Compactness of Helly spaces

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We work in set-theory without the Axiom of Choice **ZF**:

$$\begin{tikzcd}
\textbf{AC} \Leftrightarrow \textit{``Tychonov for compact spaces''} \arrow[d, shift left] \\
\text{Alaoglu} \Leftrightarrow \textit{``Tychonov for Hausdorff compact spaces''} \arrow[u, shift left, "/" marking] \arrow[d, shift left] \\
\textbf{HB}\textit{(Hahn-Banach)} \arrow[u, shift left, "/" marking] \arrow[d] \arrow[dd, bend right] \\
\textit{There exists a non-trivial measure on } \omega \arrow[d, dotted, "?"] \\
\textit{There exists a non L-measurable subset of } \mathbb{R}
\end{tikzcd}$$

References

Howard & Rubin, *Consequences of the Ax. of Choice*, AMS 1998.

Jech, *The Axiom of Choice*, NHPC 1973.

Two "analytic" Hahn-Banach properties

Given a real vector space E, a mapping $p : E \to \mathbb{R}$ is *sublinear* if for every $x, y \in E$ and every real number $\lambda \geq 0$, $p(x + y) \leq p(x) + p(y)$ and $p(\lambda x) = \lambda p(x)$.

Hahn-Banach property (*HBP*) on a real vector space E:

For every sub-linear functional $p : E \to \mathbb{R}$, every vector subspace F of E, and every linear mapping $f : F \to \mathbb{R}$ such that $f \leq p_{\restriction F}$, there exists a linear mapping $\tilde{f} : E \to \mathbb{R}$ extending f such that $\tilde{f} \leq p$.

Continuous Hahn-Banach property (*CHBP*) on a *tvs* E:

Similar statement but we assume that E is a topological vector space and that $p : E \to \mathbb{R}$ is continuous.

Models of **ZF** without Hahn-Banach

We denote by $\mathcal{CHB}$ the class of normed spaces satisfying the *continuous* Hahn-Banach property.

Spaces not in $\mathcal{CHB}$

There is a model of **ZF** (see [8, Pincus & Solovay, 1977]) where every (positive) finitely additive unitary measure on $\mathbb{N}$ is trivial (*i.e.* discrete). Equivalently, the continuous dual of $\ell^\infty(\mathbb{N})$ is $\ell^1(\mathbb{N})$. In such a model, $\ell^\infty(\mathbb{N})$ does not belong to $\mathcal{HB}$.

Question

1. Which normed spaces belong to $\mathcal{CHB}$?
2. For which topological spaces X does the normed space $C^*(X)$ belong to $\mathcal{HB}$?
3. Given a compact Hausdorff topological space K, does the normed space $C(K)$ belong to $\mathcal{HB}$?

The class $\mathcal{CHB}$ of normed spaces satisfying *CHBP*

Theorem ([4] Fossy & M. 1998, [7] M. 2009)

The following normed spaces satisfy the CHBP*:*

1. *Spaces with a dense well-ordered subset ($\ell^1(\alpha)$ for α ordinal).*
2. *Spaces with a uniformly Gâteaux differentiable equivalent norm (*e.g. *Hilbert spaces, unif. smooth spaces).*

Theorem ([5] Dodu & M. 1999)

The class $\mathcal{C}HB$ is closed under:

1. *subspaces*
2. *continuous images*
3. *closure*

Remark

$\ell^\infty(\mathbb{N}) \in \mathcal{HB}$ *iff* $\ell^1(\mathbb{R}) \in \mathcal{HB}$.

The *FEP*

Finite Extension property (*FEP*) on a normed space E:

Similar statement but we assume that E is a normed space, F is finite dimensional and p is the norm of E.

Remark

1. For every set I, $\ell^\infty(I)$ satisfies the *FEP*.
2. Gâteaux-differentiable normed spaces satisfy the *FEP* ([7] M. 2009)..

Using the axiom of *Dependent Choices* (a weak form of the Axiom of Choice), G-differentiable spaces belong to $\mathcal{HB}$ ([5] M. 1999).

Question

Do Gâteaux-differentiable (*resp.* Fréchet-differentiable) normed spaces belong to $\mathcal{HB}$ in **ZF**?

A statement equivalent to the *CHBP*

convex-compactness [3, Luxemburg 1969]

A subset A of a topological vector space E is *convex-compact* if for every family $\mathcal{C}$ of closed *convex* subsets of A satisfying the finite intersection property (FIP), the set $\cap\mathcal{C}$ meets A.

Given a normed space E, we denote by E' the continuous dual of E, and by $B_{E'}$ the closed unit ball of E'.

