$ uniformizable?

## 4. Weak compactness in a uniformly convex Banach space

### 4.1. Uniform convexity.

4.1.1. *Strict convexity.* A normed space $(E, \|.\|)$ is *strictly convex* if every segment contained in the unit sphere is a singleton: for every $x, y \in S_E$, $x \neq y \Rightarrow \|\frac{x+y}{2}\| < 1$.

4.1.2. *Uniform convexity.* This is a strong quantitative version of the strict convexity. A normed space $(E, \|.\|)$ is *uniformly convex* if for every real number $\varepsilon > 0$, there exists $\delta > 0$ such that for every $x, y \in B_E$, $\big(\|x - y\| > \varepsilon \Rightarrow \|\frac{x+y}{2}\| < 1 - \delta\big)$.

*Example* 1. Every Hilbert space is uniformly convex (see [2, p. 190-191]). Let $p \in ]1, +\infty[$. If $\mathcal{B}$ is a boolean algebra of subsets of a set $I$, and if $\nu : \mathcal{B} \to [0, +\infty]$ is non-null and finitely additive, then the normed space $L^p(\nu)$ is uniformly convex (see [4, Section 4]). In particular, for every set $I$, the normed space $\ell^p(I)$ (see Section 5.1) is uniformly convex.

**Proposition 3.** *Given a uniformly convex normed space $E$, and denoting by $d$ the metric on $E$ given by the norm, the family of closed convex subsets of $E$ satisfies the property of $d$-smallness in thin $d$-crowns centered at $0_E$.*

*Proof.* This follows directly from the definition of uniform convexity. □

4.2. **Various weak forms of the Alaoglu theorem.** Consider the following statements (the first two were introduced in [3] and [11] and are consequences of **BPI** -or rather the Alaoglu theorem-):

- **A1**: The closed unit ball (and thus every bounded subset which is closed in the convex topology) of a uniformly convex Banach space is compact in the convex topology.
- **A2**: (Hilbert) The closed unit ball (and thus every bounded weakly closed subset) of a Hilbert space is weakly compact.
- **A3**: (Hilbert with hilbertian basis) For every set $I$, the closed unit ball of $\ell^2(I)$ is weakly compact.
- **A4**: For every sequence $(F_n)_{n\in\mathbb{N}}$ of finite sets, the closed unit ball of $\ell^2(\cup_{n\in\mathbb{N}}F_n)$ is weakly compact.

Of course, $\mathbf{A1} \Rightarrow \mathbf{A2} \Rightarrow \mathbf{A3} \Rightarrow \mathbf{A4}$.

*Remark* 4. If a Hilbert space $H$ has a well orderable dense subset, then $H$ has a well orderable hilbertian basis, thus $H$ is isometrically isomorphic with some $\ell^2(I)$ where $I$ is well orderable, and in this case, the closed ball $B_H$, which is homeomorphic with a closed subset of $[-1,1]^I$, is weakly compact (use Remark 1). In particular, given an ordinal $\alpha$, the closed unit ball of $\ell^2(\alpha)$ (for example the closed unit ball of $\ell^2(\mathbb{N})$) is weakly compact.

**Theorem 3.** *(i)* $\mathbf{AC}(\mathbb{N}) \Rightarrow \mathbf{A1}$.
*(ii)* $\mathbf{A1} \not\Rightarrow \mathbf{AC}(\mathbb{N})$.
*(iii)* $\mathbf{A4} \Leftrightarrow \mathbf{AC}(\mathbb{N}, \mathbf{fin})$.

*Proof.* (i) Let $E$ be a uniformly convex Banach space. Denote by $\mathcal{C}$ the class of (strongly) closed convex subsets of $E$. Denoting by $\mathcal{T}$ the convex topology on $E$, the class $\mathcal{C}$ is a sub-basis of closed subsets of the topological space $(E, \mathcal{T})$. Now, consider the distance $d$ associated to the norm on $E$: using Proposition 3, $\mathcal{C}$ satisfies the property of $d$-smallness in thin $d$-crowns centered at $0_E$. Moreover, the metric space $(E, d)$ is complete, closed $d$-balls belong to $\mathcal{C}$ and $\mathcal{T}$ is included in the topology associated to $d$. Applying Theorem 1-(i), it follows from $\mathbf{AC}(\mathbb{N})$ that the unit closed ball of $E$ (and also every bounded subset of $E$ which is closed in the convex topology) is compact in the convex

