Typing Linear Constraints

SALVATORE RUGGIERI
Dipartimento di Informatica, Università di Pisa, Italy
FRED MESNARD
Lim-Iremia, Université de la Réunion, France

We present a type system for linear constraints over the reals intended for reasoning about input-output directionality of variables. The types model the properties of definiteness, range width or approximation, lower and upper bounds of variables in a linear constraint. Several proof procedures are presented for inferring the type of a variable and for checking validity of type assertions. We rely on theory and tools for linear programming problems, linear algebra, parameterized polyhedra and negative constraints. An application of the type system is proposed in the context of static analysis of constraint logic programs. Type assertions are at the basis of the extension of well-moding from pure logic programming. The proof procedures (both for type assertion validity and for well-moding) are implemented and their computational complexity is discussed. We report experimental results demonstrating the efficiency in practice of the proposed approach.

Categories and Subject Descriptors: I.2.2 [Artificial Intelligence]: Automatic programming; I.2.3 [Artificial Intelligence]: Deduction and theorem proving

General Terms: Languages, Verification

Additional Key Words and Phrases: Linear constraints, polyhedra, constraint logic programming, well-moding, definiteness.

1. INTRODUCTION

(Linear) constraint solving is naturally undirected, in the precise sense that there is no explicit set of input/output variables. As an example, consider the linear constraint

\[ F = \frac{9}{5} \cdot C + 32 \]

stating the Fahrenheit-Celsius conversion rule. A constraint solver provided with \( c \) and a Celsius value \( C = 30 \) returns \( F = 86 \) as the solved constraint. Conversely, a constraint solver provided with \( c \) and a Fahrenheit value \( F = 77 \) returns \( C = 25 \) as the solved constraint. This extends to intervals, lower bounds and upper bounds as well. Assume that the Celsius temperature is known with some approximation (e.g., due to measurement error of a sensor) \( C = 30 \pm 1 \), or, by adopting a different syntax, that \( 29 \leq C, C \leq 31 \). A constraint solver provided with \( c \) and a Celsius interval \( 29 \leq C, C \leq 31 \) returns as the solved constraint \( 84.2 \leq F, F \leq 87.8 \), or \( F = 86 \pm 1.8 \).

Programming with constraints allows then for declaratively modelling a problem by generating the set of constraints among the variables at hand, and then letting a constraint solver find a solved form. Constraint generation can be as simple as feeding a constraint solver with a predetermined system of linear equalities and
inequalities, as in the example above. Or it can assumes more complex forms, requiring iteration, recursion or non-determinism. Constraint logic programming languages, for instance, adopt recursion, non-determinism and intertwined constraint generation & solving.

The objective of this paper is twofold. As the first main contribution, we propose a type system that allows for reasoning on the input-output directionality of variables occurring in a linear constraint. The types considered include definiteness of a variable, namely it assumes a unique value; lower bound; upper bound; and maximum range width. These types allow for verifying that the solved form returned by a constraint solver satisfies the intended directionality, e.g., that a certain variable is definite, or that it has a variability range within a maximum threshold, or that it is upper/lower bounded. Recalling the Fahrenheit-Celsius conversion example, we will be able to conclude that if \( C \) is definite, then in the solved form of \( c \) also \( F \) is definite; or that if \( C \) is known with an approximation of \( \pm 1 \), then in the solved form of \( c \) we have that \( F \) is known with an approximation of \( \pm 1.8 \). Formally, type assertions are introduced in order to derive types implied by a constraint and a set of typed variables. The semantics of a type assertion is provided as a first order formula over the theory of the reals. However, a geometric view based on parameterized polyhedra will be considered in order to reason on type assertions. Validity of type assertions is thoroughly investigated by devising several checking and inference procedures. We proceed incrementally by first considering lower and upper bound types: the inference procedure relies on solving homogeneous linear programming problems. Next, we add definiteness by reasoning on the set of implicit equalities of the underlying linear constraint. Third, we tackle range width by computing the Minkowski’s form of a parameterized polyhedron. The procedures are sound and complete with respect to the appropriate subset of types. Moreover, the approach is extended to deal with constraints containing strict inequalities and disequalities. Finally, parameters are added to the languages of types in order to express range widths of parametric size. Recalling again the Fahrenheit-Celsius conversion example, we will be able to show that if \( C \) is known with an approximation of \( \pm s \), then in the solved form of \( c \) we have that \( F \) is known with an approximation of \( \pm 1.8s \).

The considered types can be useful for reasoning about variable directionality in programs and systems loosely coupled with constraint solver. Even more challenging is the case of programming languages that tightly integrate constraint solving. As another main contribution of this paper, we study how the type system can be adopted for reasoning over constraint logic programs, which represent an sophisticated scheme for programming with constraint. Modern constraint logic programming languages offer the notion of moding [Apt 1997] as program annotations allowing the programmer to specify the input-output behaviour of predicate arguments. Modes are at the basis of compiler optimizations, program transformations and termination analyses. As an example, consider the \texttt{MORTGAGE} program over \texttt{CLP(R)}.

\begin{verbatim}
(m1) mortgage(P,T,R,B) ← T = 0, B = P.
(m2) mortgage(P,T,R,B) ← T >= 1, NP = P + P * 0.05 - R, NT = T - 1, mortgage(NP,NT,R,B).
\end{verbatim}
The query ← mortgage(100, 5, 20, B) is intended for calculating the balance of a mortgage of 100 units after giving back 20 units per year for a period of 5 years. The answer provides an exact value (i.e., a real number) for the required balance, namely B = 17.12. Using the moding terminology, we say that given definite values for principal, time and repayment, in every (non-deterministically computed) answer we obtain a definite value for the balance. However, this is only one mode we can query the program above. The query ← 3 <= T, T <= 5, mortgage(100, T, 20, B) is intended for calculating the balance at the end of the third, fourth and fifth year. Principal and repayment are now definite, whilst time is (upper and lower) bounded. Again, for each of the three answers: T=3, B=52.71; T=4, B=35.35; and T=5, B=17.12 we get a definite value for the balance. Intuitively, this mode is more general than the previous one, since definiteness of time has been replaced by boundedness. Finally, consider the query ← 0 <= B, B <= 10, 15 <= R, R <= 20, mortgage(P, 5, R, B). It is intended for calculating the principal that a person could be granted under the condition of repaying from 15 to 20 units per year, and with the requirement that after 5 years the final balance is of at most 10 units. The answer is \( P = 0.78 \times B + 4.33 \times R \), which is not definite, but, since B and R are bounded, it is (upper and lower) bounded. This mode is not comparable to the previous ones, since we now provide a definite value for time and a range for balance and repayment, and we wish to compute a range for the principal of the answer. These examples give only a few hints about the flexibility of the constraint logic programming (CLP) scheme when compared to pure logic programming, where definiteness of variables corresponds to groundness, but upper and lower bounds have no direct counterpart. Type assertions are at the basis of moding programs in constraint logic programming languages with linear constraints over reals and rationals, as in CLP(R) [Jaffar et al. 1992; Holzbaur 1995], ECLiPSe, SICStus Prolog, SWI Prolog, and many others. We conservatively extend the notion of well-moding [Apt 1997] from pure logic programming to CLP(R), proving useful properties in support of static analysis, including persistence, call pattern characterization and answer constraint characterization. Moreover, we show how to deal with programs mixing linear constraints and logic terms, and how the notion generalizes in the presence of parametric types.

