Rado's selection Lemma implies Hahn-Banach

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Rado' selection Lemma

Rado's selection Lemma implies Hahn-Banach

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RL

 $T_2 \Rightarrow RL$

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 $RL \Rightarrow H$

RL ⇒ HB:

Notation

For every set I, we denote by $fin^*(I)$ the set of non-empty finite subsets of I.

Rado' selection Lemma

Rado's selection Lemma implies Hahn-Banach

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idea

 $\mathsf{RL}\Rightarrow \mathsf{HE}$ proof

Notation

For every set I, we denote by $fin^*(I)$ the set of non-empty finite subsets of I.

Rado (1949) Axiomatic treatment of rank in infinite sets

RL: Given a family $(X_i)_{i\in I}$ of finite sets and a family $(\sigma_F)_{F\in fin^*(I)}$ such that for every $F\in fin^*(I)$, $\sigma_F\in \prod_{i\in F} X_i$, there exists $f\in \prod_{i\in I} X_i$ which "respects" $(\sigma_F)_{F\in fin^*(I)}$:

$$\forall F \in \mathit{fin}^*(I) \ \exists G \in \mathit{fin}^*(I) \ (F \subseteq G \ \mathsf{and} \ f_{\upharpoonright G} = \sigma_G)$$

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 $\forall F \in \mathit{fin}^*(I) \ \exists G \in \mathit{fin}^*(I) \ (F \subseteq G \ \mathsf{and} \ f_{\restriction G} = \sigma_G)$

Remark

"f respects $(\sigma_F)_{F \in fin^*(I)}$ " means that the set $\{G \in fin^*(I) : f_{\upharpoonright G} = \sigma_G\}$ is cofinal in the poset $(fin^*(I), \subseteq)$.

Tychonov implies **RL**

Rado's selection Lemma implies Hahn-Banach

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 $T_2 \Rightarrow RL$

;RL ⇒ T₂

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 $RL \Rightarrow H$

RL ⇒ HB proof We work in set-theory **ZF** (without the Axiom of Choice **AC**).

Tychonov implies RL

Rado's selection Lemma implies Hahn-Banach

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 $\mathsf{RL}\Rightarrow \mathsf{HB}$ proof

We work in set-theory **ZF** (without the Axiom of Choice **AC**).

The classical proof of **RL** relies on Tychonov's axiom for families of (finite) compact Hausdorff spaces:

 \mathbf{T}_2 : For every infinite family $(X_i)_{i \in I}$ of compact Hausdorff spaces, the topological product $\prod_{i \in I} X_i$ is compact. Thus:

$$\textbf{AC} \Rightarrow \textbf{T}_2 \Rightarrow \textbf{RL}$$

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 \mathbf{T}_2 : For every infinite family $(X_i)_{i \in I}$ of compact Hausdorff spaces, the topological product $\prod_{i \in I} X_i$ is compact. Thus:

$$AC \Rightarrow T_2 \Rightarrow RL$$

Consider the following consequence of T_2 :

AC^{fin}: "Every infinite family of finite non-empty sets has a non-empty product."

Blass noticed that:

$$\mathsf{T}_2 \Leftrightarrow (\mathsf{RL} + \mathsf{AC}^\mathsf{fin})$$



Rado's selection Lemma implies Hahn-Banach

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$$T_2 \Rightarrow RL$$

$$iRL \Rightarrow T_2$$

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 $RL \Rightarrow H$

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Let *R* be the following bi-partite graph ("play-boy graph"):

$$R = \{(i+1,i) : i \in \mathbb{N}\} \cup \{(0,i) : i \in \mathbb{N}\}$$

For every $F \in \mathit{fin}^*(\mathbb{N})$, consider a R-marriage $\sigma_F : F \to \mathbb{N}$.

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Remark

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Remark

1 T_2 implies Hall's infinite marriage axiom H.

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Remark

- **I** T_2 implies Hall's infinite marriage axiom H.
- 2 In turn, **H** implies that in a vector space (or more generally in a finitary matroid), all bases are equipotent (one of the aims of Rado's paper [3]).

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Let **ZFA** be the set-theory without **AC** and with atoms: thus **ZF** is (**ZFA**+ "There are no atoms"), and **ZFA** is weaker (*i.e.* has less axioms) than **ZF**.

Theorem (P. Howard (1984), [2])

There is a model of **ZFA** where **RL** holds and T_2 does not hold.