topology of $E$.
(ii) $\mathbf{A1} \not\Rightarrow \mathbf{AC}(\mathbb{N})$ because $\mathbf{BPI} \Rightarrow \mathbf{A1}$ and $\mathbf{BPI} \not\Rightarrow \mathbf{AC}(\mathbb{N})$.
(iii) The idea of the implication $\mathbf{A4} \Rightarrow \mathbf{AC}(\mathbb{N}, \mathbf{fin})$ is in [5, th. 9 p. 16]: we sketch it for sake of completeness. Let $(F_n)_{n\in\mathbb{N}}$ be a disjoint sequence of non-empty finite sets. Let us show that $\prod_{n\in\mathbb{N}} F_n$ is non-empty. Let $I := \cup_{n\in\mathbb{N}} F_n$. Then the Hilbert spaces $H := \ell^2(I)$ and $\oplus_{\ell^2(\mathbb{N})}\ell^2(F_n)$ are isometrically isomorph. Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence of $]0,1[$ such that $\sum_{n\in\mathbb{N}} \varepsilon_n^2 = 1$. For every $n \in \mathbb{N}$, let $\tilde{F}_n := \{\varepsilon_n 1_{\{x\}} : x \in F_n\}$ where for each $x \in F_n$, $1_{\{x\}} : F_n \to \{0,1\}$ is the indicator of $\{x\}$; let $Z_n := \{x = (x_k)_{k\in\mathbb{N}} \in H : \ x_n\restriction F_n \in \tilde{F}_n\}$. Each $Z_n$ is a weakly closed subset of the ball $B_H$ ($Z_n$ is a finite union of closed convex sets). Moreover, the sequence $(Z_n)_{n\in\mathbb{N}}$ is centered. The weak compactness of $B_H$ implies that $Z := \cap_{n\in\mathbb{N}} Z_n$ is non-empty. An element of $Z$ defines an element of $\prod_{n\in\mathbb{N}} \tilde{F}_n$, and thus an element of $\prod_{n\in\mathbb{N}} F_n$ (because each $\varepsilon_n$ is $> 0$). For the converse statement, if $(F_n)_{n\in\mathbb{N}}$ is a sequence of finite sets, then the set $I := \cup_{n\in\mathbb{N}} F_n$ is finite or countable and in both cases, the closed unit ball of the Hilbert space $\ell^2(I)$ is weakly compact (see Remark 4). ☐

*Remark* 5. Theorem 3-(i) enhances our previous result $\mathbf{DC} \Rightarrow \mathbf{A1}$ which we proved in [3], where we left open the two questions: Does $\mathbf{AC}(\mathbb{N})$ imply $\mathbf{A1}$? Does $\mathbf{AC}(\mathbb{N})$ imply $\mathbf{A2}$? A proof of $\mathbf{AC}(\mathbb{N}) \Rightarrow \mathbf{A2}$ has been found by Fremlin (see [6, chap. 56, Section 566P]).

**Question 2.** Does **A2** imply **A1**? Does **A3** imply **A2**? Does $\mathbf{AC}(\mathbb{N}, \mathbf{fin})$ imply **A3**?

4.3. **Convex-compactness in ZF.** Given a vector space $E$, endowed with a topology $\mathcal{T}$, say that a subset $A$ of $E$ is *convex-compact* if, denoting by $\mathcal{C}$ the set of $\mathcal{T}$-closed convex subsets of $E$, $A$ is $\mathcal{C}$-compact; moreover, if $A$ is closely $\mathcal{C}$-compact, say that $A$ is *closely convex-compact.*

**Theorem 4.** *The closed unit ball of a uniformly convex Banach space is closely convex-compact in the convex topology.*

*Proof.* The proof is analog to the proof of Theorem 3-(i), applying Theorem 1-(ii) instead of Theorem 1-(i). ☐

## 5. $\mathbf{AC}(\mathbb{N})$ AND COMPACTNESS IN $[0,1]^I$

Given a set $I$, we endow the vector space $\mathbb{R}^I$ with the the product topology, which we denote by $\mathcal{T}_I$.

