THE HAHN-BANACH PROPERTY AND THE AXIOM OF CHOICE

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Abstract. We work in the set theory without the axiom of choice: **ZF**. Though the Hahn-Banach theorem cannot be proved in **ZF**, we prove that every Gâteaux-differentiable uniformly convex Banach space E satisfies the following continuous Hahn-Banach property: if p is a *continuous* sublinear functional on E, if F is a subspace of E, and if $f : F \to \mathbb{R}$ is a linear functional such that $f \leq p_{|F}$, then there exists a linear functional $g : E \to \mathbb{R}$, such that g extends f and $g \leq p$. We also prove that the continuous Hahn-Banach property on a topological vector space E is equivalent to the classical geometrical forms of the Hahn-Banach theorem on E. We then prove that the axiom of Dependent choices (**DC**) is equivalent to Ekeland's variational principle, and that it implies the continuous Hahn-Banach property on Gâteaux-differentiable Banach spaces. Finally, we prove that, though separable normed spaces satisfy the continuous Hahn-Banach property, they do not satisfy the whole Hahn-Banach property in **(ZF+DC)**.

1 Introduction

We work in set-theory without the axiom of choice, **ZF**, such as sketched by Jech, (see [9]). We denote by ω the first infinite ordinal. All vector spaces that we consider in this paper are vector spaces over $\mathbb{R}$, the field of reals.

Let E be a vector space. A *sublinear functional* on E is a mapping $p : E \to \mathbb{R}$ such that for every $x, y \in E$ and $\lambda \in \mathbb{R}_+$ we have $p(x + y) \leq p(x) + p(y)$ and $p(\lambda.x) = \lambda.p(x)$. We say, that E satisfies the *Hahn-Banach property* if, for every sublinear functional $p : E \to \mathbb{R}$, for every subspace F of E and every linear functional $f : F \to \mathbb{R}$ such that $f \leq p_{|F}$, there exists a linear functional $g : E \to \mathbb{R}$ that extends f and such that $g \leq p$.

The *Hahn-Banach axiom* says that:

HB. *Every vector space satisfies the Hahn-Banach property.*

It is well-known that, in **ZF**, the axiom of choice implies **HB** but, Cohen's second model of **ZF** does not satisfy **HB** (see [9]), hence the theory **ZF** does not imply the axiom **HB**. In fact, Pincus and Solovay (see [11]) have found a model of **ZF**+$\neg$**HB** which satisfies the following *axiom of dependent choices:*

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DC. *(axiom of Dependent Choices) : If R is a binary relation on a nonempty set E such that $\forall x \in E\ \exists y \in E\ xRy$, then there is a sequence $(x_n)_{n\in\omega}$ of elements of E such that $\forall n \in \omega,\ x_n R x_{n+1}$.*

hence **DC** does not imply **HB**.

On the other hand, **HB** does not imply **DC**, since Cohen's first model satisfies **HB** but this model does not even satisfy the following *countable axiom of choice:*

AC$_\omega$. *For every family $(A_n)_{n\in\omega}$ of nonempty sets we have $\prod_{n\in\omega} A_n \neq \emptyset$.*

which is a consequence of **DC**, (see [9]).

Let E be a topological vector space. We say that E satisfies the *continuous Hahn-Banach property* (in abbreviated form: **CHBP**) if, for every *continuous* sublinear functional $p : E \to \mathbb{R}$, for every subspace F of E, and every linear functional $f : F \to \mathbb{R}$ such that $f \leq p_{|F}$, there exists a linear functional $g : E \to \mathbb{R}$ that extends f and such that $g \leq p$. The theory **ZF** does not imply the following continuous axiom of Hahn-Banach:

CHB. *Every topological vector space satisfies the continuous Hahn-Banach property.*

because the axiom **HB** is equivalent to the axiom **CHB** (see [6]).

Let S be the set of pairs (p, f) where $p : E \to \mathbb{R}$ is a continuous sublinear functional on E and $f : F \to \mathbb{R}$ is a linear functional defined on a subspace F of E satisfying $f \leq p_{|F}$. We say that E satisfies the *effective continuous Hahn-Banach property* if, there is a function ϕ which is defined on the set S, and which associates to every $(p, f) \in S$ a linear functional $g = \phi(p, f)$ which is defined on E, such that g extends f and $g \leq p$; here ϕ is called a *witness* of the effective continuous Hahn-Banach property on E.

In this paper, we prove in **ZF** the following theorem:

Gâteaux-differentiable uniformly convex Banach spaces satisfy the effective continuous Hahn-Banach property.

We also define the spaces $L^p(\mu), 1 \leq p < +\infty$ when μ is a *finitely additive* measure and we prove that these spaces are uniformly convex and uniformly smooth for $1 < p < +\infty$. We show that the continuous Hahn-Banach property on a topological vector space E is equivalent to various classical geometrical Hahn-Banach properties on E, which deal with separation of convex subsets of E. We then prove that the axiom of Dependent choices is equivalent to Ekeland's variational principle, so this last statement is neither stronger nor weaker than **HB**. Using this equivalence, we prove that **DC** implies the continuous Hahn-Banach property on Gâteaux-differentiable Banach spaces. Finally, we show that, though separable normed spaces satisfy the continuous Hahn-Banach property, the theory **ZF+DC** does not imply the Hahn-Banach property on separable normed spaces. We end with some rules that imply the continuous Hahn-Banach property on some normed spaces and we observe that the **CHBP** on $l^1(\mathbb{R})$ or $l^\infty(\omega)$ cannot be proved in **ZF+DC**.

2 Uniformly convex norms

Let E be a vector space and let N be a semi-norm on E. The semi-normed space (E, N) is said to be *uniformly convex* if, for every real number $\epsilon > 0$, there exists a real number $\delta > 0$ such that, for every $x, y \in E$, if $N(x) \leq 1, N(y) \leq 1$ and if $N(x-y) > \epsilon$ then $N(\frac{x+y}{2}) < 1-\delta$.

A mapping $\delta : \mathbb{R}_+^* \to \mathbb{R}_+^*$ such that for every $x, y \in E$, if $N(x), N(y) \leq 1$ and $N(x-y) \geq \epsilon$ then $N(\frac{x+y}{2}) \leq 1-\delta(\epsilon)$ is called a *witness of the uniform convexity* of the semi-norm N on E. It is easy to see that a semi-normed space is uniformly convex with witness δ if and only if all its finite dimensional subspaces are uniformly convex with this witness δ. For every real number ϵ such that $0 < \epsilon < 2$ we put

$$\delta_N(\epsilon) = \inf\{1 - N(\frac{x+y}{2});\ N(x) = N(y) = 1;\ N(x-y) \geq \epsilon\}$$

and we call the mapping δ_N the *modulus of convexity* of (E, N). The semi-normed space is uniformly convex if and only if for every ϵ such that $0 < \epsilon < 2$ we have $\delta_N(\epsilon) > 0$ and in this case, for every witness δ of uniform convexity on (E, N) we have $\delta \leq \delta_N$; this means that δ_N is the best witness of uniform convexity on (E, N).

For example, Hilbert spaces are uniformly convex (see [1] page 190) with the modulus of convexity $\epsilon \mapsto 1 - \sqrt{1 - \frac{\epsilon^2}{4}}$.

We say that a metric space is *complete* if every Cauchy filter has a limit point in this space. For example $\mathbb{R}$ is complete. If X is a topological space, we denote by $C^*(X)$ the set of real bounded continuous functions on X, equipped with the uniform norm; it is easy to see that the normed space $C^*(X)$ is complete. A normed space which is complete (for Cauchy filters) is called a *Banach space*. Every normed space has a completion, which is defined using the following well-known results:

. *Let (X, d) be a metric space and $a \in X$. For every $x \in X$ we put $j_a(x) = (d(x, y) - d(a, y))_{y \in X}$: the mapping $j_a : X \to C^*(X)$ is isometric and the closure $\overline{j_a[X]}$ of $j_a[X]$ in $C^*(X)$ is a completion of X.*

. *If $(X, \|.\|)$ is a normed space, if $a = 0_E$, then there is a unique Banach space structure on $\overline{j_a[X]}$ such that j_a is linear.*

We define the Cauchy-completion of a semi-normed space (E, N) as the Cauchy-completion of the quotient normed space E/N.