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We do not know whether **RL** implies T_2 in **ZF**.

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There is a model of **ZFA** where **RL** holds and T_2 does not hold.

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We do not know whether **RL** implies T_2 in **ZF**.

We shall prove in **ZF** (and even in **ZFA**) that **RL** implies the Hahn-Banach axiom **HB**, a consequence of T_2 .

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Given a boolean algebra \mathcal{B} , a *measure* on \mathcal{B} is a mapping $m: \mathcal{B} \to [0,1]$ such that for every $x,y \in \mathcal{B}$:

$$x \wedge y = 0_{\mathcal{B}} \Rightarrow m(x \vee y) = m(x) + m(y)$$

If moreover $m(1_B) = 1$, the measure m is said to be *unitary*.

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Uniform probability on a finite non-trivial bool. algebra ${\cal B}$

We denote by $P_{\mathcal{B}}$ the unitary measure on \mathcal{B} giving the same measure to all atoms of \mathcal{B} .

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Given an infinite boolean algebra \mathcal{B} , there exists a unitary measure $\mu: \mathcal{B} \to [0,1]$.

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Luxembourg (see [1], 1969) proved the **HB** is equivalent (in **ZF**) to the classical forms of the Hahn-Banach property (analytic form). Notice that $T_2 \Rightarrow HB$.

RL implies HB: a first idea

Rado's selection Lemma implies Hahn-Banach

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RL ⇒ HB:

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RL ⇒ HB:



Rado's selection Lemma implies Hahn-Banach

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 $RL \Rightarrow HB$: idea

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Rado's selection Lemma implies Hahn-Banach

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Given an infinite boolean algebra \mathcal{B} , for every $F \in fin^*(\mathcal{B})$, denote by bool(F) the boolean sub-algebra of \mathcal{B} which is generated by F, and consider the mapping $\sigma_F : F \to [0,1]$ which is the restriction of $P_{bool(F)}$.

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 $RL \Rightarrow Hl$ idea

RL ⇒ HB:

Notation

For every $n \in \mathbb{N}$, let $D_n := \{ \frac{k}{n+1} : k \in \mathbb{N} \text{ and } 0 \le k \le n+1 \}$.

Notice that $\bigcup_{n\in\mathbb{N}} D_n$ is countable and dense in [0,1].

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n-approximation

For every $x \in [0,1]$, there is a unique $k \in \{0,\ldots,n\}$ such that $\frac{k}{n} \le x < \frac{k+1}{n}$; call the number $\frac{k}{n}$ the *n-approximation of x (in D_n)*.

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Let \mathcal{B} be an (infinite) boolean algebra. For every $n \in \mathbb{N}$, let $\mathcal{B}_n := \mathcal{B} \times \{n\}$. Let $\mathcal{B}_{\omega} := \bigcup_{n \in \mathbb{N}} \mathcal{B}_n$. Thus \mathcal{B}_{ω} is the union of ω copies of \mathcal{B} . We shall apply **RL** to $\prod_{n \in \mathbb{N}} \mathcal{D}_n^{\mathcal{B}_n}$.

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For every non-empty finite subset F of \mathcal{B}_{ω} , we define σ_F as follows. Since F is of the form $\bigcup_{0 \leq i \leq n} (F_i \times \{i\})$ where $n \in \mathbb{N}$ and F_n is non-empty, consider the uniform probability P on $bool_{\mathcal{B}}(\bigcup_{0 \leq i \leq n} F_i)$, and, for every $(x, i) \in F_i \times \{i\}$, let $\sigma_F((x, i))$ be the i-approximation of P(x) (in D_i).

Rado's selection Lemma implies Hahn-Banach

M. Morillo

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 $RL \Rightarrow HE$ idea

 $RL \Rightarrow HB$:

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$$(1_{\mathcal{B}}, i) \in F \Rightarrow \sigma_F((1_{\mathcal{B}}, i)) = 1$$

Rado's selection Lemma implies Hahn-Banach

M. Morillor

RI

 $T_2 \Rightarrow RL$

ЦΒ

 $RL \Rightarrow H$

idea

 $RL \Rightarrow HB$:

