Variable Ranges in Linear Constraints

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Introduction: example

We add *variable ranges* to linear constraint over the real numbers as a means to reason on the approximation of variables.

Example

- linear constraint:
  - $0 \leq x \leq 4$ means $x \in [0, 4]$.
- linear constraint with variable range:
  - $0 \leq x \leq 4, \delta_x = 1$ means x belongs to one of $[max(0, a - 1), min(4, a + 1)]$ where the real number $a \in [1, 3]$.
  - $0 \leq x \leq 4, \delta_x \geq 3$ means *false*.

Plan

Introduction: basic definitions

- *vars*(c): the set of variables in the linear constraint c.
- A *polyhedron*: the set of solution points that satisfy a linear system: $Sol(\mathbf{Ax} \leq \mathbf{b}) = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{Ax} \leq \mathbf{b}\}$.

Let $S = Sol(\mathbf{Ax} \leq \mathbf{b})$ be a non-empty polyhedron, and x a variable in $\mathbf{x}$.

- $width(S, x) = max\{x \mid \mathbf{x} \in S\} - min\{x \mid \mathbf{x} \in S\}$ if both *min* and *max* exist.
- Otherwise, $width(S, x) = \infty$.

For an empty polyhedron $S = \emptyset$, we set $width(S, x) = 0$.

LR-constraints: syntax

- Let Var be the set of linear constraint variables. We define $\Delta = \{\delta_x \mid x \in Var\}$ as the set of range variables.
- A *primitive range constraint* is an inequality $\delta_x \simeq s$, where $\simeq$ is in $\{\leq, \geq\}$ and $s \geq 0 \in \mathbb{R}$.
- A *range constraint* d is a sequence of primitive range constraints.
- A *linear-range constraint* (LR-constraint) $c \wedge d$ is a sequence of linear constraints and range constraints.
- $\delta_x = s$ is a shorthand for $\delta_x \leq s, \delta_x \geq s$.

LR-constraints: syntax

Example

Syntax

1. $0 \leq x \leq 2$
2. $1 \leq x \leq 4, \delta_x \leq 1$
3. $x = 2y, \delta_y \leq 1, \delta_x \geq 3$

LR-constraints: syntax

Example

Syntax and semantics

1. $0 \leq x \leq 2$ denotes a set of values for x ranging from 0 to 2, and hence $\delta_x = 1$.
2. $1 \leq x \leq 4, \delta_x \leq 1$
3. $x = 2y, \delta_y \leq 1, \delta_x \geq 3$

LR-constraints: syntax

Example

Syntax and semantics

1. $0 \leq x \leq 2$ denotes a set of values for x ranging from 0 to 2, and hence $\delta_x = 1$.

2. $1 \leq x \leq 4, \delta_x \leq 1$ denotes a collection of intervals for x of the form $[max(1, a - \delta), min(4, a + \delta)]$ with $a \in \mathbb{R}$ and $0 \leq \delta \leq 1$.

3. $x = 2y, \delta_y \leq 1, \delta_x \geq 3$

LR-constraints: syntax

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3. $x = 2y, \delta_y \leq 1, \delta_x \geq 3$ is unsatisfiable since the approximation of x is at most 2 (the double of the approximation of y) which contradicts $\delta_x \geq 3$.

LR-constraints: semantics

- A *model* of a LR-constraint $c \wedge d$ is a non-empty polyhedron:

$$S = Sol(c \wedge \bigwedge_{\delta_x \in vars(d)} \underline{x} \leq x \leq \overline{x})$$

  where $\underline{x}, \overline{x} \in \mathbb{R}$ such that for every $\delta_x \simeq s$ in d with $\simeq$ in $\{\leq, \geq\}$, we have $width(S, x) \simeq 2s$.
- $c \wedge d$ is *satisfiable* if there exists a model of it.
- $c \wedge d$ *entails* $\delta_x \leq s$ if every model of $c \wedge d$ is a model of $\delta_x \leq s$.

LR-constraints: semantics

- NB: when d is empty, satisfiability conservatively boils down to the condition $Sol(c) \neq \emptyset$.
- Towards providing a procedure for checking satisfiability in the general case, we first *compile* LR-constraints into parameterized linear systems.
- A *parameterized linear system of inequalities* is a linear system $\mathbf{Ax} \leq \mathbf{b} + \mathbf{Ba}$ where $\mathbf{a}$ is a vector of *parameters*.