As well as discussing the computational complexity of the procedures presented in the paper, we have implemented, in standard C++, all the type assertion checking and inference procedures, and the well-moding checking procedures. Experimental results are reported over a large collection of testbed CLP programs, which demonstrate the efficiency in practice of the proposed approach. The software developed is released as open source. This is the third main contribution of this paper.

Organization of the paper. We start by recalling in Section 1.1 basic notions for linear algebra, linear programming and constraint logic programming. In Section 2 we introduce syntax and semantics of types, type declarations and type assertions. The problems of checking validity of type assertions and of inferring the most general types for variables are formally introduced. In Section 3, we rephrase the problems in a geometric view, which is the basis for designing several checking and inference procedures. Also, extensions to strict inequalities and disequalities,
and to parametric types are reported. Finally, the computational complexity of
the procedures is discussed. In Section 4, modes and well-moding for CLP(\(\mathcal{R}\))
programs, possibly with type terms, is introduced, with type assertions as the
basic proof mechanism. The implementations of the type inference procedures
and of the well-mode checking procedures is presented in Section 5 together with
an extensive experimentation over several testbed programs. Related work and
conclusion follow in Section 6 and Section 7. For readability reasons, all proofs of
the paper are reported in Appendix A.

1.1 Preliminaries

We adhere to standard notation for linear algebra [Schrijver 1986], linear program-
ming [Murty 1983] and (constraint) logic programming [Apt 1997; Jaffar and Maher
1994; Jaffar et al. 1998].

Linear algebra. \(\mathcal{R}\) denotes the set of real numbers. Small capital letters (\(a, b,
\ldots\)) denote column vectors, while capital letters (\(A, B, \ldots\)) denote matrices. \(0\)
and \(1\) are column vectors with all elements equal to 0 and 1 respectively. \(a_i\) denotes
the \(i^{th}\) element in \(a\), \(A_i\) the \(i^{th}\) column in \(A\), and \(\text{row}(A, i)\) the row vector consisting
of the \(i^{th}\) row of \(A\). \(a^T\) (resp., \(A^T\)) denotes the transposed vector (resp., matrix) of
\(a\) (resp., \(A\)). \(c^T x\) denotes the inner product of the transposed vector \(c^T\) and \(x\). \(\Sigma v\)
is the sum of all the elements in \(v\). \(Ax \leq b\) denotes a system of linear inequalities
(or, a linear system) over the variables in \(x\). We assume that the dimensions of
vectors and matrices in inner products and linear systems are of the appropriate
size. The solution set of points that satisfy a formula/linear system \(\psi\) over \(\mathcal{R}^n\)
is defined as \(\text{Sol}(\psi) = \{x \in \mathcal{R}^n \mid \psi(x)\}\). A polyhedron is the solution set of a linear
system, namely \(\text{Sol}(Ax \leq b)\). Polyhedra are both closed and convex sets. The
vectorial sum of two sets of vectors is defined as:

\[
A + B = \{x + y \mid x \in A, y \in B\}.
\]

Linear programming. A linear programming problem consists of determining
\(\max\{c^T x \mid Ax \leq b\}\), if it exists. The problem is infeasible when \(\{x \mid Ax \leq b\} = \emptyset\).
If feasible, but \(\{c^T x \mid Ax \leq b\}\) has no upper bound, the problem is unbounded,
and we write \(\max\{c^T x \mid Ax \leq b\} = \infty\). Otherwise, it is bounded. We write
\(\max\{c^T x \mid Ax \leq b\} \in \mathcal{R}\) when the problem is feasible and bounded.

Constraint logic programming. The CLP Scheme defines a family of languages,
CLP(\(\mathcal{C}\)), that are parametric in the constraint domain \(\mathcal{C}\). We are interested here
in CLP(\(\mathcal{R}\)), namely the constraint domain over the reals\(^1\). All results apply to the
constraint domain of rationals as well, namely to CLP(\(\mathcal{Q}\)) programming languages.
A primitive linear constraint is an expression \(a_1 \cdot x_1 + \ldots + a_n \cdot x_n \simeq a_0\), where \(\simeq\) is
in \(\{\leq, =, \geq\}\), \(a_1, \ldots, a_n\) are constants in \(\mathcal{R}\) and \(x_1, \ldots, x_n\) are variables. We will
use the inner product form by rewriting it as \(c^T x \simeq a\). A linear constraint \(c\) is a
sequence of primitive linear constraints, whose interpretation is their conjunction.
We write \(\mathcal{R} \models \psi\) to denote that the first order formula \(\psi\) is true in the domain of
reals [Shoenfield 1967]. A CLP(\(\mathcal{R}\)) program is a finite set of clauses of the form

\(^1\)We assume that the underlying constraint solver is ideal, namely without rounding errors. For
CLP(\(\mathcal{R}\)) this is achieved by adopting exact arithmetic libraries. For CLP(\(\mathcal{Q}\)) this is achieved by
adopting rationals and exact integer arithmetic libraries.
A ← c, B₁, . . . , Bₙ, where A is an atom, c a linear constraint, and B₁, . . . , Bₙ (n ≥ 0) a sequence of atoms. We assume that atoms are in flat form, namely an atom is of the form p(x₁, . . . , xₙ) where p is a predicate of arity n and x₁, . . . , xₙ are (not necessarily distinct) variables. A query ← c, B₁, . . . , Bₙ consists of a linear constraint and a sequence of atoms.

2. BOUND TYPES FOR LINEAR CONSTRAINTS

2.1 Syntax and Semantics

We introduce a static typing for variables in linear constraints. The set of types BT is defined first.

**Definition 2.1 (Types).** A type is an element of \( BT = \{*, \sqcup, \sqcap, \sqsubseteq\} \cup \{\sqsubseteq r | r \in \mathbb{R}, r \geq 0\} \). We use \(!\) as a shorthand for \(\sqsubseteq 0\). Moreover, we denote by \( BT_\sqcap \) the subset \( \{*, \sqcup, \sqcap, \sqsubseteq\}\), and by \( BT; \) the subset \( \{*, \sqcup, \sqcap, \square, !\}\).