For every non-empty finite subset F of \mathcal{B}_{ω} , we define σ_F as follows. Since F is of the form $\bigcup_{0 \leq i \leq n} (F_i \times \{i\})$ where $n \in \mathbb{N}$ and F_n is non-empty, consider the uniform probability P on $bool_{\mathcal{B}}(\bigcup_{0 \leq i \leq n} F_i)$, and, for every $(x, i) \in F_i \times \{i\}$, let $\sigma_F((x, i))$ be the i-approximation of P(x) (in D_i).

$$(0_{\mathcal{B}},i) \in F \Rightarrow \sigma_F((0_{\mathcal{B}},i)) = 0$$

Rado's selection Lemma implies Hahn-Banach

M. Morillor

RL

 $T_2 \Rightarrow RL$

 $_{\mathcal{E}}\mathbf{RL}\Rightarrow\mathbf{T}_{2}$?

HR

 $RL \Rightarrow H$

idea

RL ⇒ HB: proof For every non-empty finite subset F of \mathcal{B}_{ω} , we define σ_F as follows. Since F is of the form $\bigcup_{0 \leq i \leq n} (F_i \times \{i\})$ where $n \in \mathbb{N}$ and F_n is non-empty, consider the uniform probability P on $bool_{\mathcal{B}}(\bigcup_{0 \leq i \leq n} F_i)$, and, for every $(x, i) \in F_i \times \{i\}$, let $\sigma_F((x, i))$ be the i-approximation of P(x) (in D_i).

$$(0_{\mathcal{B}},i) \in F \Rightarrow \sigma_F((0_{\mathcal{B}},i)) = 0$$

$$|\sigma_F((x,i)) - \sigma_F((x,j))| \leq \frac{1}{i} + \frac{1}{j}$$

Rado's selection Lemma implies Hahn-Banach

M. Morillo

RL

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 $RL \Rightarrow H$

RL ⇒ HB:

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4 If
$$x \wedge y = 0_{\mathcal{B}}$$
 and $(x \vee y, I) \in F$ then $|\sigma_F((x \vee y, I)) - \sigma_F((x, i)) - \sigma_F((y, k))| \le \frac{1}{I} + \frac{1}{I} + \frac{1}{k}$.

RL implies **HB**

Rado's selection Lemma implies Hahn-Banach

M. Morillor

RL

 $T_2 \Rightarrow RL$

iRL ⇒ T₂

 $RL \Rightarrow H$

 $RL \Rightarrow HB$:

Using **RL**, let $f \in \prod_{n \in \mathbb{N}} D_n^{\mathcal{B}_n}$ be a mapping respecting the family $(\sigma_F)_{\mathcal{F} \in \mathit{fin}^*(\mathcal{B}_\omega)}$. For every $n \in \mathbb{N}$, let $f_n : \mathcal{B} \to D_n$ be the mapping $x \mapsto f(x, n)$.

RL implies **HB**

Rado's selection Lemma implies Hahn-Banach

M. Morillor

RL

 $T_2 \Rightarrow RL$

 $_{\dot{c}}RL\Rightarrow T_{2}$

ΗВ

 $RL \Rightarrow H$

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Theorem

For every $x \in \mathcal{B}$, the sequence $(f_n(x))_{n \in \mathbb{N}}$ is Cauchy so the sequence $(f_n(x))_{n \in \mathbb{N}}$ converges to a real number $m(x) \in [0,1]$. The mapping $m : \mathcal{B} \to [0,1]$ is a unitary measure on \mathcal{B} .

RL implies **HB**

Rado's selection Lemma implies Hahn-Banach

M. Morillon

RL

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RL ⇒ H

idea

 $RL \Rightarrow HB$:

Using **RL**, let $f \in \prod_{n \in \mathbb{N}} D_n^{\mathcal{B}_n}$ be a mapping respecting the family $(\sigma_F)_{\mathcal{F} \in fin^*(\mathcal{B}_\omega)}$. For every $n \in \mathbb{N}$, let $f_n : \mathcal{B} \to D_n$ be the mapping $x \mapsto f(x, n)$.

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Proof.

Given $x \in \mathcal{B}$, Condition (3) implies that the sequence $(f_n(x))_{n \in \mathbb{N}}$ is Cauchy. Conditions (1) and (2) imply that $m(1_{\mathcal{B}}) = 1$ and $m(0_{\mathcal{B}}) = 0$. Condition (4) implies that m is a measure.

References

Rado's selection Lemma implies Hahn-Banach

M. Morillor

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idea

 $RL \Rightarrow HB$:



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