LR-constraints: semantics

For $c \wedge d$, we define the parameterized linear system:

$$\mathcal{S}(c \wedge d) = c \wedge d \wedge \bigwedge_{\delta_x \in vars(d)} (a_x - \delta_x \leq x \leq a_x + \delta_x, 0 \leq \delta_x)$$

where the a_x, δ_x's are parameters. *Intuition:* a_x models an approximatively known value for x, and δ_x its radius.

Example

1. $\mathcal{S}(0 \leq x \leq 2)$ is $0 \leq x \leq 2$ (a 0-parameter linear system).
2. $\mathcal{S}(1 \leq x \leq 4, \delta_x \leq 1)$ is $1 \leq x \leq 4, a_x - \delta_x \leq x \leq a_x + \delta_x,$ $0 \leq \delta_x \leq 1$.
3. $\mathcal{S}(x = 2y, \delta_y \leq 1, \delta_x \geq 3)$ is $x = 2y, a_y - \delta_y \leq y \leq$ $a_y + \delta_y, 0 \leq \delta_y \leq 1, a_x - \delta_x \leq x \leq a_x + \delta_x, \delta_x \geq 3$.

LR-constraints: semantics

A *parameterized polyhedron* is the collection of polyhedra defined by fixing the values of the parameters in a parameterized linear system: $\mathit{Sol}(\mathbf{Ax} \leq \mathbf{b} + \mathbf{Ba}, \mathbf{u}) = \{\mathbf{x} \mid \mathbf{Ax} \leq \mathbf{b} + \mathbf{Bu}\}$, where $\mathbf{u} \in \mathbb{R}^{|\mathbf{a}|}$ is an instance of $\mathbf{a}$.

Example

1. $\mathcal{S}(0 \leq x \leq 2)$ boils down to $\mathit{Sol}(0 \leq x \leq 2)$.
2. For $\mathcal{S}(1 \leq x \leq 4, \delta_x \leq 1)$, we enumerate below the polyhedra obtained by fixing $\delta_x = 1$:

$$\begin{array}{rl} \mathit{Sol}(1 \leq x \leq a_x + 1) & \text{when } a_x \leq 2 \\ \mathit{Sol}(a_x - 1 \leq x \leq a_x + 1) & \text{when } 2 \leq a_x \leq 3 \\ \mathit{Sol}(a_x - 1 \leq x \leq 4) & \text{when } 3 \leq a_x. \end{array}$$

3. $\mathcal{S}(x = 2y, \delta_y \leq 1, \delta_x \geq 3)$ is unexpectedly non-empty!
   Let $\delta_x = 3, \delta_y = a_x = a_y = 0$, then $x = y = 0$ is a solution.

LR-constraints: semantics

So we explicitly restrict to parameter instances that satisfy the lower bounds in a range constraint.

- Let $S = \mathcal{S}(c \wedge d)$. We say that a parameter instance $\mathbf{u}$ is a *solution* of d in S if for every $\delta_x \geq s$ in d, we have: $\mathit{width}(\mathit{Sol}(S, \mathbf{u}), x) \geq 2s$.

And we extend the *width*() function.

- Let $S = \mathbf{Ax} \leq \mathbf{b} + \mathbf{Ba}$ be a parameterized linear system. The width of x in S is defined as: $\mathit{parwidth}(S, x) = \mathit{max}_{\mathbf{u}}\ \mathit{width}(\mathit{Sol}(S, \mathbf{u}), x)$, if it exists. Otherwise, $\mathit{parwidth}(S, x) = \infty$.

LR-constraints: satisfiability

Theorem

A LR-constraint $c \wedge d$ is **satisfiable** *iff*

- *c is satisfiable,*
- *$d \wedge \bigwedge_{\delta_x \in vars(d)} 0 \leq \delta_x$ is satisfiable as a linear constraint,*
- *there exists a solution of d in $\mathcal{S}(c \wedge d)$.*

For $c \wedge d$ satisfiable and $s = parwidth(\mathcal{S}(c \wedge d), x)/2 \neq \infty$

- *$c \wedge d$* **entails** *$\delta_x \leq s$.*

We'll give algorithms for checking the existence of a solution of d and computing $parwidth()$.