The intuitive meaning of a type is to label variables occurring in a constraint. \(!\) is intended for typing variables that show at most one single value in every solution, a property known as definiteness; \(\square r\) is intended for typing variables that assume a range of values of width at least \( r\) or, as an alternative interpretation, that are bounded by \( r\); \(\sqsubseteq\) is intended for variables that assume a range of values of unknown width, yet lower and upper bounds for those values still exist; \(\sqcap\) (resp., \(\sqcup\)) is intended for variables that have a lower bound (resp., an upper bound); and finally, \(*\) is to be used when no upper or lower bound can be stated. Let us introduce syntactic means to assert the type of variables.

**Definition 2.2 (Types assertions).** An atomic type declaration (atd, for shorthand) is an expression \( x : \tau \), where \( x \) is a variable and \( \tau \in BT \). We define \( \text{vars}(x : \tau) = \{x\} \), and say that \( x \) is typed as \( \tau \).

A type declaration is a sequence of atd’s \( d₁, \ldots , dₙ \), with \( n \geq 0 \). We define \( \text{vars}(d₁, \ldots , dₙ) = \bigcup_{i=1..n} \text{vars}(dᵢ) \).

A type assertion is an expression \( d₁ \vdash c \rightarrow d₂ \), where \( d₁, d₂ \) are type declarations and \( c \) is a linear constraint.

Type declarations assign a type to variables. Such a typing is used in type assertions as an hypothesis (to the left of \(\vdash\)) or as a conclusion (to the right of \(\rightarrow\)). Intuitively, the type assertion \( d₁ \vdash c \rightarrow d₂ \) states that given the type declaration \( d₁ \), the type declaration \( d₂ \) holds under the linear constraint \( c \).

**Example 2.3.** The type assertion \( z ! : \vdash y - x \leq z, y + x \leq z, -y - 2x \leq 5 - z \rightarrow y : \sqcap, x : \sqcup \) intuitively states that if \( z \) has a fixed value then the set of solutions of the involved constraint is either empty or such that \( y \) has an upper bound and \( x \) has a lower bound. Figure 1 (left) shows graphically the set of solutions for \( z = 1 \).

The type assertion \( z ! : \vdash y - x \leq z, y + x \leq z, z \leq y \rightarrow y : \square, x ! : \vdash \) states that if \( z \) has a fixed value then either the set of solutions of the involved constraint is empty or both \( x \) and \( y \) assume unique values in it. Figure 1 (right) shows graphically the set of solutions for \( z = 1 \).

For a type declaration \( d \), we write \( d|ₓ \) to denote the subsequence of \( d \) consisting only of atd’s typing variables in \( x \). Analogously, we write \( d|r \) (resp., \( d|₂ \)) to denote
For the type assertion

\[ y - x \leq z \]

\[ y + x \leq z \]

it is readily checked that \( z = 1 \).

Notice that we are redundant about the type !.

**Definition 2.4 (Semantics).** We associate with an atd \( d = x : \tau \) a formula \( \phi(d) \) over fresh variables \( v(d) \), called parameters, as follows:

\[
\begin{align*}
\phi(x :) &= x = a \\
\phi(x : \square r) &= a \leq x \land x \leq a + r \\
\phi(x : \downarrow) &= a \leq x \land x \leq b \\
\phi(x : \uparrow) &= a \leq x \\
\phi(x : \sqcap) &= x \leq b \\
\phi(x : \star) &= \text{true}
\end{align*}
\]

\( \phi \) and \( v \) extend to type declarations as follows:

\[
\phi(d_1, \ldots, d_n) = \land_{i=1..n} \phi(d_i) \quad v(d_1, \ldots, d_n) = \cup_{i=1..n} v(d_i).
\]

A type assertion \( d_1 \vdash c \rightarrow d_2 \) is valid if for \( v = \text{vars}(c) \cup \text{vars}(d_1) \cup \text{vars}(d_2) \), the following formula is true in the domain of reals:

\[
\forall v(d_1) \exists v(d_2) \forall v(d_1) \forall v(d_1) \forall v(d_1) \land c \rightarrow \phi(d_2).
\] (1)

Notice that, for the \( v() \) function defined so far, we have that \( v(d_2) \setminus v(d_1) = v(d_2) \), since we consider fresh parameters. Later on in Section 3.6, the syntax will be extended, and then the general formulation of (1) will be required.

**Example 2.5.** For the type assertion \( z :: y - x \leq z, y + x \leq z, z \leq y \rightarrow y ::, x :: \), the formula to be shown is:

\[
\forall a \exists b, c \forall x, y, z \ (z = a \land y - x \leq z \land y + x \leq z \land z \leq y) \rightarrow (y = b \land x = c).
\]

The set of variables is fixed to \( v = \text{vars}(c) \cup \text{vars}(d_1) \cup \text{vars}(d_2) \) in order to take into account variables that appear in \( d_1 \) or \( d_2 \) but not in \( c \). E.g., for the (valid) type assertion \( x : \sqcap \leftarrow \text{true} \rightarrow x : \sqcap \), we have the formula \( \forall a \exists b \forall x \leq a \rightarrow x \leq b \).

A natural ordering over types is induced by the semantics above. For instance, it is readily checked that \( d \vdash c \rightarrow x :: \) implies \( d \vdash c \rightarrow \square \) for any \( d, c \) and \( x \), or

\[
\begin{align*}
\forall x &\exists y, z \ (z = a \land y - x \leq z \land y + x \leq z \land z \leq y) \rightarrow (y = b \land x = c).
\end{align*}
\]
that $d \vdash c \rightarrow x : \Box_2$ implies $d \vdash c \rightarrow x : \Box_3$. Similar implications lead to define an order $\geq_t$ over types, which models type preciseness.

**Definition 2.6.** The $\geq_t$ partial order over $\mathcal{BT}$ is defined as the reflexive and transitive closure of the following relation $\rightarrow$:

$$
\begin{align*}
! & \rightarrow \Box \\
\Box & \rightarrow \ast \\
\ast & \rightarrow \Box_r \\
\Box_r & \rightarrow \Box_s \\
\Box_s & \rightarrow \Box
\end{align*}
$$

for every $r < s$.

We write $\tau >_t \mu$ when $\tau \geq_t \mu$ and $\tau \neq \mu$. We define $\text{lub}(\emptyset) = \ast$ and for $n > 0$:

$$
\text{lub}\{\tau_1, \ldots, \tau_n\} = \min\{\tau \mid \tau \geq_t \tau_i, i = 1..n\}.
$$

Next, the $\geq_t$ relation is extended to type declarations.

**Definition 2.7.** We write $d_1 \geq_t d_2$ if for every $x : \tau$ in $d_2$ there exists $x : \mu$ in $d_1$ such that $\mu \geq_t \tau$.