We say that a normed space $(E, \|.\|)$ satisfies the *property of Projection* (in abbreviated form *property* **P**) if and only if:

P. *For every nonempty complete convex subset C of E, and every $a \in E\backslash C$, there exists $c \in C$ such that $\|c - a\| = \inf\{\|x - a\| ; x \in C\}$.*

We will use the following result (see [1] pages 194-195): uniformly convex normed spaces satisfy the property **P**, and in fact, for these spaces, the element c is unique.

Remark. In set-theory with the axiom of choice, one can prove James's sup theorem:

"A Banach space is reflexive iff every continuous linear functional attains its norm on the closed unit ball of E."

and one can deduce, (see [12]), that in **ZFC**, a Banach space satisfies the property of projection if and only if it is reflexive.

3 Gâteaux-differentiable and uniformly smooth norms

If O is an open subset of a topological vector space E, we say that a mapping $f : O \to \mathbb{R}$ is *Gâteaux-differentiable* at a point $a \in O$ if for all $h \in E$ the quotient $\frac{f(a+th)-f(a)}{t}$ admits a limit when $t \mapsto 0, t \in \mathbb{R}^*$. We note that, if f is convex, then for every $a \in O$ and $h \in E$ the previous quotient has a limit when $t \mapsto 0, t > 0$ and $t \mapsto 0, t < 0$ so, we introduce the following notations: $G_f^+(a,h) = \lim_{t\to 0^+} \frac{f(a+th)-f(a)}{t}$ and $G_f^-(a,h) = \lim_{t\to 0^-}(\frac{f(a+th)-f(a)}{t})$; morover, the functional $G_f^+(a,.)$ is continuous and sublinear, the functional $-G_f^-(a,.)$ is continuous and sublinear, for every $a \in O$ and $h \in E$ we have the following equality $G_f^+(a,h) = -G_f^-(a,-h)$ and the following extra inequality: $G_f^-(a,h) \leq G_f^+(a,h)$.

If N is a semi-norm on a vector space E that is Gâteaux-differentiable at a point a we put $G(a,h) = \lim_{t\to 0} \frac{N(a+th)-N(a)}{t}$; in this case, (see [1] page 179), the function $h \mapsto G(a,h)$ is a continuous linear functional on E, with $G(a,a) = \|a\|$ and $\|G(a,.)\| = 1$. We say that a semi-normed space (E,N) is *Gâteaux-differentiable* if and only if the semi-norm N is Gâteaux-differentiable at every point a such that $N(a) \neq 0$. It is equivalent to say that every finite dimensional subspace of E is Gâteaux-differentiable; it implies that for every $a \in E\backslash\{0\}$, the mapping $G(a,.)$ is the unique supporting functional at point a: here we say that a continuous linear functional $f : E \to \mathbb{R}$ is a *supporting functional at* point a if and only if $\|f\| = 1$ and $f(a) = \|a\|$.

The semi-normed space (E,N) is said to be *uniformly smooth* if and only if it is Gâteaux differentiable and if

$$\lim_{t\to 0^+}(\frac{N(a+th)-N(a)}{t} - \frac{N(a-th)-N(a)}{-t}) = 0$$

uniformly for $a,h \in E$ such that $N(a) = N(h) = 1$. The *modulus of smoothness of* the semi-normed space (E,N) is the function

$$\rho_N : t \mapsto \sup_{N(a)=N(b)=1} \{\frac{N(a+tb)+N(a-tb)}{2} - 1\}$$

It is easy to see that (E,N) is *uniformly smooth* if and only if $\lim_{t\to 0^+}(\frac{\rho_N(t)}{t}) = 0$. We say that a mapping $\rho : \mathbb{R}_+^* \to \mathbb{R}_+^*$ is a *witness of uniform smoothness* on the semi-normed space (E,N) if and only if $\rho_N \leq \rho$ and $\lim_{t\to 0^+}(\frac{\rho(t)}{t}) = 0$. If the space (E,N) is uniformly smooth then ρ_N is the best witness of uniform smoothness on (E,N).

We now recall *Lindenstrauss' duality formula* (see [10] page 61): if (E,N) is a finite dimensional semi-normed space, if δ_N is the modulus of convexity on E, then the modulus of smoothness on the continuous dual (E',N') is $\rho_{N'} : t \mapsto \sup_{0<\epsilon<2}\{\epsilon\frac{t}{2} - \delta_N(\epsilon)\}$; morover, if (E,N) is uniformly convex then we have $\lim_{t\mapsto 0}(\frac{\rho_{N'}(t)}{t}) = 0$ hence (see [5] page 132) the space (E',N') is uniformly smooth with modulus $\rho_{N'}$. It is easy to prove the following slight modification: if (E,N) is uniformly convex and if δ is only a witness (and not the modulus) of the uniform convexity on (E,N), then, the mapping $\rho : t \mapsto \sup_{0<\epsilon<2}\{\epsilon\frac{t}{2} - \delta(\epsilon)\}$ is such that $\rho_{N'} \leq \rho$ and, since $\lim_{\epsilon\mapsto 0}(\delta(\epsilon)) = 0$ a slight modification of the proof of [1] page 209-210 leads to $\lim_{t\mapsto 0}(\frac{\rho(t)}{t}) = 0$ so (E',N') is uniformly smooth with witness ρ.

4 The spaces $L^p(\mu), 1 < p < +\infty$ are uniformly convex and uniformly smooth

The Lebesgue measure on $\mathbb{R}$ cannot be proved to be σ-additive in **ZF** since there exists a model of **ZF** in which $\mathbb{R}$ is a countable union of countable sets (see [9] page 142-144). Anyhow, we now show that it is possible to define the spaces $L^p(\mu)$ without choice, when μ is a finitely additive measure on a Boolean algebra of sets.

4.1 Integration on a measured space.

Let $\mathcal{B}$ be a Boolean algebra; we say that a mapping μ from $\mathcal{B}$ to $\mathbb{R}_+ \cup \{+\infty\}$ is a *measure* on $\mathcal{B}$ if, for every $x, y \in \mathcal{B}$ such that $x \wedge y = 0$, we have $\mu(x \vee y) = \mu(x) + \mu(y)$. If E is a set, if $\mathcal{B}$ is a sub-algebra of the Boolean algebra $\mathcal{P}(E)$, and if μ is a measure on $\mathcal{B}$, we say that $(E, \mathcal{B}, \mu)$ is a *measured space*; we say that a function $f : E \to \mathbb{C}$ is *simple* if there exist disjoint subsets $A_1, \dots, A_m \in \mathcal{B}$ and scalars $\lambda_1, \dots, \lambda_m \in \mathbb{C}$ such that $f = \sum_{i=1}^m \lambda_i 1_{A_i}$ where 1_{A_i} is the characteristic function of the set A_i. Among all the decompositions, we distinguish the *canonical decomposition*, for which the λ_i are distinct reals.

