Ingleton's axiom and the Axiom of Choice

33rd Summer Conference on Topology and its Applications

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Bowling Green, July 17-20, 2018

The Hahn-Banach axiom

Given a vector space E over the field $\mathbb{R}$ of real numbers, a *semi-norm* on E is a mapping $N : E \to \mathbb{R}_+$ such that for every $\lambda \in \mathbb{R}$ and every $x, y \in E$, $N(\lambda.x) = |\lambda|_{\mathbb{R}} N(x)$ and $N(x+y) \leq N(x) + N(y)$, where $|.|_{\mathbb{R}}$ is the usual absolute value $x \mapsto \max(x, -x)$ on $\mathbb{R}$.

HB: *Given a $\mathbb{R}$-vector space E, a semi-norm $N : E \to \mathbb{R}_+$, a vector subspace V of E and a linear form $f : V \to \mathbb{R}$ such that for every $x \in V$, $|f(x)|_{\mathbb{R}} \leq N(x)$, there exists a linear form $\tilde{f} : E \to \mathbb{R}$ extending f such that for every $x \in E$, $|\tilde{f}(x)|_{\mathbb{R}} \leq N(x)$.*

Remark. In set-theory without the axiom of choice:
- **AC** $\Rightarrow$ **HB** $\Rightarrow$ "The Hausdorff-Banach-Tarski" paradox.
- None of these two arrows is reversible.

See Jech's book "The Axiom of Choice" or Howard and Rubin's book "Consequences of the Axiom of Choice".

The Hahn-Banach Lemma (set theory without choice)

The usual proof of **HB** can be obtained by transfinitely iterating the following Lemma (for example using Zorn's lemma or a transfinite recursion and the Axiom of Choice).

Lemma (Hahn-Banach, 1932, "one step")

Let E be a $\mathbb{R}$-vector space, let $N : E \to \mathbb{R}_+$ be a semi-norm on E, let V be a vector subspace of E and let $f : V \to \mathbb{R}$ be a linear form such that $|f|_{\mathbb{R}} \leq N_{\restriction V}$. For every $a \in E \backslash V$, there exists a linear form $\tilde{f} : V + \mathbb{R}.a \to \mathbb{R}$ extending f such that $|\tilde{f}|_{\mathbb{R}} \leq N_{\restriction V+\mathbb{R}.a}$.

A similar result to Hahn-Banach's Lemma

The following result is choiceless:

Lemma (Ingleton, 1952, "one step")

Let E be a vector space over a spherically complete ultrametric valued field $(\mathbb{K}, |.|)$, let $N : E \to \mathbb{R}_+$ be a ultrametric semi-norm, let V be a vector subspace of E and let $f : V \to \mathbb{K}$ be a linear form such that $|f| \leq N_{\restriction V}$. If $a \in E \backslash V$, then there exists a linear form $\tilde{f} : V + \mathbb{K}.a \to \mathbb{K}$ extending f such that $|\tilde{f}| \leq N_{\restriction V+\mathbb{K}.a}$.

Valued fields

An *absolute value* on a (commutative) field $\mathbb{K}$ is a mapping $|.| : \mathbb{K} \to \mathbb{R}_+$ satisfying the following properties for every λ, $\mu \in \mathbb{K}$: $|\lambda| = 0 \Leftrightarrow \lambda = 0$; $|\lambda\mu| = |\lambda||\mu|$ and $|\lambda + \mu| \leq |\lambda| + |\mu|$. Each valued field $(\mathbb{K}, |.|)$ gives rise to a metric $d : \mathbb{K} \times \mathbb{K} \to \mathbb{R}_+$ defined by $d(x, y) = |x - y|$ for every $x, y \in \mathbb{K}$. An absolute value $|.|$ on $\mathbb{K}$ is said to be *ultrametric* if the associated metric d is ultrametric, equivalently if for every λ, $\mu \in \mathbb{K}$, $|\lambda + \mu| \leq \max(|\lambda|, |\mu|)$.

- For each commutative field $\mathbb{K}$, the mapping $|.|_{triv} : \mathbb{K} \to \mathbb{R}_+$ associating to each $\lambda \in \mathbb{K}$ the real number 0 if $\lambda = 0$ and 1 otherwise is a ultrametric absolute value, called the *trivial* absolute value on $\mathbb{K}$. If $\mathbb{K}$ is finite, $|.|_{triv}$ is the only absolute value on $\mathbb{K}$.
- For each prime number p, the mapping $x \mapsto |x|_p := p^{-v_p(x)}$ is a ultrametric absolute value on the field $\mathbb{Q}$ of rational numbers, where $v_p : \mathbb{Q} \to \mathbb{Z} \cup \{+\infty\}$ is the p-adic valuation on $\mathbb{Q}$.
- Every *non trivial* absolute value on $\mathbb{Q}$ is of the form $|.|_{\mathbb{R}}^{\tau}$ where $0 < \tau < 1$, or of the form $|.|_p^{\tau}$ for some prime number p and some $\tau > 0$ (Ostrowski's theorem).