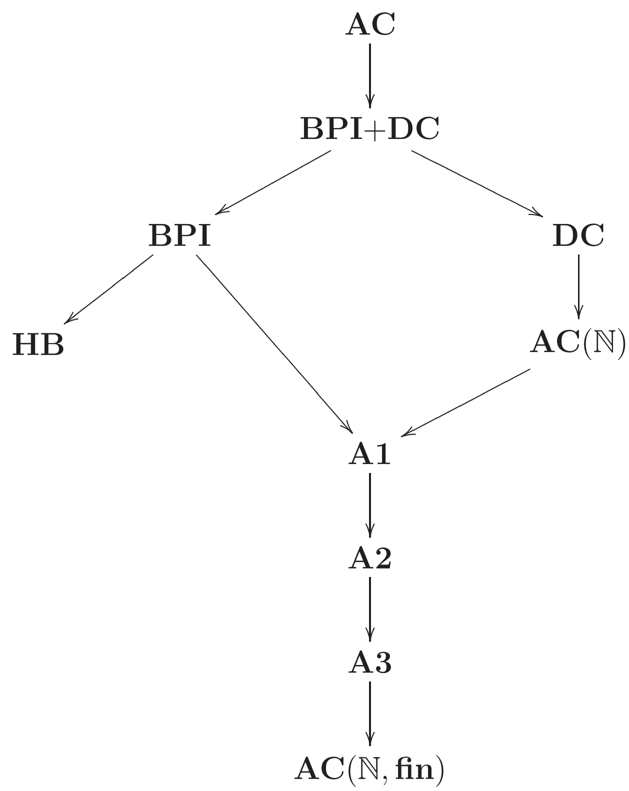


FIGURE 1. Some weak forms of **AC**

5.1. **Spaces** $\ell^p(I)$**,** $1 \leq p \leq \infty$ **or** $p = 0$**.** Denote by $\ell^\infty(I)$ the following vector space endowed with the "sup" norm $N_\infty$:

$$\ell^\infty(I) := \{x = (x_i)_{i \in I} : \sup_{i \in I} |x_i| < +\infty\}$$

Denote by $\ell^0(I)$ the following closed vector space of $\ell^\infty(I)$ endowed with the norm $N_\infty$:

$$\ell^0(I) := \{x = (x_i)_{i \in I} \in \ell^\infty(I) : \forall \varepsilon > 0 \, \exists F \in \mathcal{P}_f(I) \, \forall i \in I \backslash F \, |x_i| \leq \varepsilon\}$$

For every $p \in [1, +\infty[$, denote by $\ell^p(I)$ the following vector space endowed with the $N_p$-norm:

$$\ell^p(I) := \{x = (x_i)_{i \in I} \in \mathbb{R}^I : \sum_{i \in I} |x_i|^p < +\infty\}$$

Recall that the continuous dual of $\ell^0(I)$ is (canonically isometrically isomorphic with) $\ell^1(I)$. Given some $p \in ]1, +\infty[$, the continuous dual of $\ell^p(I)$ is (canonically isometrically isomorphic with) $\ell^q(I)$ where $q$ is the conjuguate of $p$. The following Lemma is easy:

**Lemma 1.** *Let $I$ be a set.*

*(i) The topology induced by $\mathcal{T}_I$ on the subset $\ell^1(I)$ (*resp. *$\ell^\infty(I)$) is included in the weak* topology $\sigma(\ell^1(I), \ell^0(I))$ of $\ell^1(I)$ (*resp. *the weak* topology $\sigma(\ell^\infty(I), \ell^1(I))$ of $\ell^\infty(I)$). Moreover, the two topologies induce the same topology on bounded subsets of $\ell^1(I)$ (*resp. *$\ell^\infty(I)$).*

*(ii) The topology induced by $\mathcal{T}_I$ on the subset $\ell^0(I)$ is included in the weak topology $\sigma(\ell^0(I), \ell^1(I))$. Moreover, the two topologies induce the same topology on bounded subsets of $\ell^0(I)$.*

*(iii) Given some $p \in ]1, +\infty[$, the topology induced by $\mathcal{T}_I$ on the subset $\ell^p(I)$ is included in the weak topology $\sigma(\ell^p(I), \ell^q(I))$ where $q$ is the conjugate of $p$. Moreover, the two topologies induce the same topology on bounded subsets of $\ell^p(I)$.*

5.2. **Closed subspaces of $[0,1]^I$ included in $\ell^p(I)$, $1 \leq p < +\infty$.** Given a set $I$, denote by $[0,1]^I_\sigma$ the set of elements $x = (x_i)_{i \in I} \in [0,1]^I$ such that the support $\{i \in I : x_i \neq 0\}$ of $x$ is countable. Say that a closed subset $F$ of $[0,1]^I$ is *Corson* if $F \subseteq [0,1]^I_\sigma$.