Though the algebra $\mathcal{B}$ is not stable for countable unions and μ is not countably additive, it is possible to define the integral of a simple function as usual:

-if $f : E \to \mathbb{R}_+$ is a positive simple function, and if $f = \sum_{i=1}^m \lambda_i 1_{A_i} = \sum_{i=1}^n \nu_i 1_{B_i}$ are two decompositions of f with $\lambda_1, \dots, \lambda_m, \nu_1, \dots, \nu_n \in \mathbb{R}_+$, we have $\sum_{i=1}^m \lambda_i \mu(A_i) = \sum_{i=1}^n \nu_i \mu(B_i)$ and we define $\int f.d\mu = \sum_{i=1}^m \lambda_i \mu(A_i)$ with the usual conventions: $0 \times (+\infty) = 0$, if $\lambda > 0$ then $\lambda \times (+\infty) = +\infty$ and $(+\infty) \times (+\infty) = +\infty$;

-if $f : E \to \mathbb{R}$ is a simple function, and if $\int \mid f \mid .d\mu < +\infty$ we put $\int f.d\mu = \int f^+.d\mu - \int f^-.d\mu$ where $f^+ = \sup(f, 0)$ and $f^- = \sup(-f, 0)$.

The set $\mathcal{E}(E, \mathcal{B}, \mu)$ of simple μ-integrable functions is a vector subspace of $\mathbb{R}^E$ and the mapping $f \mapsto \int f.d\mu$ is linear on $\mathcal{E}(E, \mathcal{B}, \mu)$; hence, for every real number $p \geq 1$, the function $N_p : f \mapsto (\int \mid f \mid^p .d\mu)^{\frac{1}{p}}$ is a semi-norm on $\mathcal{E}(E, \mathcal{B}, \mu)$; we call $L^p(E, \mathcal{B}, \mu)$ the Banach space which is the Cauchy-completion of the semi-normed space $(\mathcal{E}(E, \mathcal{B}, \mu), N_p)$ and we denote by $\mid . \mid_p$ the norm on the Banach space $L^p(E, \mathcal{B}, \mu)$.

If $p > 1$ and if $q = \frac{p}{p-1}$ then for all $a, b \in \mathbb{C}$ we have:

-if $2 \leq p < +\infty$ then $\mid a + b \mid^p + \mid a - b \mid^p \leq 2^{p-1}(\mid a \mid^p + \mid b \mid^p)$

-if $1 < p \leq 2$ then $\mid a + b \mid^p + \mid a - b \mid^p \leq 2(\mid a \mid^p + \mid b \mid^p)$

hence the following Clarkson's inequalities hold for all $f, g \in \mathcal{E}(E, \mathcal{B}, \mu)$, and by density, they hold for all $f, g \in L^p(E, \mathcal{B}, \mu)$:

-if $2 \leq p < +\infty$ then $\mid \frac{f+g}{2} \mid_p^p + \mid \frac{f-g}{2} \mid_p^p \leq \frac{|f|_p^p + |g|_p^p}{2}$

-if $1 < p \leq 2$ then $\mid \frac{f+g}{2} \mid_p^q + \mid \frac{f-g}{2} \mid_p^q \leq (\frac{|f|_p^p + |g|_p^p}{2})^{\frac{1}{p-1}}$

hence the Banach space $L^p(\mu)$ is uniformly convex: for $2 \leq p < +\infty$ we obtain the following witness of uniform convexity: $\delta_p(\epsilon) = 1 - (1 - (\frac{\epsilon}{2})^p)^{\frac{1}{p}}$ and for $1 < p \leq 2$ we obtain the following witness: $\delta_p(\epsilon) = 1 - (1 - (\frac{\epsilon}{2})^q)^{\frac{1}{q}}$. These two formula lead to the same formula $\delta_p(\epsilon) = 1 - (1 - (\frac{\epsilon}{2})^r)^{\frac{1}{r}}$ where $r = max(p, q)$. We note that if $p = 2$ the two formula give $\delta_2(\epsilon) = 1 - \sqrt{1 - \frac{\epsilon^2}{4}}$ which is the modulus of convexity for Hilbert spaces.

4.2 Examples of measured spaces.

-For every set I, the Banach space $l^p(I)$ is uniformly convex and a witness is δ_p, because the counting measure on I is a (countably additive) measure on the algebra $\mathcal{P}(I)$.

-Let $\mathbb{B}$ be the Boolean algebra $\mathcal{P}(\{0,1\})$ and let μ be the uniform probability on $\mathbb{B}$; for every set I, let $\mathbb{B}_I$ be the coproduct $\otimes_{i\in I}\mathbb{B}$ and let μ be the coproduct measure $\otimes_{i\in I}\mu_i$ on $\mathbb{B}_I$; then the normed space $L^p(\{0,1\}^I, \mathbb{B}_I, \mu_I)$ is uniformly convex.

-If p is a real number such that $p \geq 1$, if $m \in \omega$ and $1 \leq m$, for all positive real numbers $\alpha_1, \ldots, \alpha_m$, we call $l_p^m(\alpha_1, \ldots, \alpha_m)$ the space $\mathbb{R}^m$ equipped with the following semi-norm $N : (x_1, \ldots, x_m) \mapsto \sum_{i=1}^m \alpha_i \mid x_i \mid^p$. The semi-normed space $l_p^m(\alpha_1, \ldots, \alpha_m)$ is in fact the space $\mathcal{E}(m, \mathcal{P}(m), \mu)$ associated to the measured space $(m, \mathcal{P}(m), \mu)$ where μ is the measure on the set $m = \{0, \ldots, m-1\}$ such that $\mu(\{i\}) = \alpha_i$. Hence the semi-normed space $l_p^m(\alpha_1, \ldots, \alpha_m)$ is uniformly convex and δ_p witnesses this uniform convexity.

4.3 The spaces $L^p(\mu), 1 < p < +\infty$ are uniformly smooth.

-Let p be a real number such that $p > 1$, let $m \in \omega$ such that $1 \leq m$, and let $\alpha_1, \ldots, \alpha_m \in \mathbb{R}_+$. Let $q = \frac{p}{p-1}$. The continuous dual of the semi-normed space $l_q^m(\alpha_1, \ldots, \alpha_m)$ is the semi-normed space $l_p^m(\alpha_1, \ldots, \alpha_m)$ so, by Lindenstrauss' duality formula, the space $l_p^m(\alpha_1, \ldots, \alpha_m)$ is uniformly smooth and the function $\rho_p : t \mapsto \sup_{0<\epsilon<2}\{\epsilon\frac{t}{2} - \delta_q(\epsilon)\}$ is a witness of this uniform smoothness. With some calculus, we obtain that $\rho_p(t) = -1 + (1+t^s)^{\frac{1}{s}}$ with $s = min(p,q)$.

-Hilbert spaces are Gâteaux-differentiable: just take $G(a,h) =< \frac{a}{\|a\|}, h >$; in fact, the continuous dual of a finite dimensional Hilbert space is a Hilbert space so every Hilbert space is uniformly smooth with modulus of smoothness $\rho_2(t) = \sup_{0<\epsilon<2}\{\epsilon\frac{t}{2} - \delta_2(\epsilon)\} = -1 + \sqrt{1+t^2}$.

-If p is a real number such that $1 < p < +\infty$, if $(E, \mathcal{B}, \mu)$ is a measured space, then the semi-normed space $\mathcal{E}(E, \mathcal{B}, \mu)$ is Gâteaux-differentiable and, for all $a, h \in \mathcal{E}(E, \mathcal{B}, \mu)$, if $N_p(a) \neq 0$ we have $G(a,h) = (N_p(a))^{1-p}\ \mu(\mid a \mid^{p-1}\ sgn(a)\ h)$ where sgn is the *signum* function. If $\mathcal{B}$ is finite then $(\mathcal{E}(E, \mathcal{B}, \mu), N_p)$ is isometric to $l_p^m(\alpha_1, \ldots, \alpha_m)$ where m is the number of atoms of $\mathcal{B}$ and $\alpha_1, \ldots, \alpha_m$ are the measures of these atoms. If F is a finite dimensional subspace of $(\mathcal{E}(E, \mathcal{B}, \mu), N_p)$ then F is isometric to a subspace of $(\mathcal{E}(E, \mathcal{A}, \mu), N_p)$ where $\mathcal{A}$ is the finite subalgebra of $\mathcal{B}$ generated by the elements of $\mathcal{B}$ which occur in the decompositions of elements of F. Hence every finite dimensional subspace of $\mathcal{E}(E, \mathcal{B}, \mu)$ is isomorphic to some subspace of $l_p^m(\alpha_1, \ldots, \alpha_m)$, hence the semi-normed space $\mathcal{E}(E, \mathcal{B}, \mu)$ is uniformly smooth with a smoothness witness $t \to \rho_p(t) = -1 + (1+t^s)^{\frac{1}{s}}$ with $s = min(p,q)$ so its completion $L^p(E, \mathcal{B}, \mu)$ is also uniformly smooth with the same witness of smoothness.