LR-constraints: satisfiability

Example

Let $c \wedge d$ be $0 \leq x \leq 10, 0 \leq y \leq x, \delta_x = 3, \delta_y \geq 4$. **Satisfiable?**
$S = \mathcal{S}(c \wedge d)$, where $a_x, a_y, \delta_x, \delta_y$ are parameters, is:

$$0 \leq x \leq 10, 0 \leq y \leq x, \delta_x = 3, 4 \leq \delta_y,$$
$$a_x - \delta_x \leq x \leq a_x + \delta_x, a_y - \delta_y \leq y \leq a_y + \delta_y$$

- c is satisfiable,
- $\delta_x = 3, \delta_y \geq 4, 0 \leq \delta_x, 0 \leq \delta_y$ is satisfiable,
- let us find a parameter instance $\mathbf{u}$ such that:

  $width(Sol(S, \mathbf{u}), x) \geq 6$, since $\delta_x \geq 3$ is in d, and
  $width(Sol(S, \mathbf{u}), y) \geq 8$, since $\delta_y \geq 4$ is in d.

  By defining $\mathbf{u}$ as: $a_x = 7, \delta_x = 3, a_y = 5, \delta_y = 5$, we have $width(Sol(S, \mathbf{u}), x) = 6$ and $width(Sol(S, \mathbf{u}), y) = 10$.
  Hence $\mathbf{u}$ is a solution for d in S.

Yes, $c \wedge d$ is satisfiable.

Interlude

Minkowski, Motzkin, 1953:

Theorem (Minkowski's decomposition thm)

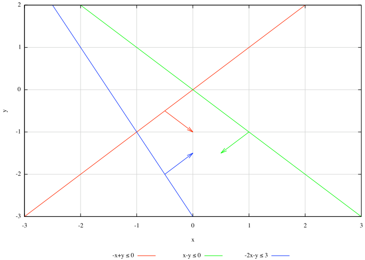
There exists an effective procedure that given $\mathbf{Ax} \leq \mathbf{b}$ *decides whether or not the polyhedron* $Sol(\mathbf{Ax} \leq \mathbf{b})$ *is empty and, if not, it yields a generating matrix* $\mathbf{R}$ *and a vertex matrix* $\mathbf{V}$ *such that:*

- $Sol(\mathbf{Ax} \leq \mathbf{b}) = \{\mathbf{x} \mid \mathbf{x} = \mathbf{R}\boldsymbol{\lambda}, \boldsymbol{\lambda} \geq 0\} + \{\mathbf{x} \mid \mathbf{x} = \mathbf{V}\gamma, \gamma \geq 0, \Sigma\gamma = 1\}$,
- $Sol(\mathbf{Ax} \leq \mathbf{0}) = \{\mathbf{x} \mid \mathbf{x} = \mathbf{R}\boldsymbol{\lambda}, \boldsymbol{\lambda} \geq 0\}$.

Interlude

Example

$$\begin{pmatrix} -1 & 1 \\ 1 & 1 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$$



$$\mathbf{R} = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} \qquad \mathbf{V} = \begin{pmatrix} 0 & -1 \\ 0 & -1 \end{pmatrix}$$

Interlude

Loechner and Wilde, 1997:

Theorem (Minkowski's thm for parameterized polyhedra)

Every parameterized polyhedron can be expressed by a generating matrix $\mathbf{R}$ *and finitely many pairs*

$$(\mathbf{v}^{\mathbf{a}}(1), \mathbf{C}_1\mathbf{a} \leq \mathbf{c}_1), \ldots, (\mathbf{v}^{\mathbf{a}}(k), \mathbf{C}_k\mathbf{a} \leq \mathbf{c}_k)$$

where, for $i = 1..k$, $\mathbf{v}^{\mathbf{a}}(i)$ *is a vector parametric in* $\mathbf{a}$, $Sol(\mathbf{C}_i\mathbf{a} \leq \mathbf{c}_i)$ $\neq \emptyset$, *and such that:*

- $Sol(\mathbf{Ax} \leq \mathbf{b} + \mathbf{Ba}, \mathbf{u}) = \{\mathbf{x} \mid \mathbf{x} = \mathbf{R}\boldsymbol{\lambda}, \boldsymbol{\lambda} \geq 0\}$ $+ ConvexHull(\{\mathbf{v}^{\mathbf{u}}(i) \mid i = 1..k, \mathbf{C}_i\mathbf{u} \leq \mathbf{c}_i\})$,
- $Sol(\mathbf{Ax} \leq \mathbf{0}) = \{\mathbf{x} \mid \mathbf{x} = \mathbf{R}\boldsymbol{\lambda}, \boldsymbol{\lambda} \geq 0\}$.