Using the notation $\geq_t$, the intuition behind the ordering can be formalized in a monotonicity lemma.

**Lemma 2.8 (Monotonicity).** Assume that $d_1 \vdash c \rightarrow d_2$ is valid. If $d'_1 \geq_t d_1$, $d_2 \geq_t d'_2$ and $R \models c' \rightarrow c$ for a linear constraint $c'$, then $d'_1 \vdash c' \rightarrow d'_2$ is valid.

Normal forms for type declarations are introduced by assigning to each variable the least upper bound of its types. When the least upper bound is $\ast$, the type assignment provides no actual information and then it can be discarded. Normal forms are unique modulo reordering of atd’s.

**Definition 2.9.** We define $\text{nf}(d)$ as any type declaration $d'$ such that $x : \tau$ is in $d'$ iff $\tau = \text{lub}\{\mu \mid x : \mu \text{ is in } d\}$ and $\tau \neq \ast$.

**Example 2.10.** Notice that $x : \Box \geq_t x : \cap, x : \cup$ holds, while $x : \cap, x : \cup \geq_t x : \Box$ does not hold. Actually, $\geq_t$ does not capture semantic implication. We have to move to normal forms to conclude that $\text{nf}(x : \cap, x : \cup) = x : \Box \geq_t x : \Box$.

Normal forms precisely characterize validity when it only depends on type declarations, i.e. for the constraint $\text{true}$.

**Lemma 2.11.** $d_1 \vdash \text{true} \rightarrow d_2$ is valid iff $\text{nf}(d_1) \geq_t \text{nf}(d_2)$.

Finally, transitivity of the $\vdash$ relation is readily checked.

**Lemma 2.12 (Transitivity).** Assume that $d_1 \vdash c \rightarrow d_2$ and $d_1, d_2 \vdash c \rightarrow d_3$ are valid. Then $d_1 \vdash c \rightarrow d_3$ is valid.

### 2.2 Checking and Inferring Type Assertions

Let us concentrate now on the problem of checking the validity of a type assertion. In principle, formulas as in (1) can be checked by real quantifier elimination methods [Dolzmann et al. 1998b; Renegar 1992]. Quantifier elimination traces back to Tarski’s decision procedure [Van Den Dries 1988] for first order formula over real polynomials. The core of the procedure in the case of linear polynomials over
real numbers is the Fourier-Motzkin projection method [Schrijver 1986]. Although in the worst case quantifier elimination is doubly exponential [Basu et al. 1996; Davenport and Heintz 1988], approaches efficient-in-practice have been proposed and successfully applied to theorem proving and program verification. We mention partial cylindrical algebraic decomposition as provided in the QEPCAD/QEPCAD-B systems [Collins and Hong 1991; Brown 2003] and available in the Mathematica tool [Strzebonski 2000] and virtual substitution of test terms [Dolzmann et al. 1998a] as provided in the REDLOG system [Dolzmann and Sturm 1997] and specialized for low-degree polynomials.

While quantifier elimination represents a direct solution to the checking problem and it allows for generalizing to the non-linear case, we observe that formulas in (1) represent a quite restricted class. We will be looking for a specialized and efficient approach to check them. In addition, provided with d and c, we are interested in the problem of inferring type declarations d' defined over a given set of variables v such that d ⊢ c → d' is valid. More precisely, we are interested in the largest one (w.r.t. the ≥ order), if it exists. The inference problem cannot be directly tackled by real quantifier elimination.

**Definition 2.13 (Checking and Inference Problems).** The type checking problem consists of deciding whether a given type assertion d_1 ⊢ c → d_2 is valid.

The type inference problem consists of computing, given d_1, c and a set of variables v, a type declaration d_2 such that for every d'_2 with vars(d'_2) ⊆ v, d_2 ≥_t d'_2 iff d_1 ⊢ c → d'_2 is valid.

A procedure is sound for the type inference problem if the type declaration d_2 returned is such that d_1 ⊢ c → d_2 is valid. For a subset of types B ⊆ BT, we say that a procedure is complete for B if d_2 is such that d_2 ≥_t d'_2 for every valid d_1 ⊢ c → d'_2 with vars(d'_2) ⊆ v and all types in d'_2 belong to B.

The next result shows that a solution exists for the type inference problem and there is a standard representation obtained as the normal form of any solution. Notice, however, that the proof of the lemma is non-constructive.

**Lemma 2.14.** Given d_1, c and a set of variables v, there exists at least one solution d_2 to the type inference problem. Moreover, for any two solutions d_2 and d'_2, we have that nf(d_2) = nf(d'_2) (modulo reordering of atd’s).

Assume d_1 be x : !, and c be y = x, z ≤ w and v be y, z. A solution to the inference problem is y : !, z : *. By adding less precise typings we can obtain other solutions, such as y : !, y : !, z : *. All of them have the same normal form y : !.

**Example 2.15.** Consider the type assertion x : □2 ⊢ x − 1 ≤ z ≤ x − z : τ. What is the most general type inferrible for τ? Since x ranges in [a, a + 2] for some a ∈ R, by simply varying x we have that z ranges in the interval [a − 1, a + 2], hence τ = □3 is the most accurate type inferrible for z. In other words, τ = □3 makes the assertion above valid (soundness), and □3 ≥_t τ holds for every τ such that the type assertion above is valid (completeness).

An inference procedure returning z : □ is sound, albeit it cannot be complete for the full BT type system.

---

Notice that this implies that d_1 ⊢ c → d'_2 is valid for every d'_2 such that d_2 ≥_t d'_2.
Input: a type assertion \(d_1 \vdash c \rightarrow d_2\)

Step 0. \(d = \chi(d_1, c, \text{vars}(d_2))\)

Step 1. If \(d \geq d_2\) Then output “valid”

Else output “not valid”

Fig. 2. \(\text{Check}(\chi)\) procedure.

Notice that type inference can be tricky, especially when \(\Box_r\) types are involved.

Example 2.16. For \(x: \Box_2 \vdash -x \leq z \leq x : \tau\), we can only infer \(\tau = \Box\). In fact, assume that \(x\) ranges in \([a, a+2]\) for \(a \geq 0\). Then \(z\) ranges in \([-a-2, a+2]\), whose width (i.e., \(2(a+2)\)) is unbounded since \(a \geq 0\).

As another example, let \(c\) be \(x-w \leq z, z \leq x+w\). From \(x: \Box_2, w: \Box_3 \vdash c \rightarrow z : \tau, w : \mu\), we can only infer \(\tau = \Box, \mu = \Box_3\). However, by adding \(w \leq 0\) to \(c\), we can infer \(\tau = \Box_2, \mu = !\).