5 Gâteaux-differentiability + property **P** imply the Mazur property

A topological vector space E is said to satisfy the *Mazur property* (in abbreviated form **MP**) if, for every nonempty closed convex subset C of E, and every point $a \in E \backslash C$, there exists a continuous linear functional $f : E \to \mathbb{R}$ such that $f(a) < \inf_C f$.

We recall that the *weak topology* on E is the topology which is generated by the sets $\{x \in E / f(x) < \alpha\}$ where $f \in E'$ and $\alpha \in \mathbb{R}$. If E satisfies the *Mazur property*, then every closed convex subset C of E is weakly closed. The converse is also true thanks to the following lemma:

Lemma 1. *Let E be a topological vector space, let C be a convex subset of E and let $a \in E\backslash C$. If there are $f_1, ..., f_m \in E'\backslash\{0\}$ and $\alpha_1, ..., \alpha_m \in \mathbb{R}$ such that the weak open set $O = \bigcap_{i=1}^m \{x \in E / f_i(x) < \alpha_i\}$ satisfies $O \cap C = \emptyset$ and $a \in O$ then there exists $f \in E'$ such that $f(a) < \inf_C f$.*

proof. Let $V = \bigcap_{i=1}^m Ker(f_i)$, let F be a finite dimensional subspace of E such that $V \oplus F = E$ and let $p : E \to F$ be the projection on F with kernel V. Let $K = p[C]$ and $U = p[O]$; the convex subsets K and U are disjoint in F and U is open in F hence, since F is finite dimensional, there is $g \in F'$ such that $g[U] < g[K]$ and we deduce that $g(p(a)) < \inf_K g$. Let $f = g \circ p$; we now have $f(a) < \inf_C f$. $\square$

The space E is said to satisfy the *effective Mazur property* if, there is a mapping which associates to every nonempty closed convex subset C of E, and every point $a \in E\backslash C$ continuous linear functional $f : E \to \mathbb{R}$ such that $f(a) < \inf_C f$.

It has been proved by Ishihara (see [8]), in a constructive way, that if E is a *separable* uniformly convex Banach space, if f is a continuous linear functional which is defined on a Gâteaux-differentiable subspace of E, then there is a unique functional $g : E \to \mathbb{R}$ which extends f and such that $\|g\| = \|f\|$. Following Ishihara's idea, we now prove, that Gâteaux-differentiable uniformly convex Banach spaces satisfy the effective Mazur property.

Lemma 2. *Let $(E, \|.\|)$ be a normed space, let C be a nonempty closed convex subset of E and $a \in E\backslash C$, let $\rho = \inf\{\|x - a\| ; x \in C\}$. If there exists $x \in C$ such that $\|x - a\| = \rho$ and if $\|.\|$ is Gâteaux-differentiable at point $x - a$, then the affine functional $f : E \to \mathbb{R}$ which associates to every $z \in E$ the number $G(x - a, z - a)$ satisfies $f[\overline{B}(a, \rho)] \leq \rho, f(a) = 0,$ and $f[C] \geq \rho$.*

proof. Since $G(x-a, .)$ is a linear functional with norm 1, it is clear that f is affine and that $f[\overline{B}(a, \rho)] \leq \rho$ and $f(a) = 0$. Let $z \in C$. For every real number $t \in [0, 1]$ we have $x + t(z - x) = (1 - t)x + tz \in C$ hence $\|x + t(z - x) - a\| \geq \rho$ and since the norm $\|.\|$ is Gâteaux-differentiable at $x - a$, we get: $f(z) = G(x - a, x - a) + G(x - a, z - x) = \|x - a\| + \lim_{t\to 0^+} \frac{\|x-a+t(z-x)\|-\|x-a\|}{t} \geq \|x - a\| = \rho$ hence we deduce that $f[C] \geq \rho$. $\square$

Theorem 1. *i) If a Gâteaux-differentiable normed space satisfies property* **P** *then it also satisfies the Mazur property.*

ii) Gâteaux-differentiable uniformly convex Banach spaces satisfy the effective Mazur property.

proof. i) Let E be a Gâteaux-differentiable normed space which satisfies property **P**. Let C be a nonempty closed convex subset of E and $a \in E\backslash C$. Using property **P**, let $x \in C$ such that $\inf\{\|z - a\| ;\ z \in C\} = \|x - a\|$. The linear functional $z \mapsto G(x-a, z)$ satisfies $f(a) = G(x-a, a)$ hence $f[C] \ \geq\ \rho+G(x-a, a)\ >\ G(x-a, a)$

ii) Let E be a Gâteaux-differentiable uniformly convex Banach space, let C be a nonempty closed convex subset of E and $a \in E\backslash C$: since C is complete (for Cauchy filters) let x be the unique element in C such that $\inf\{\|z - a\| ; z \in C\} = \|x - a\|$. Let f be the linear functional $G(x - a, .)$. Then $g = f_{|E}$ is a continuous linear functional on E such that $g(a) < \inf_C g$. $\square$

6 The geometrical Hahn-Banach properties and the continuous Hahn-Banach property

We now show that the classical geometrical forms of the Hahn-Banach axiom on a topological vector space E are equivalent to the *continuous* Hahn-Banach property on E.

Lemma 3. *Let E be a topological vector space. i) The continuous Hahn-Banach property on E is equivalent to the following property:*

For every continuous sublinear functional $p : E \to \mathbb{R}$, there exists a linear functional $f : E \to \mathbb{R}$ such that $f \leq p$.

ii) The effective **CHBP** *on E is equivalent to the following form:*

There is a mapping which associates to every continuous sublinear functional $p : E \to \mathbb{R}$ a linear functional $f : E \to \mathbb{R}$ such that $f \leq p$.

proof. see [6]. $\square$

Lemma 4. *Let E be a vector space, let $p : E \to \mathbb{R}$ be a sublinear functional on E, and let $f : E \to \mathbb{R}$ be a linear functional on E. Let $O = \{x \in E/p(x) < -1\}$, let $C = \{x \in E\, ; p(x) \leq -1\}$ and let $\alpha = \sup_C f$. If $O \neq \emptyset$ and if $\forall x \in O\ f(x) < 0$ then $\alpha < 0$ and $f \leq -\alpha p$.*

proof. First we note that $\alpha \leq 0$ because if $x \in C$, then $2x \in O$ hence $f(x) < 0$. For every $x \in E$ the following implications hold: $p(x) \leq -1 \;\Rightarrow\; f(x) \leq \alpha$; $\forall t > 0\ (p(x) \leq -t \;\Rightarrow\; f(x) \leq \alpha t)$ Hence we obtain:

$$\forall x \in E\ (p(x) < 0 \;\Rightarrow\; f(x) \leq -\alpha p(x))$$

Let $\epsilon > 0$. Let $z \in C$ such that $f(z) \geq \alpha - \epsilon$. Let $a = \frac{z}{-p(z)}$. We have $p(a) = -1$ and $f(a) \geq \frac{\alpha-\epsilon}{-p(z)} \geq \alpha - \epsilon$ because $\alpha - \epsilon < 0$ and $p(z) \leq -1$. For every $x \in E$ such that $p(x) \leq 0$ we have: $p(x + a) \leq p(x) + p(a) \leq -1$ hence $x + a \in C$ so $f(x+a) \leq \alpha$ therefore $f(x) \leq -f(a)+\alpha \leq \epsilon$. This last inequality is true for every $\epsilon > 0$ hence $f(x) \leq 0$. Hence we deduce:

$$\forall x \in E\ (p(x) \leq 0 \;\Rightarrow\; f(x) \leq 0)$$

Let $x \in E$ such that $p(x) = 1$: we have $p(x+a) \leq 1+p(a) = 0$ hence $f(x+a) \leq 0$ so $f(x) \leq -f(a) \leq \epsilon - \alpha$; this last inequality is true for every $\epsilon > 0$ whence $f(x) \leq -\alpha$. Hence we obtain:

$$\forall x \in E\ (p(x) > 0 \;\Rightarrow\; f(x) \leq -\alpha p(x))$$

and finally:

$$\forall x \in E\ f(x) \leq -\alpha p(x)$$

We have $\alpha \neq 0$ because if $\alpha = 0$ then $f \leq 0$ hence $f = 0$ and this is contradictory! $\square$