Spherically complete ultrametric valued fields

A ultrametric valued field $(\mathbb{K}, |.|)$ is *spherically complete* if every chain of balls with "large inequalities" (*i.e.* of the form $\{x \in \mathbb{K} : |x - a| \leq r\}$ where $a \in \mathbb{K}$ and $r \in \mathbb{R}_+$) of the metric space $(\mathbb{K}, d)$ has a non-empty intersection.

Examples

-Each commutative field $\mathbb{K}$ endowed with the trivial absolute value is spherically complete.
-For each prime number p, the valued field $(\mathbb{Q}, |.|_p)$ is not spherically complete, however, the Cauchy completion $\mathbb{Q}_p$ of $(\mathbb{Q}, |.|_p)$ is spherically complete (because the unit ball of $\mathbb{Q}_p$ is compact).

Semi-normed vector spaces over a valued field

Given a vector space E over a valued field $(\mathbb{K}, |.|)$, a *semi-norm* on E is a mapping $N : E \to \mathbb{R}_+$ satisfying for every $x, y \in E$ and $\lambda \in \mathbb{K}$ the properties $N(\lambda.x) = |\lambda| N(x)$ and $N(x + y) \leq N(x) + N(y)$.

For a ultrametric valued field $(\mathbb{K}, |.|)$, the semi-norm N is *ultrametric* if the semi-metric associated to N is ultrametric, equivalently if for every $x, y \in E$, $N(x + y) \leq \max(N(x), N(y))$.

AC implies Ingleton's statement

From Ingleton's Lemma and the Axiom of Choice, it follows for each spherically complete ultrametric valued field $(\mathbb{K}, |.|)$:

Ingleton's statement

$\mathbf{I}_{\mathbb{K},|.|}$: *"Let E be a $\mathbb{K}$-vector space, let $N : E \to \mathbb{R}_+$ be a ultrametric semi-norm, let V be a vector subspace of E and let $f : V \to \mathbb{K}$ be a linear form such that $|f| \leq N_{\restriction V}$. Then there exists a linear form $\tilde{f} : E \to \mathbb{K}$ extending f such that $|\tilde{f}| \leq N$."*

• A.C.M. van Rooij (1992) asked whether the "full Ingleton theorem" (*i.e.* the conjonction of all statements $\mathbf{I}_{\mathbb{K},|.|}$) implies **AC**.
• We shall show that in set theory **ZFA** (set theory without choice weakened to allow "atoms"), the "full Ingleton theorem" + **HB** does not imply **AC** (unless **ZFA** is inconsistent).

A model of **ZFA**$+\neg$**AC** with "multiple choices"

Levy (1962) built a model of **ZFA** in which there exists a sequence $(F_n)_{n\in\mathbb{N}}$ of finite sets such that for every $n \in \mathbb{N}$, $\#F_n = n+1$ and $\prod_{n\in\mathbb{N}} F_n = \varnothing$: such a model does not satisfy **AC**.

However, Levy showed that this model satisfies the following consequences of **AC**:

• **MC**: ("Multiple Choice") *"For every family $(A_i)_{i\in I}$ of non-empty sets, there exists a family $(B_i)_{i\in I}$ of non-empty finite sets such that for every $i \in I$, $B_i \subseteq A_i$."*

For every prime number $p \geq 2$, the following refined statement:

• **MC**(p): *"For every family $(A_i)_{i\in I}$ of nonempty sets, there exists a family $(B_i)_{i\in I}$ of finite sets such that for every $i \in I$, $B_i \subseteq A_i$ and $\#B_i$ is not a multiple of p."*

Remark. In set-theory **ZFA**, **MC** does not imply **AC**. In set-theory **ZF** (without atoms), **MC** implies **AC**.

$\mathbf{MC}+\forall^{Prime} p\, \mathbf{MC}(p)$ implies **HB** + "Full Ingleton"

We shall prove the following Lemma:

Extension Lemma

Let $(\mathbb{K}, |.|)$ be a spherically complete ultrametric valued field or the usual valued field $\mathbb{R}$. Let E be a $\mathbb{K}$-vector space endowed with a semi-norm N which is assumed to be ultrametric if $\mathbb{K} \neq \mathbb{R}$. Then $\mathbf{MC}+\forall^{Prime} p\, \mathbf{MC}(p)$ implies the existence of a mapping associating to each ordered pair (V, f) where V is a proper vector subspace of E and $f : V \to \mathbb{K}$ is a linear form such that $|f| \leq N_{\restriction V}$, an ordered pair (V', f') such that V' is a vector subspace of E strictly including V and $f' : V' \to \mathbb{K}$ is a linear mapping extending f with $|f'| \leq N_{\restriction V'}$.