A solution to the inference problem can be easily turned into a solution to the checking problem. Given \(d_1 \vdash c \rightarrow d_2\) to be checked for validity, we first compute the type declaration \(d_2\) returned by the type inference procedure, and then we check whether \(d_2 \geq d_2\) holds. This meta-procedure is summarized in Fig. 2.

Lemma 2.17. Let \(\chi\) be a type inference procedure that is sound, and complete for \(B \subseteq BT\). \(\text{Check}(\chi)\) is a decision procedure for the type checking problem w.r.t type assertions \(d_1 \vdash c \rightarrow d_2\) such that all types in \(d_2\) belong to \(B\).

3. INFERENCE PROCEDURES FOR TYPE ASSERTIONS

3.1 First Intuitions

Our approach switches from the logical view of constraints-as-formulas to a geometric view of constraints-as-polyhedra. Consider a linear constraint \(c\) and a type declaration \(d\). We observe that \(c\) can be equivalently represented as a linear system of inequalities \(A_c v \leq b_c\) where \(v = \text{vars}(c) \cup \text{vars}(d)\). The set of solutions of \(c\) coincides then with the polyhedron represented by \(A_c v \leq b_c\), which we call the geometric representation of \(c\). Analogously, the linear constraint \(\phi(d)\) can be represented as \(A_d v \leq B_d a_d\), where \(a_d\) is the symbolic vector of parameters in \(v(d)\). The resulting system \(\phi(d) \land c\) is a parameterized system of linear inequalities \(P\), where variables in \(v(d)\) play the role of parameters:

\[
\begin{pmatrix}
  A_c \\
  A_d
\end{pmatrix}
\begin{pmatrix}
  v \\
  a_d
\end{pmatrix}
\leq
\begin{pmatrix}
  b_c \\
  0
\end{pmatrix} +
\begin{pmatrix}
  0 \\
  B_d
\end{pmatrix} a_d
\]

The notion of parameterized polyhedra models the solutions of parameterized linear systems.

Definition 3.1 (Parameterized Polyhedron). A parameterized polyhedron is a collection of polyhedra defined by fixing the value for parameters in a parameterized system of linear inequalities: \(\text{Sol}(Ax \leq b + Ba, u) = \{x | Ax \leq b + Bu\}\).

\(\text{Sol}()\) is now a binary function. In addition to a system of parameterized linear inequalities, an assignment to parameters is required.
Example 3.2. Let $d$ be $z$ and $c$ be $y - x \leq z, y + x \leq z, -y - 2x \leq 5 - z$. We have that $\phi(d)$ is $z = a$, and then the parameterized system for $\phi(d) \land c$ is:

$$
\begin{pmatrix}
-1 & 1 & -1 \\
1 & 1 & -1 \\
-2 & -1 & 1 \\
0 & 0 & 1 \\
0 & 0 & -1 \\
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z \\
\end{pmatrix}
\leq
\begin{pmatrix}
0 \\
0 \\
5 \\
0 \\
0 \\
\end{pmatrix} +
\begin{pmatrix}
0 \\
0 \\
0 \\
-1 \\
\end{pmatrix} a
$$

Under this interpretation, validity of $d \vdash c \rightarrow x : \tau$ has an intuitive geometric interpretation. Assume that $x = v_i, d \vdash c \rightarrow x : \tau$ is valid if for every $u \in \mathcal{R}^{(d)}$, the set of solution points $\mathcal{S}_u = \text{Sol}(\mathcal{P}, u)$ is empty or, called $M_u$, max{v | v ∈ $\mathcal{S}_u$} and $m_u = \min{v_i | v \in \mathcal{S}_u}$, the following statements hold:

- if $\tau = \square_r$ then $M_u \in \mathcal{R}, m_u \in \mathcal{R}$, and $M_u - m_u \leq r$, namely $x$ assumes a range of values of width at most $r$.
- as a special case of the previous one, if $\tau = !$ then $x$ assumes a single value;
- if $\tau = \square$ then $M_u \in \mathcal{R}$ and $m_u \in \mathcal{R}$ namely both an upper and a lower bound exist for $x$, but there exists no maximal range width independently from $u$;
- if $\tau = \square_r$ then $m_u \in \mathcal{R}$, namely a lower bound exists for $x$;
- if $\tau = \square_r$ then $M_u \in \mathcal{R}$, namely an upper bound exists for $x$;
- if $\tau = \star$ then we have nothing to show.

Unfortunately, this procedure is not effective, since there are infinitely many $\mathcal{S}_u$ to be checked. In the next two subsections, we will develop approaches for turning the intuitions above into effective procedures.

Example 3.3. Let us consider an example explaining the difference between $\square_r$ and $\square$. The type assertion $x : ! \vdash 0 \leq z \leq x : \square$ is valid, since for every $a \in \mathcal{R}$, $z$ is lower bounded (by 0) and upper bounded (by $x = a$) in $\mathcal{S}_u = \text{Sol}((x = a, 0 \leq z \leq x), (a))$. However, the width of the variability range of $z$ is $a$. Thus, for any $r \geq 0$ by choosing $a = r + 1$ the type assertion $x : ! \vdash 0 \leq z \leq x : \square_r$ is invalidated.

3.2 A Linear Programming Approach for $BT_{\square}$

In this section we develop an inference algorithm which does not explicitly take into account parameters. We will be able to reason on type assertions over $BT_{\square}$. First of all, let us consider the case of unsatisfiable constraints.

Lemma 3.4. Consider the parameterized linear system $\mathcal{P} = \phi(d) \land c$ in (2). There exists a parameter instance $u$ such that $\text{Sol}(\mathcal{P}, u) \neq \emptyset$ if and only if $\text{Sol}(A \mathbf{v} \leq b_i) \neq \emptyset$.

As a consequence, if $\text{Sol}(A \mathbf{v} \leq b_i) = \emptyset$ (i.e., $c$ is unsatisfiable) then there is no chance to obtain a non-empty polyhedron by some instantiation of the parameters in $\phi(d)$. In this case, we can infer assertions of the form $d \vdash c \rightarrow !$, for every variable $x$. Hence, from now on, we will concentrate on satisfiable constraints.

As it will be recalled later on, a non-empty polyhedron $\text{Sol}(A \mathbf{x} \leq \mathbf{b})$ can be decomposed into the vectorial sum of its characteristic cone $\text{Sol}(A \mathbf{x} \leq \mathbf{0})$ with a polytope, a polyhedra bounded along every dimension. Therefore, the existence of
Typing Linear Constraints

\[
\begin{bmatrix}
-1 & 1 & -1 \\
1 & 1 & -1 \\
-2 & -1 & 1 \\
0 & 0 & 1 \\
0 & 0 & -1 \\
\end{bmatrix} \begin{bmatrix}
x \\
y \\
z \\
\end{bmatrix} \leq \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix}
\]

\[y \cdot x \leq 0 \]
\[y - x \leq 0 \]
\[-y - 2x \leq 0 \]

Fig. 3. The homogeneous version of the parameterized linear system in Example 3.2.

an upper/lower bound for a linear function over a polyhedron depends only on its characteristic cone. It is immediate to observe that for every parameter instance \( u \), the polyhedra \( \text{sol}(P, u) \) share the same characteristic cone. As a consequence, proving the existence of an upper bound relies only on the homogeneous version of \( P \), which is not anymore parameterized.