We now consider the following properties on a topological vector space E:

$\mathbf{HB}_1$. *If C is an affine subspace of E and if O is an nonempty open convex subset in E such that $C \cap O = \emptyset$ then there exists a linear functional f on E such that $\forall x \in O\ f(x) < \inf_C f$.*

$\mathbf{HB}_2$. *If C is a nonempty convex subset of E and if O is a nonempty open convex subset in E such that $C \cap O = \emptyset$ then there exists a linear functional f on E such that $\forall x \in O\ f(x) < \inf_C f$.*

$\mathbf{HB}_3$. *If $a \in E$ and if O is a nonempty open convex subset in E such that $a \notin O$ then there exists a linear functional f on E such that $\forall x \in O\ f(x) < f(a)$.*

$\mathbf{HB}_4$. *If C and O are two nonempty disjoint convex subsets of E, and if O is open then there exist a linear functional f on E such that $f[O] < f[C]$.*

$\mathbf{HB}_5$. *If C and O are two nonempty disjoint open convex subsets of E, then there exist a linear functional f on E such that $f[O] < f[C]$.*

$\mathbf{HB}_6$. *If C is a nonempty closed convex subset of E and if K is a nonempty compact convex subset of E then there exists a linear functional f on E such that $\sup_K f < \inf_C f$.*

Theorem 2. *On every topological vector space E, the following properties are equivalent:* **CHBP**, $\mathbf{HB}_1$, $\mathbf{HB}_2$, $\mathbf{HB}_3$, $\mathbf{HB}_4$, $\mathbf{HB}_5$.

proof. **CHBP** $\Rightarrow$ $\mathbf{HB}_1$ and $\mathbf{HB}_1 \Rightarrow \mathbf{HB}_2$: see [4], EVT II p. 39-41. The sublinear functional p used by Bourbaki is dominated by 1 on the nonempty open set O, hence p is continuous (see [4], EVTII.20, proposition 21). The proof of $\mathbf{HB}_2 \Rightarrow \mathbf{HB}_3$ is obvious. $\mathbf{HB}_3 \Rightarrow \mathbf{HB}_4$: we apply $\mathbf{HB}_3$ to the open convex set $O - C$ which does not contain 0. The implication $\mathbf{HB}_4 \Rightarrow \mathbf{HB}_5$ is obvious.

$\mathbf{HB}_5 \Rightarrow$ **CHBP**: Let $p : E \to \mathbb{R}$ be a continuous sublinear functional. We want to define a linear functional $f : E \to \mathbb{R}$ such that $f \leq p$. If $p \geq 0$ then we take $f = 0$ else, let $O = \{x \in E/p(x) < -1\}$ and $C = \{x \in E/p(x) > -1\}$; the open convex sets are nonempty and disjoint so let $f : E \to \mathbb{R}$ be a linear functional such that $f[O] < f[C]$; we deduce that $\forall x \in O\ f(x) < 0$. We put $C = \{x \in E; p(x) \leq -1\}$ and $\alpha = \sup_C f$ and thanks to lemma 4 we obtain $\alpha < 0$ and $\frac{f}{-\alpha} \leq p$. □

Theorem 3. *Let E be a topological vector space. If E satisfies the (resp. effective) Mazur property, then E satisfies the (resp. effective) continuous Hahn-Banach property.*

proof. The proof is similar: let $p : E \to \mathbb{R}$ be a continuous sublinear functional; if $p \geq 0$ then we take $f = 0$ and we are done! Else, $C = \{x \in E\ /\ p(x) \leq -1\}$ is a nonempty closed convex subset of E such that $0 \notin C$; thanks to the Mazur property on E, we take a linear functional $f : E \to \mathbb{R}$ such that $\sup_C f = -1$ and we get $f \leq p$ thanks to lemma 4. □

Corollary 1. *Gâteaux-differentiable uniformly convex Banach spaces satisfy the effective continuous Hahn-Banach property.*

proof. See theorem 1 and theorem 3. □

In particular, if p is a real number such that $1 < p < +\infty$, if $(E, \mathcal{B}, \mu)$ is a measured space, then $L^p(E, \mathcal{B}, \mu)$ satisfies the effective **CHBP** because it is both uniformly convex and uniformly smooth. We will see in section 10 that, if $\mu(1_\mathcal{B}) < +\infty$ then $L^1(E, \mathcal{B}, \mu)$ also satisfies the effective **CHBP**, though it is neither uniformly convex nor Gâteaux- differentiable.

Corollary 2. *If E is a locally convex topological vector space, then the following properties are equivalent on E:* **CHBP**, $\mathbf{HB}_6$, **MP**.

proof. **CHBP** $\Rightarrow$ $\mathbf{HB}_6$: see [4], EVT II.41, proposition 4: here we use the fact that E is locally convex. The implication $\mathbf{HB}_6 \Rightarrow$ **MP** is clear and **MP** $\Rightarrow$ **CHBP** comes from theorem 3. $\square$

If a topological vector space satisfies the Mazur property, then all its subspaces also satisfy this property. Therefore, and thanks to corollary 2, we obtain the following result:

Corollary 3. *If E is a locally convex topological vector space that satisfies the continuous Hahn-Banach property, then all its subspaces also satisfy this property.*

In particular, if a normed space satisfies the **CHBP** then its subspaces also satisfy the **CHBP**.

7 Ekeland's variational principle

We first prove that **DC** is equivalent to *Ekeland's variational principle* and that it implies the *principle of uniform boundedness* on Banach spaces. We will use these two principles to prove the continuous Hahn-Banach property on Gâteaux-differentiable Banach spaces.

Let us consider the following weak form of the axiom of dependent choices:

DMC. *(axiom of Dependent Multiple Choice) If E is a nonempty set and if R is a binary relation on E which satisfies the condition $\forall x \in E\ \exists y \in E\ xRy$, then there is a sequence $(F_n)_{n\in\omega}$ of finite nonempty subsets of E such that $\forall n \in \omega\ \forall x \in F_n\ \exists y \in F_{n+1}\ xRy$.*

It is known that the axiom **DC** is equivalent to the statement "Complete metric spaces are Baire spaces" (see [2]) and the axiom **DMC**, which was introduced by Blass (see [3]), is equivalent (see [6]) to the following statement:

BC. *(Baire for Compact Hausdorff spaces) Compact Hausdorff spaces are Baire.*

It is easy to see that **DC** implies **DMC**, and it is known that, in set theory without the axiom of foundation (**ZFA**), the axiom **DMC** does not imply **DC** because Fraenkel's second model of **ZFA** satisfies **DMC** but does not satisfy **DC** (see [9]). As far as we know, the question "Does **DMC** imply **DC** in set theory **ZF**" is open.