Interlude

Example

$a + b \geq y, y \geq a, y \geq b, x = a$

$$\begin{pmatrix} 1 & 0 \\ \text{-1} & 0 \\ 0 & 1 \\ 0 & \text{-1} \\ 0 & \text{-1} \end{pmatrix} \begin{pmatrix} \text{x} \\ \text{y} \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ \text{-1} & 0 \\ 1 & 1 \\ \text{-1} & 0 \\ 0 & \text{-1} \end{pmatrix} \begin{pmatrix} \text{a} \\ \text{b} \end{pmatrix}$$

$\mathbf{R} = \mathbf{0}$

$(\begin{pmatrix} \text{a} \\ \text{b} \end{pmatrix}, b \geq a \geq 0)$ $(\begin{pmatrix} \text{a} \\ \text{a} \end{pmatrix}, a \geq b \geq 0)$ $(\begin{pmatrix} \text{a} \\ \text{a+b} \end{pmatrix}, a, b \geq 0)$

Computing widths: abs()

The maximum absolute value of a linear expression over the solutions of a non-empty polyhedron S:

- $abs(S, \mathbf{c}^T\mathbf{x} + \alpha) = max\{|\mathbf{c}^T\mathbf{x}_0 + \alpha| \mid \mathbf{x}_0 \in S\}$ if it exists.
- Otherwise, $abs(S, \mathbf{c}^T\mathbf{x} + \alpha) = \infty$.

A direct implementation of the $abs()$ function:

- $M = max\{\mathbf{c}^T\mathbf{x} + \alpha \mid \mathbf{x} \in S\}$,
- $m = min\{\mathbf{c}^T\mathbf{x} + \alpha \mid \mathbf{x} \in S\}$,
- $abs(S, \mathbf{c}^T\mathbf{x} + \alpha) = r \in \mathbb{R}$ iff $M, m \in \mathbb{R}$ and $max\{M, -m\} = r$.

Computing widths

Theorem

Consider the Minkowski's form of the parameterized $S = \mathcal{S}(c \wedge d)$.

- $parwidth(S, \mathbf{x}_i) = r \in \mathbb{R}$ *iff* $row(\mathbf{R}, i) = \mathbf{0}$ *and*

$$r = max(\{0\} \cup \{s \mid 1 \leq m < n \leq k, Sol(P_{m,n}) \neq \emptyset, \\ s = abs(Sol(P_{m,n}), \mathbf{v^a}(m)_i - \mathbf{v^a}(n)_i)\}),$$

  where $P_{m,n} = \mathbf{C}_m\mathbf{a} \leq \mathbf{c}_m, \mathbf{C}_n\mathbf{a} \leq \mathbf{c}_n$.

- *There exists a solution of* d *in* S *iff the following constraint over parameters is satisfiable:*

$$\bigwedge_{\substack{\delta_{\mathbf{x}_i} \geq s \in d,\\ s>0,\\ row(\mathbf{R},i)=\mathbf{0}}} \bigvee_{1 \leq m < n \leq k} (P_{m,n} \wedge |\mathbf{v^a}(m)_i - \mathbf{v^a}(n)_i| \geq 2s)$$

Example

Let $c \wedge d$ be $0 \leq x \leq 10, 0 \leq y \leq x, \delta_x = 3, \delta_y \geq 4$. The generating matrix $\mathbf{R}$ has no ray. Parameterized vertices for $\mathcal{S}(c \wedge d)$, together with their domains:

The additional constraint $4 \leq \delta_y, \delta_x = 3$ must be added to the domain of every vertex.

$\mathbf{v^a}(1) = (a_x + 3, a_y + \delta_y)$
if $0 \leq a_y + \delta_y \leq a_x + 3$,
$a_x + 3 \leq 10$

$\mathbf{v^a}(2) = (a_x + 3, a_y - \delta_y)$
if $0 \leq a_y - \delta_y \leq a_x + 3$,
$a_x + 3 \leq 10$

$\mathbf{v^a}(3) = (a_x - 3, a_y + \delta_y)$
if $0 \leq a_y + \delta_y \leq a_x - 3$,
$a_x - 3 \leq 10$

$\mathbf{v^a}(4) = (a_x - 3, a_y - \delta_y)$
if $0 \leq a_y - \delta_y \leq a_x - 3$,
$a_x - 3 \leq 10$

