Rado's selection Lemma implies Hahn-Banach

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Rado (1949) Axiomatic treatment of rank in infinite sets

RL: *Given a family* $(X_i)_{i\in I}$ *of finite sets and a family* $(\sigma_F)_{F\in fin^*(I)}$ *such that for every* $F \in fin^*(I)$, $\sigma_F \in \prod_{i\in F} X_i$, *there exists* $f \in \prod_{i\in I} X_i$ *which "respects"* $(\sigma_F)_{F\in fin^*(I)}$:

$$\forall F \in fin^*(I)\ \exists G \in fin^*(I)\ \big(F \subseteq G \text{ and } f_{\restriction G} = \sigma_G\big)$$

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$$\forall F \in fin^*(I)\ \exists G \in fin^*(I)\ \left(F \subseteq G \text{ and } f_{\restriction G} = \sigma_G\right)$$

Remark

"f respects $(\sigma_F)_{F \in fin^*(I)}$" means that the set $\{G \in fin^*(I) : f_{\restriction G} = \sigma_G\}$ is cofinal in the *poset* $(fin^*(I), \subseteq)$.

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The classical proof of **RL** relies on Tychonov's axiom for families of (finite) compact Hausdorff spaces:
$\mathbf{T}_2$: *For every infinite family* $(X_i)_{i\in I}$ *of compact Hausdorff spaces, the topological product* $\prod_{i\in I} X_i$ *is compact.* Thus:

$$\mathbf{AC} \Rightarrow \mathbf{T}_2 \Rightarrow \mathbf{RL}$$

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$$\mathbf{AC} \Rightarrow \mathbf{T}_2 \Rightarrow \mathbf{RL}$$

Consider the following consequence of $\mathbf{T}_2$:
$\mathbf{AC}^{\mathbf{fin}}$: *"Every infinite family of finite non-empty sets has a non-empty product."*
Blass noticed that:

$$\mathbf{T}_2 \Leftrightarrow (\mathbf{RL} + \mathbf{AC}^{\mathbf{fin}})$$

RL does not hold for *infinite* sets

Let R be the following bi-partite graph ("play-boy graph"):

$$R = \{(i+1, i) : i \in \mathbb{N}\} \cup \{(0, i) : i \in \mathbb{N}\}$$

For every $F \in \mathit{fin}^*(\mathbb{N})$, consider a R-marriage $\sigma_F : F \to \mathbb{N}$.

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Remark

1. $\mathbf{T}_2$ implies Hall's infinite marriage axiom **H**.
2. In turn, **H** implies that in a vector space (or more generally in a finitary matroid), all bases are equipotent (one of the aims of Rado's paper [3]).

Does **RL** imply $\mathbf{T}_2$ in **ZF**?

Let **ZFA** be the set-theory without **AC** and with atoms: thus **ZF** is (**ZFA**+"There are no atoms"), and **ZFA** is weaker (*i.e.* has less axioms) than **ZF**.

Theorem (P. Howard (1984), [2])

There is a model of **ZFA** *where* **RL** *holds and* $\mathbf{T}_2$ *does not hold.*

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We do not know whether **RL** implies $\mathbf{T}_2$ in **ZF**.

We shall prove in **ZF** (and even in **ZFA**) that **RL** implies the Hahn-Banach axiom **HB**, a consequence of $\mathbf{T}_2$.

The Hahn-Banach axiom **HB**

Given a boolean algebra $\mathcal{B}$, a *measure* on $\mathcal{B}$ is a mapping $m : \mathcal{B} \to [0,1]$ such that for every $x, y \in \mathcal{B}$:

$$x \wedge y = 0_{\mathcal{B}} \Rightarrow m(x \vee y) = m(x) + m(y)$$

If moreover $m(1_{\mathcal{B}}) = 1$, the measure m is said to be *unitary*.

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Luxembourg (see [1], 1969) proved the **HB** is equivalent (in **ZF**) to the classical forms of the Hahn-Banach property (analytic form). Notice that $\mathbf{T}_2 \Rightarrow$ **HB**.