Theorem. Given a normed space E:

$E \in \mathcal{CHB} \Leftrightarrow E$ satisfies the *FEP* and $B_{E'}$ is weak* convex-compact.

Proof. See [3, Luxemburg 1969] or [5, 1999].

Questions

Given a compact Hausdorff space K, and denoting by $\mathcal{M}(K)$ the continuous dual of $C(K)$:

1. Is the closed unit ball $B(K)$ of $\mathcal{M}(K)$ weak* convex-compact?
2. For which compact Hausdorff spaces K is $B(K)$ weak* compact?

Spaces $C(K)$ for a linearly ordered compact space K

Our aim is to prove the following results:

Theorem

Let $(X, \leq)$ be a linearly ordered set (for short, say a “line”). The following closed subset of $[0,1]^X$ is compact:

$$H(X,[0,1]) := \bigcap_{x,y \in X;\; x \leq y} \{ f \in [0,1]^X : f(x) \leq f(y) \}$$

Corollary

If K is a complete line, then the dual ball $B(K)$ of $C(K)$ is weak compact.*

Corollary

If K is a complete line, then the normed space $C(K)$ satisfies the CHBP.

The Helly space $H(X, L)$

Lexicographic lines

Given an ordinal α, the lexicographic line $L_\alpha := \{0, 1\}^\alpha$ endowed with the order topology is complete and thus (Loeb-)compact.

Given two lines X, L, we denote by $H(X, L)$ the set of ascending mappings $u : X \to L$. We endow L with the order topology, and $H(X, L)$ with the topology induced by the product space L^X.

Notation

Given $x \in X$ and $t \in L$, we define the following subset of $H(X, L)$:

$$F(x, t) := \{u \in H(X, L) : u(x) = t\}$$

If X is a line, then $H(X, L_2)$ is (Loeb-)compact

Proof.

Let $\mathcal{F}$ be a filter of the lattice $\mathcal{L}$ of closed subsets of $H(X, L_2)$. Let $\mathcal{S}(\mathcal{F}) := \{G \in \mathcal{L} : \forall F \in \mathcal{F}\ F \cap G \neq \varnothing\}$. We define the following subset of X:

$$I := \{x \in X : F(x, 0) \in \mathcal{S}(\mathcal{F})\}$$

Then I is an initial section of X, and for every $x \in X \backslash I$, $F_{x,1} \in \mathcal{F}$. Let $u : X \to L_2$ be the indicator function of $X \backslash I$. Let us check that $u \in \cap\mathcal{F}$. Let $F \in \mathcal{F}$. Let us show that every neighbourhood N of u meets F. We may assume that $N = \cap_{x \in A} F(x, 0) \cap \cap_{x \in B} F(x, 1)$ where A (*resp.* B) is a finite subset of I (*resp.* $X \backslash I$). If $A = \varnothing$ then $N = \cap_{x \in B} F(x, 1) \in \mathcal{F}$ so N meets F, else $\cap_{x \in A} F(x, 0) = F(\max A, 0) \in \mathcal{S}(\mathcal{F})$ so $N \cap F = F(\max A, 0) \cap \left(\cap_{x \in B} F(x, 1) \cap F \right) \neq \varnothing$. □

More generally:

If a line L is finite, then $H(X, L)$ is Loeb-compact, a witness of Loeb-compactness beeing definable from the lines X and L.

L_ω is the projective limit of the sequence $(L_n)_{n\in\omega}$

$\tau_{n,m} : L_n \to L_m$ for $m \leq n \leq \omega$

$\tau_{n,m}$ is the "truncature mapping" $\sigma \mapsto \sigma_{\restriction m}$. Notice that $\tau_{n,m}$ is ascending, continuous and onto.

$$\begin{tikzcd}
L_\omega \arrow[d, "\tau_{\omega,n}"'] \arrow[dr, "\tau_{\omega,m}"] & \\
L_n \arrow[r, "\tau_{n,m}"'] & L_m
\end{tikzcd}$$

lim

Given a sequence $(\sigma_k)_{k\in\omega}$ such that for every $m \leq n \in \omega$, $\sigma_n \in L_n$ and $\tau_{n,m}(\sigma_n) = \sigma_m$, we denote by $\lim((\sigma_n)_{n\in\omega})$ the (unique) element σ of L_ω such that for every $n \in \omega$, $\tau_{\omega,n}(\sigma) = \sigma_n$.

If $(X, \leq)$ is a line, then $H(X, L_\omega)$ is Loeb-compact

Proof.