**Lemma 3.5.** Consider the parameterized polyhedron \( P \) in (2). Let \( H \) be its homogeneous version: \( A_c v \leq 0, A_d v \leq 0 \), and assume that \( \text{sol}(A_c v \leq b_c) \neq \emptyset \).

We have that \( \max \{ c^T v \mid v \in \text{sol}(H) \} = 0 \) iff for every parameter instance \( u \), \( \text{sol}(P, u) = \emptyset \) or \( \max \{ c^T v \mid v \in \text{sol}(P, u) \} \in \mathbb{R} \).

When \( c \) is always 0 except for the \( i^{th} \) position where it is 1, we have \( c^T v = v_i \). Lemma 3.5 solves then the problem of deciding whether \( d \vdash c \rightarrow v_i \). By reasoning similarly for types \( \sqcup \) and \( \sqcap \), we can state an effective procedure, called \( \text{LPInfer} \) and summarized in Fig. 4.

**Example 3.6.** The homogeneous version of the parameterized linear system in Example 3.2 and its graphical representation are reported in Figure 3.

It is readily checked that \( x \) has a lower bound and \( y \) has an upper bound.

Notice that, in general, the homogeneous version of \( \phi(x : !) \), \( \phi(x : \sqcap) \) and \( \phi(x : \sqcup) \) collapse to \( x = 0 \). Also, the homogeneous version of \( \phi(x : \sqcap) \) is \( x \geq 0 \), and the one of \( \phi(x : \sqcup) \) is \( x \leq 0 \).

Soundness and a relative form of completeness of \( \text{LPInfer} \) follow. As a consequence of Lemma 2.17, \( \text{Check}(\text{LPInfer}) \) is a decision procedure for \( d_1 \vdash c \rightarrow d_2 \) when \( d_2 \) is defined in the \( \mathcal{BT}_\square \) type system.

**Theorem 3.7** (\( \text{LPInfer} \) - soundness and completeness). \( \text{LPInfer} \) is sound for the type inference problem, and it is complete for \( \mathcal{BT}_\square \).

The \( \text{LPInfer} \) procedure is not tied to any underlying linear programming solver. One straight choice is to adopt the Simplex-based approach [Murty 1983; Schrijver 1986], with at most \( 2|v \cap v_c| + 1 \) calls to the Simplex algorithm, namely one call to test satisfiability at Step 2 and two calls per variable to test upper and lower bounds at Step 4 (a). Due to the approach that we will follow later on for dealing with parameters, we present here an instantiation of \( \text{LPInfer} \) which consists of directly computing the generating matrix and the vertex matrix of polyhedra. This is an alternative representation of polyhedra, known as the explicit representation.
Define \( \phi \) and \( \phi(d) \) as follows.

\begin{align*}
\text{Step 0.} & \quad \text{Define } v_c = \text{vars}(c), \quad n = n(d_1), \quad d = n_v.
\text{Step 1.} & \quad \text{Let } A_v c \leq b_v \text{ be the geometric representation of } c, \quad \text{and } A_d v_c \leq B_d a_d \text{ the geometric representation of } \phi(d).
\text{Step 2.} & \quad \text{If } \text{Sol}(A_v c \leq b_v) = \emptyset \text{ Then for every } x \in v, \text{ output } "x!:"
\text{Else} & \quad \text{Step 3.} \quad \text{For every } x \in v \setminus v_c \text{ and } x : \tau \in n, \text{ output } "x : \tau";
\quad \text{Step 4.} \quad \text{For every } x \in v \cap v_c:
\quad \text{Step 4 (a).} \quad \text{Let } M = \max\{x \mid A_v c \leq 0, A_d v_c \leq 0\}, \quad m = \max\{-x \mid A_v c \leq 0, A_d v_c \leq 0\}.
\quad \text{Step 4 (b).} \quad \text{Output } "x : \bigoplus" \text{ if } M = 0 \text{ and } m = 0;
\quad \text{Step 4 (c).} \quad \text{Output } "x : \bigotimes" \text{ if } M = \infty \text{ and } m = 0;
\quad \text{Step 4 (d).} \quad \text{Output } "x : \sqcap" \text{ if } M = 0 \text{ and } m = \infty;
\quad \text{Step 4 (e).} \quad \text{Output } "x : \ast" \text{ if } M = \infty \text{ and } m = \infty.
\end{align*}

A column of \( R \) is called a ray for any \( x_0 \in \text{Sol}(A x \leq b) \) and ray \( r \), it turns out that \( r \lambda + x_0 \in \text{Sol}(A x \leq b) \) for every \( \lambda \geq 0 \). A column of \( V \) is called a vertex. The set Convex Hull \( (V) \) = \( \{ x \mid x = V \gamma, \gamma \geq 0, \Sigma \gamma = 1 \} \), where \( V \) is a matrix or a finite set of vectors, is the convex hull of the vertices, namely the smallest convex set which contains all vertices. A procedure to extract minimal \( R \) and \( V \) is the double description method, also known as the Motzkin-Chernikova-Le Verge algorithm [Motzkin et al. 1953; Chernikova 1965; Le Verge 1992].

Turning back to the LPInfer procedure, the satisfiability test at Step 2 is performed as part of the construction of the explicit representation of the polyhedron. The maximization problems at Step 4 (a) can easily be solved given the explicit representation.

**Theorem 3.8 (Motzkin-Chernikova-Le Verge algorithm)**

There exists an effective procedure that given \( A x \leq b \) decides whether or not the polyhedron \( \text{Sol}(A x \leq b) \) is empty and, if not, it yields a generating matrix \( R \) and a vertex matrix \( V \) such that:

\[
\text{Sol}(A x \leq b) = \{ x \mid x = R \lambda, \lambda \geq 0 \} + \{ x \mid x = V \gamma, \gamma \geq 0, \Sigma \gamma = 1 \},
\]

and \( \text{Sol}(A x \leq 0) = \{ x \mid x = R \lambda, \lambda \geq 0 \} \).