On the other hand, the axiom **HB** does not imply **DMC** since the stronger axiom **BPI**, the Boolean Prime Ideal axiom, does not imply **BC**: in fact, **BPI+BC** is equivalent to **BPI+DC** (see [7]) and **BPI** does not imply **DC** since Cohen's first model satisfies **BPI+**$\neg$ **DC**.

If every Cauchy sequence of a metric space (E, d) has a limit point we say that (E, d) is *sequentially complete*. We now recall Ekeland's variational principle:

Ek. *If* (E, d) *is a nonempty sequentially complete metric space, if* $f : E \to \mathbb{R}$ *is lower semi-continuous and bounded bellow, if* $\epsilon \in \mathbb{R}_+^*$, *then there exists* $a \in E$ *such that:* $\forall x \in E\ f(a) \leq f(x) + \epsilon d(x, a)$.

and we introduce the following weaker axiom:

WEk. *If* C *is a nonempty sequentially complete convex subset of a normed space* $(E, \|.\|)$ *and if* $f : C \to \mathbb{R}$ *is lower semi-continuous, convex and bounded bellow, if* $\epsilon \in \mathbb{R}_+^*$, *then there exists* $a \in C$ *such that:* $\forall x \in E\ f(a) \leq f(x) + \epsilon d(x, a)$.

Lemma 5. *i)* **DC** *implies* **Ek**.
ii) **DMC** *implies* **WEk**.

proof. i) We follow the classical proof and we define by recursion a sequence $(x_n)_{n\in\omega}$ of points of E and a sequence $(E_n)_{n\in\omega}$ of subsets of E. Let $x_0 \in E$ and $E_0 = E$. Using axiom **DC**, we define a sequence $(E_n, x_n)_{n\in\omega}$ such that for every $n \in \omega$ the following three conditions are satisfied: $E_{n+1} = \{y \in E / f(y) \leq f(x_n) - \epsilon d(x_n, y)\}$, $x_{n+1} \in E_{n+1}$ and $f(x_{n+1}) \leq \frac{f(x_n)+\inf\{f(x);\, x\in E_{n+1}\}}{2}$. The sequence $(f(x_n))_{n\in\omega}$ is decreasing and it is bounded below hence it converges; since we have $d(x_n, x_{n+1}) \leq \frac{f(x_n)-f(x_{n+1})}{\epsilon}$ we deduce that for every $n, p \in \omega$ we have: $d(x_n, x_{n+p}) \leq \frac{f(x_n)-f(x_{n+p})}{\epsilon}$, hence the sequence $(x_n)_{n\in\omega}$ is a Cauchy sequence: let $x = \lim_{n\to+\infty}(x_n)$. We have $\bigcap_{n\in\omega} E_n = \{x\}$: in fact, $x \in \bigcap_{n\in\omega} E_n$ because the E_n are closed and the sequence $(E_n)_{n\in\omega}$ is decreasing; and if $y \in \bigcap_{n\in\omega} E_n$, then we have for every $n \in \omega$:

$$2f(x_{n+1}) - f(x_n) \leq \inf_{E_{n+1}} f \leq f(y) \leq f(x_n) - \epsilon d(x_n, y)$$

It follows that:

$$\lim_{n\to+\infty} f(x_n) \leq f(y) \leq \lim_{n\to+\infty} f(x_n) - \epsilon d(x, y)$$

hence $d(x, y) = 0$ and $x = y$. Let $y \in E$ such that $y \neq x$; then, there exists $N \in \omega$ such that for all $n \geq N$, $f(x_n) < f(y) + \epsilon d(x_n, y)$; hence for $n \to +\infty$ we get $f(x) \leq f(y) + \epsilon d(x, y)$.

ii) We first take $x_0 \in E$ and $E_0 = E$, then, using **DMC**, for every $n \in \omega$, we put $E_{n+1} = \{y \in E / f(y) \leq f(x_n) - \epsilon d(x_n, y)\}$ and we take a finite nonempty subset $F \subseteq E_{n+1}$ such that for all $y \in F$ we have $f(y) \leq \frac{f(x_n)+\inf\{f(x);\, x\in E_{n+1}\}}{2}$; then we put $x_{n+1} = \frac{\sum_{y\in F} y}{|F|}$ where $|F|$ is the cardinal of F; we get $x_{n+1} \in E_{n+1}$ and $f(x_{n+1}) \leq \frac{f(x_n)+\inf\{f(x);\, x\in E_{n+1}\}}{2}$ because f and $d(x_n, .)$ are convex; the end of the proof goes through without changes. $\square$

In the spirit of "reverse mathematics", we now prove that the axiom **Ek** implies **DC**.

Theorem 4. Ek $\Leftrightarrow$ DC

proof. We follow Blair (see [2]). Let R be a binary relation on a nonempty set E such that $\forall x \in E\ \exists y \in E\ xRy$. We denote by δ the discrete distance on E; we know that E^ω equipped with the usual distance: $d((x_n)_{n\in\omega}, (y_n)_{n\in\omega}) = \sum_{n\in\omega} \frac{\delta(x_n,y_n)}{2^n}$ is a complete metric space. Let α be a set such that $\alpha \notin E$ and $Z = E \cup \{\alpha\}$.

Let $P = \{(x_n)_{n\in\omega} \in Z^\omega; x_0 \neq \alpha \textit{ and } (\forall n \in \omega\ x_n = \alpha \textit{ or } x_n R x_{n+1})\}$: since P is a closed subset of E, P is a complete metric space.

We suppose that for every $(x_n)_{n\in\omega} \in P$, there exists $n \in \omega$ such that $x_n = \alpha$; for every $x = (x_n)_{n\in\omega} \in P$, we put $f(x) = \frac{1}{\lambda(x)}$ where $\lambda(x)$ is the first element of the set $\{n \in \omega / x_n = \alpha\}$. Using Ekeland's variational principle with $\epsilon = 1$, since f is continuous and bounded below by 0, let $m = (m_n)_{n\in\omega} \in P$ such that $\forall x \in P\ f(m) \leq f(x) + d(m, x)$; let $y \in P$ such that for all $k \in \omega$ such that $k < \lambda(m)$ we have $y(k) = m(k)$ and $\lambda(y) > \frac{2^{\lambda(m)}\lambda(m)}{2^{\lambda(m)}-\lambda(m)}$; then we have $f(m) > f(y) + d(m, y)$ hence a contradiction! □

8 The axiom DC implies the Mazur property on Gâteaux-differentiable Banach spaces

On a Gâteaux-differentiable normed space E, one can extend each linear functional f which is defined on a finite dimensional subspace F of E, to a linear functional g defined on E, with $\|g\| = \|f\|$: in fact, if $f \neq 0$, by compacity of the closed unit ball Γ of F, there exists $a \in \Gamma$ such that $f(a) = \|f\|$, and since $\|.\|$ is differentiable at a, we have $f = \|f\|\, G(a, .)_{|E}$, and the linear functional $g = \|f\|\, G(a, .)$ is defined on E, extends f, and $\|g\| = \|f\|$.

Our aim is now to prove the Mazur property for Gâteaux-differentiable Banach spaces, using the axiom of Dependent choices. We say that a normed space E satisfies the *bounded Mazur property if* for every nonempty *bounded* closed convex subset C of E and every $a \in E \backslash C$ there is $f \in E'$ such that $f(a) < \inf_C f$; using lemma 1, the bounded Mazur property on E is equivalent to say that every bounded closed convex subset of E is weakly closed.