Let $\mathcal{F}$ be a filter of the lattice $\mathcal{L}$ of closed subsets of $H(X, L_\omega)$. We define an ascending sequence $(\mathcal{F}_n)_{n\in\omega}$ of filters of $\mathcal{L}$ such that $\mathcal{F}_0 = \mathcal{F}$ and, denoting by $\mathcal{F}_n^*$ the filter $\tau_{\omega,n}[\mathcal{F}_n]$, $\cap\mathcal{F}_n^*$ is a singleton $\{u_n\}$ of L_n. Having defined $(\mathcal{F}_i)_{i\leq n}$ for some $n \in \omega$, then $\mathcal{F}_n^* = \{Z \in \mathcal{L}_{L_n} : \tau_{\omega,n}^{-1}[Z] \in \mathcal{F}_n\}$; using the Loeb-compactness of $H(X, L_n)$, let $u_n \in \cap\mathcal{F}_n$; then $F_n := \tau_{\omega,n}^{-1}[u_n] \in \mathcal{S}(\mathcal{F}_n)$; let $\mathcal{F}_{n+1}$ be the filter generated by $\mathcal{F}_n$ and F_n. By definition of $(u_n)_{n\in\omega}$, for every $n \in \omega$, $\tau_{n+1,n}(u_{n+1}) = u_n$, thus we may define $u := \lim((u_n)_{n\in\omega})$. Then $u \in \cap\mathcal{F}$. □

If $(X, \leq)$ is a line, then $H(X, [0,1])$ is Loeb-compact

Proof.

Consider a continuous ascending onto mapping $\pi : L_\omega \to [0,1]$. Then the mapping $\pi^* : H(X, L_\omega) \to H(X, [0,1])$ associating to every $u \in H(X, L_\omega)$ the mapping $\pi \circ u$ is continuous and onto. Thus $H(X, [0,1]) = \pi^*[H(X, L_\omega)]$ is Loeb-compact. $\square$

Gap-full lines

Given a line $(X, \leq)$, let D_X be the set of elements $x \in X$ such that $x = \sup X$ or x has a successor in X. For every $x \in D_X$, the indicator function f_x of the interval $\leftarrow, x]$ is continuous.

Definition

A line $(X, \leq)$ is *gap-full* if the set D_X is dense in X.

Proposition

Let $(X, \leq)$ be a gap-full complete line. Then the vector space generated by the subset $\{f_x : x \in D_X\}$ of $C(X)$ is dense in $C(X)$.

Proof.

It is a consequence of the Stone-Weierstrass Theorem (which is choiceless). □

The continuous dual of $C(X)$ for a gap-full line X

If the line X has a first element a, let

$$H_0(X,[0,1]) := \{u \in H(X,[0,1]) : u(a) = 0\}$$

Given a compact Hausdorff space K, denote by $\mathcal{M}_+(K)$ the set of positive linear forms on $C(K)$. Let $B_+(K) := B(K) \cap \mathcal{M}_+(K)$ be the "positive dual ball" of K.

Theorem (Stieltjes measures on a gap-full complete line)

Let $(X, \leq)$ be a gap-full complete line. We endow $B(X)$ with the weak topology.*

1. *For every $u \in H_0(X,[0,1])$, there exists a unique element $\mu_u \in \mathcal{M}(X)$ such that for every $x \in D_X$, $\mu_u(f_x) = u(x)$. Moreover $\mu_u \in B_+(X)$.*
2. *Let $\Phi : H_0(X,[0,1]) \to B^+(X)$ be the mapping $u \mapsto \mu_u$. Then Φ is continuous and onto.*

Thus $B^+(X)$ (and also $B(X)$) is (Loeb-)compact.

Splitted lines

Given a complete line $(X, \leq)$, we denote by $S(X)$ the "splitted" line $X \times_{lex} L_2$ endowed with the lexicographic order: this space is gap-full. We denote by $\pi_X : S(X) \twoheadrightarrow X$ the mapping $(x, \varepsilon) \mapsto x$.

Lemma

Let $(X, \leq)$ be a complete line.

1. *The mapping $\pi_X^* : C(X) \hookrightarrow C(S(X))$ associating to each $f \in C(X)$ the mapping $f \circ \pi_X$ is linear and isometrical.*
2. *The transposed mapping ${\pi_X^*}^t : \mathcal{M}(S(X)) \twoheadrightarrow \mathcal{M}(X)$ is linear, weak*-continous and onto.*
3. *The induced mapping ${\pi_X^*}^t : B_+(S(X)) \twoheadrightarrow B_+(X)$ is continuous and onto.*

Theorem

Given a complete line X, $B(X)$ is weak (Loeb)-compact.*

Corollary

Given a complete line X, $C(X) \in \mathcal{H}B$.

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