The "full Ingleton theorem" follows from this Lemma in set theory **ZFA**.

Proof of the Lemma in $\mathbf{ZFA}+\mathbf{MC}+\forall^{Prime} p\ \mathbf{MC}(p)$

With **MC**, let Φ be a mapping associating to each non-empty subset X of $E \cup \mathbb{K}^E$ a finite non-empty subset of X. Given a proper vector subspace V of E and a linear form $f : V \to \mathbb{K}$ satisfying $|f| \leq N_{\restriction V}$, let $F := \Phi(E \backslash V)$ and let $V_F := \mathrm{span}(V \cup F)$. Using Hahn-Banach's lemma (for $\mathbb{K} = \mathbb{R}$) or Ingleton's lemma (otherwise), the set $\mathcal{G}$ of linear forms $g : V_F \to \mathbb{K}$ extending f such that $|g| \leq N_{\restriction V_F}$ is non-empty.

For the first two cases below, we let $G := \Phi(\mathcal{G})$.
- *Case $\mathbb{K} = \mathbb{R}$.* Consider the linear form $\tilde{f} := \frac{1}{\#G} \sum_{g \in G} g$ on V_F: then $\tilde{f}$ extends f and $|\tilde{f}|_{\mathbb{R}} \leq N_{\restriction V_F}$ (whence $\mathbf{MC} \Rightarrow \mathbf{HB}$).
- *Case $\mathbb{K}$ has characteristic zero and the restriction $|.|_{\restriction \mathbb{Q}}$ is the trivial absolute value.* Consider the same linear form $\tilde{f} := \frac{1}{\#G} \sum_{g \in G} g$. Then $|\#G| = 1$ thus for every $x \in V_F$, $|\tilde{f}(x)| = \frac{1}{|\#G|} |\sum_{g \in G} g(x)| = |\sum_{g \in G} g(x)| \leq \max_{g \in G} |g(x)| \leq N(x)$ whence $|\tilde{f}| \leq N_{\restriction V_F}$.

Proof of the Lemma in $\mathbf{ZFA}+\mathbf{MC}+\forall^{Prime} p\, \mathbf{MC}(p)$: cont'd

- *Other cases.*

-Subcase a): The characteristic of the field $\mathbb{K}$ is zero; then $\mathbb{K}$ extends the field $\mathbb{Q}$ of rational numbers; $|.|_{\restriction\mathbb{Q}}$ is non-trivial. Using Ostrowski's theorem, the absolute value induced by $|.|$ on $\mathbb{Q}$ is equivalent to the p-adic absolute value for some prime number p.
-Subcase b): The characteristic of $\mathbb{K}$ is not zero. Then, this characteristic is a prime number p.
With $\mathbf{MC}(p)$, let Φ_p be a mapping associating to each non-empty subset X of $\mathbb{K}^E$ a finite subset G of X such that p does not divide $\#G$. Let $G := \Phi_p(\mathcal{G})$: then G is a finite subset of $\mathcal{G}$ such that p does not divide $\#G$. Let $n := \#G$. Then $|n| = 1$: in Subcase a), $|n| = |n|_p = 1$ because p does not divide n; in Subcase b), $n \in \mathbb{F}_p \backslash \{0\} \subseteq \mathbb{K}$ thus $|n| = 1$.
Now we consider the linear form $\tilde{f} := \frac{1}{n}\sum_{g\in G} g$: this linear form extends f, and for every $x \in V_F$, $|\tilde{f}(x)| = \frac{1}{|n|}|\sum_{g\in G} g(x)| = |\sum_{g\in G} g(x)| \leq \max_{g\in G}(|g(x)|) \leq N(x)$, whence $|\tilde{f}| \leq N_{\restriction V_F}$.

Some questions

-Are there links in set-theory without choice between the statements $\mathbf{I}_{\mathbb{K}}$ obtained for various spherically complete ultrametric valued fields $\mathbb{K}$?
-Does the conjonction of the statements $\mathbf{I}_{\mathbb{Q}_p}$ for p prime number imply $\mathbf{I}_{\mathbb{Q},|.|_{triv}}$ or **HB**?
-Given two different prime numbers p and q, are the statements $\mathbf{I}_{\mathbb{Q}_p}$ and $\mathbf{I}_{\mathbb{Q}_q}$ equivalent?

Remark

For each ultrametric spherically complete valued field $(\mathbb{K}, |.|)$, the statement $\mathbf{I}_{(\mathbb{K},|.|)}$ is equivalent to the following one (see MM-2017):

"For every vector subspace F of an ultrametric semi-normed $\mathbb{K}$-vector space (E, N), there exists an isometric linear extender $T : BL(F, \mathbb{K}) \to BL(E, \mathbb{K})$."

Here, given a vector subspace V of E, $BL(V, \mathbb{K})$ denotes the set of linear bounded mappings from V to $\mathbb{K}$.

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