A column of \( R \) is called a ray for any \( x_0 \in \text{Sol}(A x \leq b) \) and ray \( r \), it turns out that \( r \lambda + x_0 \in \text{Sol}(A x \leq b) \) for every \( \lambda \geq 0 \). A column of \( V \) is called a vertex. The set Convex Hull \( (V) \) = \( \{ x \mid x = V \gamma, \gamma \geq 0, \Sigma \gamma = 1 \} \), where \( V \) is a matrix or a finite set of vectors, is the convex hull of the vertices, namely the smallest convex set which contains all vertices. A procedure to extract minimal \( R \) and \( V \) is the double description method, also known as the Motzkin-Chernikova-Le Verge algorithm [Motzkin et al. 1953; Chernikova 1965; Le Verge 1992].

Turning back to the LPInfer procedure, the satisfiability test at Step 2 is performed as part of the construction of the explicit representation of the polyhedron. The maximization problems at Step 4 (a) can easily be solved given the explicit representation.

**Lemma 3.9.** Consider a characteristic cone \( \text{Sol}(A x \leq 0) \), and let \( R \) be its generating matrix. We have that \( \max\{c^T x \mid A x \leq 0\} = 0 \) iff \( c^T R \leq 0 \).

Since in our context \( c \) is always zero except for the \( i \)th element, which is 1 or \(-1\), we can conclude that a variable \( v_i \) (the \( i \)th variable in \( v \)) is bounded from above by 0 (resp., bounded from below by 0) iff all values in row \( (R, i) \) are non-positive (resp., non-negative).
The Minkowski’s form of the homogeneous system in Example 3.6 is:

\[
\begin{pmatrix}
  1 & 1 \\
  -2 & -1 \\
  0 & 0
\end{pmatrix}
\begin{pmatrix}
  \lambda_1 \\
  \lambda_2
\end{pmatrix}, \lambda_1 \geq 0, \lambda_2 \geq 0
\]

Intuitively, the two columns in the generating matrix \( R \) correspond to vectors lying on the border of the cone in the graph of Example 3.6. Using Lemma 3.9, it is readily checked that when \( c^T \) is one of \((-1,0,0), (0,1,0), (0,0,1) \) or \((0,0,-1)\) then \( c^T R \leq 0 \), i.e. \( x \) is bounded from below, \( y \) from above, and \( z \) from both.

### 3.3 An Implicit Equality Approach for \( B \mathcal{T} \)

The LP\( \text{Infer} \) procedure is sound, and it is complete except for the \( \square_r \) types. Thus, we can build on the results of the last subsection by concentrating on inference/checking of type assertions \( d \vdash c \rightarrow x : \square \) In this section, we restrict to consider \( \square_0 \), i.e. the \( ! \) type. In such a case, a solution to the type inference problem without taking into account parameters is still possible.

**Example 3.11.** Consider \( x : \square \vdash z = x, z - y = 2, z + y = 0 \rightarrow x :!, y :!, z :!. \) Starting from the involved constraint, by Gauss-Jordan elimination, we derive: \( x = z, y = z - 2, 2z = 2 \) and then \( z = 1, y = -1, x = 1 \). Hence the type assertion is valid.

Notice that we made no use of \( x : \square \) in proving validity of the type assertion. This fact can be generalized in a result that separates type inference/checking for \( B \mathcal{T} \) into proofs for the \( B \mathcal{T} \square \) subset and for the \( ! \) type.

**Theorem 3.12 (Definiteness I).** \( d \vdash c \rightarrow x :! \) is valid iff \( d_0 \vdash c \rightarrow x :! \) is valid.

Another intuition from Example 3.11 is that we can use Gauss-Jordan elimination to infer definiteness of variables.

**Example 3.13.** Consider \( x :! \vdash c \rightarrow y :!, z :!, \) where \( c \) is \( z - y = x, z + y = x, k + h = x + 1, x = 0 \). We have that \( \phi(x :!) \land c \) is \( x = a, z - y = x, z + y = x, k + h = x + 1, x = 0 \). By Gauss-Jordan elimination, we replace \( z \) by \( x + y \) to obtain \( x = a, z = x + y, y = 0, k + h = x, x = 0 \). Finally, we replace back \( y \) by \( 0 \) to obtain \( x = a, z = x, y = 0, h = x + 1 - k, x = 0 \). Hence \( z = x = a \) and \( y = 0 \) imply that the type assertion is valid.

Notice that Gauss-Jordan elimination provided us with a constructive proof of definiteness, i.e., we can express definite variables as linear expressions over the parameters. In general, consider a constraint \( c \) with equalities only, i.e., with geometric representation \( A, v = b_c \). Since \( \phi(d_0) \) is of the form \( x = a \), the overall system \( \phi(d_0) \land c \) is a linear system of equalities \( x = a, A, v = b_c \). By Gauss-Jordan elimination of variables in \( v \setminus x \), the system can be transformed into the following form:

\[
x = a, Iw = b' + A'z + B'x, 0 = b'' + B''x
\]

where \( I \) is a diagonal matrix, \( w \) and \( z \) are a partition of variables in \( v \setminus x \) with \( z \) free to assume any value, and \( 0 = b'' + B''x \) is the condition of satisfiability of the
system. It is immediate to observe that, given this form, a variable $x$ is definite in the original system iff it is in $x$ or it is in $w$ and it is defined in terms of $b'$ and $x$ only. Moreover, notice that we can rule out $x = a$ from the system, since we implicitly assume that variables in $x$ are definite by eliminating only variables in $v \setminus x$. Also, we can rule out $b$, since $b'$ and $b''$ play no role in definiteness.

**Lemma 3.14.** Let $c$ be a satisfiable linear constraint whose geometric representation is $A_c v_c = b_c$, where $v_c = \text{vars}(c)$, and $x \in v_c$.

$d || x \vdash c \rightarrow !$ is valid iff $x$ is in $x = \text{vars}(d)$, or, called $Iw = A'z + B'x, 0 = B''x$ the system obtained by Gauss-Jordan elimination from $A_c v_c = 0$ of variables in $v_c \setminus x$, there exists $i$ such that $x = w_i$ and row$(A', i) = 0$.

**Example 3.15.** Continuing Example 3.13, Gauss-Jordan elimination of $z, y, k, h$ yields the system:

$$x = a, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} z \\ y \\ h \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} k \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} x \end{pmatrix}.$$

The satisfiability condition of the system is $x = 0$. $z$ and $y$ are defined in terms of $x$ only, hence they are defined. $k$ is free to assume any value, hence it cannot be definite. Finally, $h$ is defined in terms of $x$ and $k$, hence it can assume infinite values as for $k$.

So far we considered equality constraints only. In the presence of inequalities, as in $x ! \vdash x \leq y, y \leq x \rightarrow y !$, Gaussian elimination alone is not enough. We recall the following well-known result (see [Stuckey 1991] or the survey [Greenberg 1996]).