$\mathbf{v^a}(5) = (a_y + \delta_y, a_y + \delta_y)$
if $0 \leq a_y + \delta_y \leq a_x + 3$,
$a_x - 3 \leq a_y + \delta_y \leq 10$

$\mathbf{v^a}(6) = (a_y - \delta_y, a_y - \delta_y)$
if $0 \leq a_y - \delta_y \leq 10$,
$a_x - 3 \leq a_y - \delta_y \leq a_x + 3$

$\mathbf{v^a}(7) = (a_x + 3, a_x + 3)$
if $0 \leq a_x + 3 \leq 10$,
$a_y - \delta_y \leq a_x + 3 \leq a_y + \delta_y$

$\mathbf{v^a}(8) = (a_x - 3, a_x - 3)$
if $0 \leq a_x - 3 \leq 10$,
$a_y - \delta_y \leq a_x - 3 \leq a_y + \delta_y$

$\mathbf{v^a}(9) = (a_x + 3, 0)$
if $0 \leq a_x + 3 \leq 10$,
$0 \leq a_y + \delta_y, a_y - \delta_y \leq 0$

$\mathbf{v^a}(10) = (a_x - 3, 0)$
if $0 \leq a_x - 3 \leq 10$,
$0 \leq a_y + \delta_y, a_y - \delta_y \leq 0$

$\mathbf{v^a}(11) = (0, 0)$
if $-3 \leq a_x \leq 3$,
$0 \leq a_y + \delta_y, a_y - \delta_y \leq 0$

$\mathbf{v^a}(12) = (10, a_y + \delta_y)$
if $7 \leq a_x \leq 13$,
$0 \leq a_y + \delta_y \leq 10$

$\mathbf{v^a}(13) = (10, a_y - \delta_y)$
if $7 \leq a_x \leq 13$,
$0 \leq a_y - \delta_y \leq 10$

$\mathbf{v^a}(14) = (10, 10)$
if $7 \leq a_x \leq 13$,
$10 \leq a_y + \delta_y, a_y - \delta_y \leq 10$

$\mathbf{v^a}(15) = (10, 0)$
if $7 \leq a_x \leq 13$,
$0 \leq a_y + \delta_y, a_y - \delta_y \leq 0$

Example (Continued)

- $c \wedge d$ entails $\delta_y \leq 5$
  - The intersection of the domains 2 and 14, namely $P_{2,14}$ is:

$$0 \leq a_y - \delta_y \leq a_x + 3, a_x + 3 \leq 10, 4 \leq \delta_y, \delta_x = 3,$$
$$7 \leq a_x \leq 13, 10 \leq a_y + \delta_y, a_y - \delta_y \leq 10,$$

    i.e., $a_x = 7, 0 \leq a_y - \delta_y \leq 10, 10 \leq a_y + \delta_y, 4 \leq \delta_y, \delta_x = 3$.
    For $i = 2$, $\mathbf{x}_i$ is y and $\mathbf{v}^{\mathbf{a}}(2)_i - \mathbf{v}^{\mathbf{a}}(14)_i$ is $(a_y - \delta_y) - 10$.
    Its absolute value over $P_{2,14}$ is 10, obtained by $\mathbf{u}$: $a_x = 7$, $\delta_x = 3$, $a_y = 5, \delta_y = 5$.
    Since the absolute value over all other pairs of vertices cannot be greater than 10 due to the original constraint $0 \leq x \leq 10$, $0 \leq y \leq x$, $parwidth(\mathcal{S}(c \wedge d), y) = 10$.
- there exists a solution of d in S
  - In addition to $|\mathbf{v}^{\mathbf{u}}(2)_2 - \mathbf{v}^{\mathbf{u}}(14)_2| = 10 \geq 8$ covering $\delta_y \geq 4$, $\mathbf{u}$ is a solution of $P_{10,14}$ and $|\mathbf{v}^{\mathbf{u}}(10)_1 - \mathbf{v}^{\mathbf{u}}(14)_1| = 6 \geq 6$, covering $\delta_x \geq 3$. Hence $\mathbf{u}$ a solution of d.

Conclusion

- Variable ranges in linear constraints: syntax, semantics, correct and complete algorithms for satisfiability and entailment based on linear parametric programming.
- In the paper, a corresponding conservative extension of CLP($\mathcal{R}$).
- Future work:
  - disequalities $\delta_x \neq s$,
  - generic inequalities, e.g., $\delta_x \leq \delta_y$,
  - extension of the entailment procedure to lower bounds,
  - experimental evaluation.