**Proposition 4.** *Let $I$ be a set. Let $F$ be a closed subset of $[0,1]^I$.*

*(i) If $F \subseteq \ell^0(I)$, then* $\mathbf{AC}(\mathbb{N}, \mathbf{fin})$ *implies that $F$ is Corson.*

*(ii) If $F$ is Corson, then* $\mathbf{AC}(\mathbb{N})$ *implies that $F$ is sequentially compact.*

*(iii) If $F \subseteq \ell^0(I)$, then* $\mathbf{AC}(\mathbb{N})$ *implies that $F$ is sequentially compact.*

*Proof.* (i) assume that $F \subseteq \ell^0(I)$. Given some $x = (x_i)_{i \in I} \in F$, the support $J := \{i \in I : \ x_i \neq 0\}$ of $x$ is a countable union of finite sets. Using $\mathbf{AC}(\mathbb{N}, \mathbf{fin})$, $J$ is countable. Thus $F$ is Corson.
(ii) Let $(x^n)_{n \in \mathbb{N}}$ be a sequence of $F$. For every $n \in \mathbb{N}$, denote by $J_n$ the *support* of $x^n$: $J_n := \{i \in I : x^n_i \neq 0\}$. Using $\mathbf{AC}(\mathbb{N})$, the set $J := \cup_{n \in \mathbb{N}} J_n$ is countable. So $K := [0,1]^J \times \{0\}^{I \setminus J}$ is compact and metrizable so $K$ is sequentially compact: extract from $(x^n)_{n \in \mathbb{N}}$ a convergent subsequence $(x^n)_{n \in A}$ where $A$ is some infinite subset of $\mathbb{N}$.
(iii) Use (i) and (ii). □

**Corollary 1.** *Let $F$ be a closed subset of $[0,1]^I$. If there exists $p \in [1, +\infty[$ such that $F$ is a bounded subset of $\ell^p(I)$, then* $\mathbf{AC}(\mathbb{N})$ *implies that $F$ is compact in $[0,1]^I$.*

*Proof.* Let $r \in ]p, +\infty[$. Then $N_r \leq N_p$, so $F$ is a bounded subset of $\ell^r(I)$. Since $F$ is weakly closed and bounded in $\ell^r(I)$, Theorem 3-(i) implies that, using $\mathbf{AC}(\mathbb{N})$, $F$ is compact in the weak topology $\sigma(\ell^r(I), \ell^{r'}(I))$ where $r'$ is the conjuguate of $r$. It follows from Lemma 1-(iii) that $F$ is compact for the topology $\mathcal{T}_I$. □

**Question 3.** What is the power of the statement *"The closed unit ball of $\ell^2(\mathbb{R})$ is weakly compact"*? This statement is a consequence of $\mathbf{AC}(\mathbb{N})$. Are there models of **ZF** which do not satisfy this statement?

## 6. DC and compactness in $[0,1]^I$

### 6.1. Eberlein's criterion of compactness.

6.1.1. *$\vartheta$-sequences.* Let $E$ be a normed space, and denote by $d$ the metric given by the norm on $E$. Given a subset $F$ of $E$, and some real number $\vartheta > 0$, a *$\vartheta$-sequence of $F$* is a sequence $(a_n)_{n\in\mathbb{N}}$ of $F$ satisfying for every $n \in \mathbb{N}$:

$$d\big(\mathrm{span}\{a_i : i < n\}, \mathrm{conv}\{a_i : i \geq n\}\big) \geq \vartheta$$

In [11], we proved in (**ZF**+**DC**) the following result:

**Theorem** (**DC**)**.** *Let $E$ be a Banach space. Denote by $\mathcal{T}$ the convex topology on $E$. Let $F$ be a* convex *subset of $E$ which is $d$-bounded and $\mathcal{T}$-closed. If $F$ is not $\mathcal{T}$-compact, then there exists some real number $\vartheta > 0$ and a $\vartheta$-sequence of $F$.*

If we delete the hypothesis "$F$ is convex", our next result allows us to build in (**ZF**+**DC**) "pseudo $\vartheta$-sequences". Say that a sequence $(a_n)_{n\in\mathbb{N}}$ of $F$ is a *pseudo $\vartheta$-sequence* if for every $n \in \mathbb{N}$:

$$d\big(\mathrm{span}\{a_i : i < n\}, (F \cap \mathrm{conv}\{a_i : i \geq n\})\big) \geq \vartheta$$