We now consider the two following axioms:

SM. *Gâteaux-differentiable Banach spaces satisfy the Mazur property.*

SBM. *Gâteaux-differentiable Banach space satisfy the bounded Mazur property.*

Lemma 6. WEk ⇒ SBM

proof. Let $(E, \|.\|)$ be a Gâteaux-differentiable normed space: we prove the Mazur property on E. Let C be a nonempty closed bounded convex subset of E and let $a \in E\backslash C$. Let $\rho = d(a, C)$; let R be the diameter of the set C. For every $z \in C$ we put $f(z) = \|z - a\|$. Let $\epsilon = \frac{\rho}{2R}$. Using **WEk**, let $m \in C$ such that

$$\forall z \in C\ \ \|m - a\| \leq \|z - a\| + \epsilon\, \|z - m\|$$

Let $g = G(m - a, .)$ the Gâteaux-differential at point $m - a$. If $z \in C, h \in E$ and $t \in \mathbb{R}_+^*$ then

$$\|m + t(z - m) - a\| - \|m - a\| \geq -\epsilon t\, \|z - m\|$$

hence

$$\frac{\|m - a + t(z - m)\| - \|m - a\|}{t} \geq -\epsilon\, \|z - m\| \geq -\frac{\rho}{2}$$

so it follows that $g(z - m) \geq -\frac{\rho}{2}$ whence

$$g(z - a) = g(m - a) + g(z - m) \geq \frac{\rho}{2} > 0$$

we deduce that $g[C] \geq g(a) + \frac{\rho}{2} > g(a)$. □

We now recall the following *principle of uniform boundedness*:

. *If* $(u_i)_{i\in I}$ *is a family of continuous linear functionals on a normed space* E*, if* E *is a Baire space, if* C *is an open bounded subset of* E*, and if for every* $x \in C$ *we have* $\sup\{| u_i(x) |; i \in I\} < +\infty$ *then* $\sup_{x\in C; i\in I}\{| u_i(x) |; i \in I\} < +\infty$.

The proof is classical: let $u = \sup(| u_i |)$; the function u is sublinear and lower continuous; for every $n \in \omega$ we put $C_n = \{x \in C/u(x) \leq n\}$; we have $C = \bigcup_{n\in\omega} C_n$ and each C_n is closed; since C is open in a Baire space, C is also a Baire space so, let $n_0 \in \omega, a \in C$ and $\rho > 0$ such that $\forall x \in E$ ($\|x - a\| < \rho \Rightarrow u(x) \leq n_0$); since u is sublinear we deduce that there is $M \in \mathbb{R}$ such that $\forall x \in E$ $u(x) \leq M \|x\|$; since C is bounded we deduce that $\sup_{x\in C; i\in I}\{| u_i(x) |; i \in I\} < +\infty$.

If (E, d) is a metric space, if $a \in E$ and $\rho \in \mathbb{R}^*_+$, we denote by $B(a, \rho)$ the open ball $\{z \in E; d(a, z) < \rho\}$. We say that a subset A of a topological vector space E is *weakly bounded* if and only if for every $f \in E'$, the set $f[A]$ is bounded in $\mathbb{R}$.

Lemma 7. *Let* E *be a normed space. If for every* $x \in E$ *we have*

$$\|x\| = \sup\{u(x); u \in E' \text{ and } \|u\| < 1\}$$

and if E' *is a Baire space then every subset of* E *which is weakly bounded is strongly bounded.*

proof. Let A be a nonempty subset of E which is weakly bounded. Let $I = \bigcup_{a\in A} B(a, 1)$; the set I is open in E and weakly bounded in E. Let $C = \{u \in E'; \|u\| < 1\}$: since C is open in the Baire space E', the space C is also a Baire space. We now have

$$\sup\{\|i\| ; i \in I\} = \sup_{u\in C; i\in I}\{u(i)\} = \sup_{u\in C; i\in I}\{\delta_i(u)\}$$

where $\delta_i : E' \to \mathbb{R}$ is the evaluating form at i; using the principle of uniform boundedness, we deduce that I is strongly bounded so A is also strongly bounded. $\square$

Lemma 8. *Let* E *be a normed space such that each weakly bounded subset of* E *is strongly bounded. If* C *is a subset of* E*, and if for every* $n \in \omega$ *the set* $C_n = \{z \in C/ \|z\| \leq n\}$ *is weakly closed, then* C *is weakly closed.*

proof. Let $(I, \leq)$ be a poset and let $(x_i)_{i\in I}$ be a family of C such that $(x_i)_{i\in I}$ weakly converges to a point $x \in E$. For every $u \in E'$, the family $(u(x_i))_{i\in I}$ is weakly bounded hence strongly bounded: let $n \in \omega$ such that for every $i \in I$ we have $x_i \in C_n$; since C_n is weakly closed we deduce that $x \in C_n$ hence $x \in C$. $\square$

We deduce the following theorem:

Theorem 5. DC $\Rightarrow$ **SM**.

proof. Let C be a closed convex subset of a Gâteaux-differentiable Banach space E. Using **WEk**, which is a consequence of **DC**, for every $n \in \omega$ the set $\{z \in C/ \|z\| \leq n\}$ is weakly closed; using **DC**, the Banach space E' is a Baire space, and since E is Gâteaux-differentiable, for every $x \in E\backslash\{0\}$ there is a supporting functional at x hence $\|x\| = \sup\{u(x);\ u \in E' \text{ and } \|u\| < 1\}$ so, using the two previous lemmas, the set C is weakly closed. $\square$

Since **HB** implies the Mazur property on every normed space, and since **HB** does not imply **DC**, we deduce that **SM** does not imply **DC**. We do not know the answers to the following questions:

Question 1. *Does a Gâteaux-differentiable Banach space satisfy the Mazur property in* **ZF***? the bounded Mazur property?*

Question 2. *Does the bounded Mazur property on a normed space imply the Mazur property on this space?*

9 The Hahn-Banach property on separable spaces implies non-trivial measures on ω

We now show that there is a great difference between the continuous Hahn-Banach property, which holds on every separable normed space, and the Hahn-Banach property, which cannot be proved in **ZF+DC**, for separable spaces.

If a topological vector space E has a dense subset which is well-orderable, then E satisfies the continuous Hahn-Banach property, and the proof relies on some transfinite recursion and the following classical lemma:

Lemma 9. *If E is a vector space, if $p : E \to \mathbb{R}$ is a sublinear functional, if F is a subspace of E, if $a \in E \backslash F$, if $m = \sup_{x \in F}\{f(x) - p(x-a)\}$ and $M = \inf_{x \in F}\{p(x+a) - f(x)\}$ then $m \leq M$ and for every real $\alpha \in [m, M]$, the linear functional $g : E \oplus \mathbb{R}a \to \mathbb{R}$, which extends f and such that $g(a) = \alpha$, satisfies the inequality $g \leq p$.*

In particular, separable normed spaces satisfy the continuous Hahn-Banach property.

Let $\mathcal{B}$ be a Boolean algebra. We say that an element $a \in \mathcal{B}$ is an *atom of the Boolean algebra* $\mathcal{B}$ if a is a minimal element of $\mathcal{B}\backslash\{0_{\mathcal{B}}\}$. Let m be a measure on $\mathcal{B}$; if $a \in \mathcal{B}$ we say that a is an *atom of the measure* m if $m(a) > 0$ and for every $x \in \mathcal{B}, x < a \Rightarrow m(x) = 0$. We say that m is *purely atomic* if there is a nonempty set D of atoms of $\mathcal{B}$, and $(\lambda_i)_i \in l^1(D)$ such that $\forall x \in \mathcal{B}\ m(x) = \sum_{i \in D} \lambda_i \epsilon_i(x)$ where $\epsilon_i(x) = 1$ if $i \leq x$ else $\epsilon_i(x) = 0$. We say that the measure m is *unitary* if $m(1_{\mathcal{B}}) = 1$. For example, if $a = (a_i)_{i \in I} \in l^1(I)$ and if $\sum_{i \in I} \mid a_i \mid = 1$ then the mapping $m : \mathcal{P}(I) \to [0, 1]$ which associates to every subset A of I the real number $\sum_{i \in A} \mid a_i \mid$ is a unitary measure on I, which is purely atomic. The authors Pincus and Solovay (see [11]) have described a model of **ZF+DC** in which all unitary measures on ω are purely atomic; in this model, all ultrafilters on ω are trivial.