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The finite sets D_n

Notation

For every $n \in \mathbb{N}$, let $D_n := \{\frac{k}{n+1} : k \in \mathbb{N} \text{ and } 0 \leq k \leq n+1\}$.

Notice that $\cup_{n \in \mathbb{N}} D_n$ is countable and dense in $[0, 1]$.

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n-approximation

For every $x \in [0,1]$, there is a unique $k \in \{0, \ldots, n\}$ such that $\frac{k}{n} \leq x < \frac{k+1}{n}$; call the number $\frac{k}{n}$ the *n-approximation of x (in D_n)*.

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Let $\mathcal{B}$ be an (infinite) boolean algebra. For every $n \in \mathbb{N}$, let $\mathcal{B}_n := \mathcal{B} \times \{n\}$. Let $\mathcal{B}_\omega := \cup_{n\in\mathbb{N}} \mathcal{B}_n$. Thus $\mathcal{B}_\omega$ is the union of ω copies of $\mathcal{B}$. We shall apply **RL** to $\prod_{n\in\mathbb{N}} D_n^{\mathcal{B}_n}$.

Definition of σ_F for $F \in fin^*(\mathcal{B}_\omega)$

For every non-empty finite subset F of $\mathcal{B}_\omega$, we define σ_F as follows. Since F is of the form $\cup_{0 \leq i \leq n}(F_i \times \{i\})$ where $n \in \mathbb{N}$ and F_n is non-empty, consider the uniform probability P on $bool_{\mathcal{B}}(\cup_{0 \leq i \leq n} F_i)$, and, for every $(x, i) \in F_i \times \{i\}$, let $\sigma_F((x, i))$ be the i-approximation of $P(x)$ (in D_i).

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For every $F \in fin^*(\mathcal{B}_\omega)$, for every $(x, i), (x, j), (y, k) \in F$:

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3. $|\sigma_F((x, i)) - \sigma_F((x, j))| \leq \frac{1}{i} + \frac{1}{j}$
4. If $x \wedge y = 0_{\mathcal{B}}$ and $(x \vee y, l) \in F$ then $|\sigma_F((x \vee y, l)) - \sigma_F((x, i)) - \sigma_F((y, k))| \leq \frac{1}{l} + \frac{1}{i} + \frac{1}{k}$.

RL implies HB

Using **RL**, let $f \in \prod_{n\in\mathbb{N}} D_n^{\mathcal{B}_n}$ be a mapping respecting the family $(\sigma_F)_{\mathcal{F}\in fin^*(\mathcal{B}_\omega)}$. For every $n \in \mathbb{N}$, let $f_n : \mathcal{B} \to D_n$ be the mapping $x \mapsto f(x, n)$.

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Theorem

For every $x \in \mathcal{B}$, the sequence $(f_n(x))_{n \in \mathbb{N}}$ is Cauchy so the sequence $(f_n(x))_{n \in \mathbb{N}}$ converges to a real number $m(x) \in [0, 1]$. The mapping $m : \mathcal{B} \to [0, 1]$ is a unitary measure on $\mathcal{B}$.

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Using **RL**, let $f \in \prod_{n \in \mathbb{N}} D_n^{\mathcal{B}_n}$ be a mapping respecting the family $(\sigma_F)_{\mathcal{F} \in fin^*(\mathcal{B}_\omega)}$. For every $n \in \mathbb{N}$, let $f_n : \mathcal{B} \to D_n$ be the mapping $x \mapsto f(x, n)$.

Theorem

For every $x \in \mathcal{B}$, the sequence $(f_n(x))_{n \in \mathbb{N}}$ is Cauchy so the sequence $(f_n(x))_{n \in \mathbb{N}}$ converges to a real number $m(x) \in [0, 1]$. The mapping $m : \mathcal{B} \to [0, 1]$ is a unitary measure on $\mathcal{B}$.

Proof.

Given $x \in \mathcal{B}$, Condition (3) implies that the sequence $(f_n(x))_{n \in \mathbb{N}}$ is Cauchy. Conditions (1) and (2) imply that $m(1_{\mathcal{B}}) = 1$ and $m(0_{\mathcal{B}}) = 0$. Condition (4) implies that m is a measure. □

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