6.1.2. *Building pseudo-sequences with* **DC***.* We first recall the following result for saturating filters *w.r.t.* some numeric constraint:

**Proposition** (**DC**)**.** *Let $E$ be a set, let $\mathcal{L}$ be a lattice of subsets of $E$, with smallest element $\varnothing$ and greatest element $E$. Let $\rho : \mathcal{L} \to \mathbb{R}_+$ be some mapping. let $\tilde{\rho} : \mathcal{P}(\mathcal{L}) \to \mathbb{R}_+$ be the mapping associating to every subset $\mathcal{A}$ of $\mathcal{L}$ the real number* $\inf\{\rho(A) : A \in \mathcal{A}\}$*. Let $\mathcal{F}$ be a filter of $\mathcal{L}$. Then there exists a filter $\mathcal{G}$ of $\mathcal{L}$ including $\mathcal{F}$ such that*

$$\tilde{\rho}(\mathcal{G}) = \tilde{\rho}(\mathcal{S}_{\mathcal{L}}(\mathcal{G}))$$

*Proof.* See [11]. □

*Remark* 6. The previous Proposition is easy to prove in **ZFC**: consider a maximal filter of $\mathcal{L}$ including $\mathcal{F}$.

**Notation 1.** Given a metric space $(X, d)$ and some subset $A$ of $X$, we denote by $d_A$ the (1-Lipschitzian hence) continuous mapping $d_A : X \to \mathbb{R}$ associating to every $x \in X$ the real number $d(x, A) := \inf\{d(x, a) :$

$a \in A\}$. Moreover, given some real number $\vartheta > 0$, we denote by $A_\vartheta$ the following closed subset of $X$:

$$A_\vartheta := \{x \in X : d(x, A) \leq \vartheta\}$$

Notice that if $A$ is a convex subset of a normed space, then for every $\vartheta > 0$ the set $A_\vartheta$ is convex (because the mapping $d_A$ is convex).

**Theorem 5** (**DC**). *Let $E$ be a Banach space. Let $d$ be the distance given by the norm on $E$. Let $\mathcal{T}$ be the convex topology on $E$. Let $F$ be a $d$-bounded subset of $E$, which is $\mathcal{T}$-closed (thus $d$-closed, thus $d$-complete). If $F$ is not $\mathcal{T}$-compact, then there exists some real number $\vartheta > 0$, and a sequence $(a_n)_{n\in\mathbb{N}}$ of $F$, such that for every $n \in \mathbb{N}$:*

$$d\big(\operatorname{span}\{a_k : k \leq n\}, (\overline{\operatorname{conv}}^{\mathcal{T}}\{a_k : k > n\} \cap F)\big) \geq \vartheta$$

*Proof.* Let $\mathcal{C}$ be the set of $\mathcal{T}$-closed (*i.e.* $d$-closed) convex subsets of $E$. Let $\mathcal{L}_c$ be the lattice generated by $\mathcal{C}$. Let $\mathcal{L}_1$ be the lattice induced by $\mathcal{L}_c$ on $F$: $\mathcal{L}_1 = \{A \cap F : A \in \mathcal{L}_c\}$. Since $F$ is not $\mathcal{T}$-compact, let $\mathcal{F}$ be a filter of $\mathcal{L}_1$ containing $F$ such that $\cap\mathcal{F} = \varnothing$. Let $\rho$ be the "diameter" function (*w.r.t.* the distance $d$), which is defined for $d$-bounded subsets of $E$, and in particular on $\mathcal{L}_1$. Using the previous Proposition for the "diameter" function $\rho$, **DC** implies the existence of a filter $\mathcal{G}$ of $\mathcal{L}_1$ including $\mathcal{F}$ such that

$$r := \tilde{\rho}(\mathcal{S}_{\mathcal{L}_1}(\mathcal{G})) = \tilde{\rho}(\mathcal{G})$$