We now consider the following axiom:

HB$_\omega$. *Every separable normed space satisfies the Hahn-Banach property.*

Theorem 6. *The axiom* **HB$_\omega$** *implies that there exists a unitary measure on the set ω, without any atom.*

proof. We consider the following norm M on $l^\infty(\omega)$; for every $x = (x_k)_{k \in \omega} \in l^\infty(\omega)$ we put $M(x) = \sum_{k \in \omega} \frac{|x_k|}{(k+1)^2}$. The normed space $(l^\infty(\omega), M)$ is separable because $\mathbb{Q}^{(\omega)}$ is dense in $(l^\infty(\omega), M)$. For every $x = (x_k)_{k \in \omega} \in l^\infty(\omega)$ we put $p(x) = \overline{\lim}(x_k)$:

we obtain a sublinear functional $p : l^\infty(\omega) \to \mathbb{R}$; using axiom $\mathbf{HB}_\omega$, let $f : l^\infty(\omega) \to \mathbb{R}$ be a linear functional such that $f \leq p$. Since $p(1) = 1$ and $p(-1) = -1$ we have $f(1) = 1$. For every finite subset $A \subseteq \omega$ we have $p(1_A) = 0$ hence $f(1_A) = 0$. Hence f is in the continuous dual of $l^\infty(\omega)$ but $f \not\in l^1(\omega)$. For every subset $A \subseteq \omega$ we put $m(A) = f(1_A)$. Then the measure $m : \mathcal{P}(\omega) \to [0, 1]$ is unitary and has has no atoms. □

Corollary 4. *The Hahn-Banach property for separable normed spaces is not provable in* **ZF+DC**.

proof. The vector space $l^\infty(\omega)$ equipped with the previous norm M is separable; we take a model of **ZF+DC** where all unitary measures on ω are purely atomic: the normed space $(l^\infty(\omega), M)$ does not satisfy the Hahn-Banach property in this model. □

10 Which normed spaces satisfy the continuous Hahn-Banach property?

Topological vector spaces that satisfy the continuous Hahn-Banach property, obey the following rules:

-a/ Gâteaux-differentiable uniformly convex Banach spaces satisfy the **CHBP**;

-b/ Topological vector spaces which have a dense well-orderable subset satisfy the **CHBP**;

-c/ If a dense subspace of a topological vector space E satisfies the **CHBP**, then E also satisfies the **CHBP**;

-d/ If a topological vector space satisfies the Mazur property, then all its subspaces satisfy this property, hence if a locally convex topological vector space satisfies the **CHBP** then all its subspaces satisfy this property.

-e/ If two topological vector spaces E and F satisfy the **CHBP** then the topological vector space $E \oplus F$ also satisfies this property.

Here is an other rule:

Theorem 7. *Let E, F be two vector spaces and let $u : E \to F$ be a linear functional.*

i) If E satisfies the Hahn-Banach property then $u[E]$ also satisfies this property.

ii) If E and F are topological vector spaces, if E satisfies the continuous Hahn-Banach property and if u is continuous then $u[E]$ also satisfies this property.

proof. i) Let $p : u[E] \to \mathbb{R}$ be a sublinear functional. Then $q = p \circ u$ is a sublinear functional; thanks to the Hahn-Banach property on E, let $g : E \to \mathbb{R}$ be a linear functional such that $g \leq q$; for every $x \in Ker(u)$ we have $g(x) \leq q(x) = 0$ hence $g_{|\mathrm{Ker(u)}} = 0$ hence there exists a unique linear functional $f : u[E] \to \mathbb{R}$ such that $f \circ u = g$; it is clear that $f \leq p$.
ii) Proof similar to i). □

It follows that, if a normed space E satisfies the **CHBP**, and if F is a closed subspace of E, then the normed space E/F also satisfies the **CHBP**.

Let $(E, \mathcal{T})$ be a topological vector space, and let X be a dense subspace of E. If there is a topology $\mathcal{T}'$ on X such that $(X, \mathcal{T}')$ is a topological vector space that satisfies the **CHBP** (resp. effective **CHBP**), and if $\mathcal{T}_{|X} \subseteq \mathcal{T}'$, then the topological vector space $(E, \mathcal{T})$ also satisfies this property. In particular, for every set I, the space $c_0(I) = \{x \in \mathbb{R}^I \; ; \; \forall \epsilon > 0 \; \exists F \in \mathcal{P}_f(I) \; \forall i \in I \backslash F \; \mid x_i \mid \leq \epsilon\}$ equipped with the uniform norm $\|.\|$ satisfies the continuous Hahn-Banach property, because, the set $l^2(I)$ is dense in $c_0(I)$, and the Banach space $(l^2(I), N_2)$ is Gâteaux-differentiable and uniformly convex, and the norm N_2 is such that $\|.\| \leq N_2$.

For the same reasons, if $\mu(1_{\mathcal{B}}) < +\infty$ then $L^1(E, \mathcal{B}, \mu)$ also satisfies the effective **CHBP** because $\forall f \in \mathcal{E}(E, \mathcal{B}, \mu) \; N_1(f) \leq \sqrt{\mu(1_{\mathcal{B}})} N_2(f)$ so $L^2(E, \mathcal{B}, \mu)$ is a dense subspace of $L^1(E, \mathcal{B}, \mu)$ and $L^2(E, \mathcal{B}, \mu)$ satisfies the effective **CHBP**.

Corollary 5. *Let $(E, \|.\|)$ be a Banach space. If D is a dense subset of the closed unit ball of E, and if the normed space $l^1(D)$ satisfies the continuous Hahn-Banach property, then E also satisfies this property.*

proof. If $(\lambda_e)_{e \in S} \in l^1(S)$ then $\sum_{e \in S} \| \lambda_e.e \| < +\infty$; hence, since E is complete, we can define: $u((\lambda_e)_{e \in S}) = \sum_{e \in S} \lambda_e.e$. It is clear that $u : l^1(S) \to E$ is linear; it is also continuous since $\|u((\lambda_e)_{e \in S})\| = \left\|\sum_{e \in S} \lambda_e.e\right\| \leq \sum_{e \in S} \mid \lambda_e \mid$. Hence $\overline{u[l^1(D)]} = E$ and E satisfies the continuous Hahn-Banach property. $\square$

In particular, if E is a set such that $l^1(E)$ satisfies the continuous Hahn-Banach property, and if $\|.\|$ is a norm on E such that $(E, \|.\|)$ is a Banach space, then E also satisfies the **CHBP**. We know (see [11]), that in some models of **ZF+DC**, the continuous dual of $l^\infty(\omega)$ is $l^1(\omega)$, so the continuous Hahn-Banach property on $l^\infty(\omega)$ cannot be proved in **ZF+DC**; but there is a bijection between the sets $l^\infty(\omega)$ and $\mathbb{R}$, so the continuous Hahn-Banach property on $l^1(\mathbb{R})$ cannot be proved in **ZF+DC**.

If α is an ordinal which is not countable, it is known, (see [5] page 59), that there is no Gâteaux-differentiable norm on $l^1(\alpha)$ which is equivalent to the norm of $l^1(\alpha)$, but $l^1(\alpha)$ satisfies the continuous Hahn-Banach property because $\mathbb{Q}^{(\alpha)}$ is a dense well-orderable subset of $l^1(\alpha)$, hence there are normed spaces which satisfy the **CHBP**, but which do not have an equivalent Gâteaux-differentiable norm.

We have seen that, in some models of **ZF**, there are separable normed spaces without the Hahn-Banach property, so we ask the following question:

Question 3. *Is there a model of* **ZF** *in which there is a separable Banach space which does not satisfy the Hahn-Banach property?*

A positive answer to the next question would solve the question 3:

Question 4. *If a vector space satisfies the Hahn-Banach property, do its subspaces also satisfy this property?*

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