If $r = 0$, then, since the metric space $(F, d)$ is complete, $\cap\mathcal{G}$ is a singleton $\{a\}$, and $a \in \cap\mathcal{F}$: contradictory! Thus $r > 0$. Let $0 < \vartheta < r$. We will now build a sequence $(K_n)_{n\in\mathbb{N}}$ of $\mathcal{C}$, and a sequence $(a_n)_{n\in\mathbb{N}}$ of $F$, such that for every $n \in \mathbb{N}$,

$$a_n \in \cap_{i\leq n} K_i \text{ and } (\operatorname{span}\{a_i : i < n\})_\vartheta \cap (K_n \cap F) = \varnothing \tag{3}$$

It will follow that for every $n \in \mathbb{N}$,

$$d\big(\operatorname{span}\{a_k : k < n\}, (\overline{\operatorname{conv}}^{\mathcal{T}}\{a_k : k \geq n\} \cap F)\big) \geq \vartheta$$

• $B(0,\vartheta) \notin \mathcal{S}_{\mathcal{L}_1}(\mathcal{G})$: thus there exists $G \in \mathcal{G}$ satisfying $B(0,\vartheta) \cap G = \varnothing$; since $G \in \mathcal{L}_1$, $G$ is of the form $\cup_{i\in I}(C_i \cap F)$ where $I$ is finite and each $C_i$ belongs to $\mathcal{C}$; using Section 2.1.2-(ii), let $i \in I$ such that $(C_i \cap F) \in \mathcal{S}(\mathcal{G})$. let $K_0$ be the convex set $C_i$. Let $\mathcal{G}_0$ be the filter of $\mathcal{L}_1$ generated by $\mathcal{G}$ and $K_0$. Let $a_0 \in K_0 \cap F$.

• $(\mathbb{R}a_0)_\vartheta \notin \mathcal{S}_{\mathcal{L}_1}(\mathcal{G}_0)$: let $G \in \mathcal{G}_0$ such that $(\operatorname{span}\{a_0\})_\vartheta \cap G = \varnothing$. since $G \in \mathcal{L}_1$, $G$ is of the form $\cup_{i\in I}(C_i \cap F)$ where $I$ is finite and each $C_i$ belongs to $\mathcal{C}$; let $i \in I$ such that $(C_i \cap F) \in \mathcal{S}(\mathcal{G}_0)$. Let $K_1$ be the set $C_i$. Let $\mathcal{G}_1$ be the filter of $\mathcal{L}_1$ generated by $\mathcal{G}_0$ and $K_1$. Let $a_1 \in (K_0 \cap K_1 \cap F)$.

• $(\mathrm{span}\{a_0, a_1\})_\vartheta \notin \mathcal{S}_{\mathcal{L}_1}(\mathcal{G}_1)$: let $K_2 \in \mathcal{C}$ such that $K_2 \in \mathcal{S}(\mathcal{G}_1)$ and $(\mathrm{span}\{a_0, a_1\})_\vartheta \cap (K_2 \cap F) = \varnothing$. Let $\mathcal{G}_2$ be the filter of $\mathcal{L}_1$ generated by $\mathcal{G}_1$ and $K_2$. Let $a_2 \in (K_0 \cap K_1 \cap K_2) \cap F$.

• ...

Using **DC**, we construct a sequence $(a_n, C_n)_{n\in\mathbb{N}}$ of $F \times \mathcal{C}$ satisfying (3). □

**Corollary 2.** *Let $E$ be a Banach space. Let $d$ be the metric given by the norm on $E$. Let $F$ be a $d$-bounded subset of $E$ which is closed for the convex topology $\mathcal{T}$ on $E$. Consider the three following statements:*

*(i) $F$ is compact for the convex topology $\mathcal{T}$.*
*(ii) $F$ is sequentially compact for $\mathcal{T}$.*
*(iii) For every $\vartheta > 0$, $F$ does not contain any pseudo $\vartheta$-sequence.*

*Then* (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). *Moreover, in* (**ZF**+**DC**)*,* (iii) $\Rightarrow$ (i).

*Proof.* (i) $\Rightarrow$ (ii) It is sufficient to prove this implication when $E$ is a *separable* Banach space. In this case, the normed space satisfies the *CHB* property (see Section 1.2.3), thus the convex topology $\mathcal{T}$ and the weak topology on $E$ are equal. Moreover, there is a norm $N$ on $E$ which induces on the closed unit ball of $E$ a topology which is included in the weak topology of $E$ (see for example [8, Lemme I.4 p. 2]). This implies that the topology given by the norm $N$ and the topology $\mathcal{T}$ are equal on $K$; thus the $\mathcal{T}$-compact space $K$ is metrisable whence $K$ is sequentially compact.
(ii) $\Rightarrow$ (iii) Assume that the subset $K$ is sequentially compact in the topology $\mathcal{T}$. Let $\vartheta > 0$. Seeking for a contradiction, assume that $F$ has a pseudo $\vartheta$-sequence $(a_n)_{n\in\mathbb{N}}$. Extract some sequence $(a_n)_{n\in A}$ which converges to some $l \in F$ in the topology $\mathcal{T}$. Then, for every $n \in \mathbb{N}$, $l \in \overline{\mathrm{conv}}^{\mathcal{T}}\{a_i : i \geq n\}$. Let $V := \mathrm{span}\left(\{a_i : i \in \mathbb{N}\} \cup \{l\}\right)$. Let $(u_n)_n$ be a convex block-sequence of $(a_n)_{n\in A}$ which strongly converges to $l$. Since the normed space $V$ is separable, for each $n \in \mathbb{N}$, choose some $f_n$ in the unit sphere of $V'$ such that $f_n$ is null on $\{a_i : i < n\}$ and $f_n(l) \geq \vartheta$. Let $n_0 \in \mathbb{N}$ such that $d(u_{n_0}, l) < \frac{\vartheta}{2}$. Let $N \geq n_0$ such that $u_{n_0} \in \mathrm{span}\{a_i : i < N\}$; then $f_N(u_{n_0}) = 0$ and $f_N(l) \geq \vartheta$ thus, since $\|f_N\| = 1$, $d(l, u_{N_0}) \geq \vartheta$: this is contradictory!
The implication (iii) $\Rightarrow$ (i) holds in **ZF**+**DC** thanks to Theorem 5. □

## 6.2. Closed subsets of $[0,1]^I$ included in $\ell^0(I)$.

**Corollary 3.** *Let $F$ be a closed subset of $[0,1]^I$. Assume that $F$ is a (bounded) subset of $\ell^0(I)$. Then* **DC** *implies that $F$ is compact.*

*Proof.* The normed space $\ell^0(I)$ satisfies the *CHB* property (see Section 1.2.3), thus the weak topology $\sigma(\ell^0(I), \ell^1(I))$ and the convex topology $\mathcal{T}$ on $\ell^0(I)$ are equal on $\ell^0(I)$. Since $F$ is a bounded subset of $\ell^0(I)$, the topology $\mathcal{T}$ and the product topology $\mathcal{T}_I$ induce the same topology on $F$ (see Lemma 1-(ii)). The subset $F$ of $\ell^0(I)$ is bounded and $\mathcal{T}$-closed; using Proposition 4-(iii), $F$ is sequentially compact for the topology $\mathcal{T}_I$ *i.e.* for $\mathcal{T}$. Using **DC**, Corollary 2 implies that $F$ is compact for $\mathcal{T}$ *i.e.* for $\mathcal{T}_I$. □

**Question 4.** Let $I$ be a set and $F$ be some closed subset of $[0,1]^I$. If $F \subseteq \ell^0(I)$, does $\mathbf{AC}(\mathbb{N})$ imply that $F$ is compact?

**Question 5.** Let $F$ be a closed subset of $[0,1]^I$ which is contained in $\mathbb{R}^{(I)}$ (the vector subspace of elements $x \in \mathbb{R}^I$ which have a *finite* support). Then $F \subseteq \ell^0(I)$ thus, using Corollary 3, **DC** implies that $F$ is compact. Does $\mathbf{AC}(\mathbb{N})$ imply that $F$ is compact?

**Question 6.** Let $I$ be a set and $F$ be some closed subset of $[0,1]^I$.

(i) If $F$ is Corson, does **DC** imply that $F$ is compact? Does $\mathbf{AC}(\mathbb{N})$ imply that $F$ is compact? ($\mathbf{DC}_{\aleph_1}$ implies that $F$ is compact).

(ii) More generally, which closed subsets of $[0,1]^I$ can be proved compact in **ZF**+**DC**? in **ZF**+$\mathbf{AC}(\mathbb{N})$?

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ERMIT, Département de Mathématiques et Informatique, Université de La Réunion, 15 avenue René Cassin - BP 7151 - 97715 Saint-Denis Messag. Cedex 9 FRANCE

*E-mail address*, Marianne Morillon: mar@univ-reunion.fr
*URL*: http://personnel.univ-reunion.fr/mar