**Theorem 3.16.** (Implicit equalities) Assume that $\text{Sol}(Ax \leq b) \neq \emptyset$. There exists an effective procedure that given $Ax \leq b$ yields an equivalent system $A^w x = b^w$, $A^+ x \leq b^+$ such that for every $c^T x \leq b$ from $A^+ x \leq b^+$ there exists $x_0 \in \text{Sol}(Ax \leq b)$ such that $c^T x_0 < b$.

As an example, the system $x \leq y, y \leq x$ is equivalent to $x = y$, hence $x ! \vdash x \leq y, y \leq x \rightarrow y !$ is valid iff $x ! \vdash x = y \rightarrow y !$ is valid, which can be shown by Gaussian elimination. An inequality $c^T x \leq b$ such that $c^T x_0 = b$ for every $x_0 \in \text{Sol}(Ax \leq b)$ is called an implicit equality.

**Definition 3.17.** We denote by $\text{ie}(Ax \leq b)$ the linear system $A^w x = b^w$ obtained by any fixed procedure implementing Theorem 3.16.

Intuitively, $\text{ie}(c)$ contains all the information about (implicit) equalities in $c$. If $x$ is constant in every solution of $c$, then the same holds for $\text{ie}(c)$. Stated otherwise, we can reason about definiteness by restricting to $\text{ie}(c)$ only.

**Lemma 3.18.** Let $c$ be a satisfiable linear constraint. $d || c \rightarrow x !$ is valid iff $d || \text{ie}(c) \rightarrow x !$ is valid.

Summarizing, Fig. 5 reports a procedure, called IEInfer, which first computes $\text{ie}(c)$ and then transform it by Gauss-Jordan elimination. Soundness and completeness of IEInfer is stated next.
Input: a type declaration \(d_1\), a linear constraint \(c\), and a set of variables \(v\)

Step 0. Define \(v_c = \text{vars}(c)\), \(d = d_1\|\neg\), \(x = \text{vars}(d)\)
Step 1. Let \(A_{v_c} v_c \leq b_c\) be the geometric representation of \(c\)
Step 2. Let \(A_{v_c} v_c = b_c = \text{ie}(A_{v_c} v_c \leq b_c)\)
Step 3. Rewrite \(A_{v_c} v_c = b_c = \text{ie}(A_{v_c} v_c \leq b_c)\) into \(I_w = A'_z + B'x, 0 = B''x\) by Gauss-Jordan elimination of \(v_c \setminus x\)
Step 4. For every \(x : \tau\) in output from \(\text{LPInfer}(d_1, c, v)\)
Step 5. If \(\tau \neq \square\) or \(x \notin v_c\) Then output “\(x : \tau\)”;
Step 6. Else
   (a). Output “\(x : !\)” if \(x\) is in \(x\) or there exists \(i\) such that \(x = w_i\), and \(\text{row}(A', i) = 0\)
   (b). Output “\(x : \square\)” otherwise.

Theorem 3.19 (IEInfer - soundness and completeness). IEInfer is sound for the type inference problem, and it is complete for BT.

The IEInfer procedure is parametric to a specific implementation of the \(\text{ie()}\) function for deriving the set of (implicit) equalities of \(c\). Again, we have two alternatives. One is a Simplex-based algorithm for detecting implicit equalities, such as the one proposed in [Stuckey 1991] and refined in [Refalo 1998]. Another choice is to rely, again, on the Minkowski’s form of polyhedra.

Lemma 3.20. Assume \(\text{Sol}(A x \leq b) \neq \emptyset\), and let \(R\) and \(V\) be the generating and vertex matrices of \(A x \leq b\). The system \(A x = b\) can be calculated as the system of equalities \(c^T x = b\), where \(c^T x \leq b\) is from \(A x \leq b\) and such that \(c^T R = 0\) and \(c^T V = b_1^T\).

The condition \(c^T R = 0\), known as the saturation of the inequality by the rays, is typically checked by double description method implementations in order to remove redundant inequalities and to simplify implicit inequalities into equalities. In practice, those implementations maintain and transform both the Minkowski’s form and the explicit form (this is the “double” description) of Theorem 3.16. In addition, they reduces to consider homogeneous systems by transforming forth and back \(A x \leq b\) into the form \(A' x' \leq 0\) with \(x' \subseteq x'\) (see [Goldman 1956; Wilde 1993] for details on the transformation). For homogeneous systems, the additional condition \(c^T V = b_1^T\) of Lemma 3.20 is trivially satisfied, since \(b = 0\) and \(V = 0\).

The implementation that later on we will adopt, the polylib library [Loechner 2007], performs both the homogeneous transformation and the saturation condition check. In addition to the Minkowski’s form of the original system, it returns an equivalent system where all implicit equalities are detected. As a consequence, there is no need to apply Lemma 3.20 on the output generating and vertex matrices to detect implicit equalities.

3.4 A Parameterized Polyhedra Approach for BT

In this section, we reason on the full type system \(BT\) by explicitly dealing with the parameterized system in (2) through an extension of the Minkowski’s decomposition theorem. As done in the last section, we would like to build on the existing LPInfer procedure. A direct extension of Theorem 3.12, however, does not hold for \(\square\) in general.
Example 3.21. The type assertion \( x : \Box_2 \vdash y = x/2 \rightarrow y : \Box_1 \) is valid. Intuitively, if \( x \in [a, a+2] \) then \( x/2 \in [b, b+1] \) for \( b = a/2 \). However, \( \vdash y = x/2 \rightarrow y : \Box_1 \) is not valid.

Nevertheless, a similar result can be stated allowing for separating type inference/checking for \( \mathcal{BT} \) into proofs for \( \mathcal{BT}_0 \) and for its complement.

Theorem 3.22 (Definiteness II). Let \( \mathcal{B} = \mathcal{BT} \setminus \mathcal{BT}_0 = \{ \Box s \mid s \in \mathcal{R}, s \geq 0 \} \). We have that: \( \vdash c \rightarrow x : \Box_r \) is valid iff \( \mathcal{B}[s] \vdash c \rightarrow x : \Box_r \) is valid.

Let us consider now an example which illustrates the Fourier-Motzkin elimination method for linear inequalities applied in the presence of parameters [Keerthi and Sridharan 1990].

Example 3.23. Consider the constraint \( c \) defined as \( y + x \leq z, y - x \leq z, z \leq y, 0 \leq z, w \leq z, \) and the type declaration \( z :! \). We start by isolating variable \( y \) in \( \phi(z :!) \land c \), as shown at (a) in the figure below.

\[
\begin{align*}
  y & \leq z - x \\
  y & \leq z + x \\
  z & \leq y \\
  z & \leq 0 \\
  w & \leq z \\
  z & = a
\end{align*}
\]

Bounds for variable \( y \) can then be summarized as: \( * \) \( z \leq y \leq \min\{z - x, z + x\} \). Moreover, the bounds \( e_1 \leq e_2 \) are implied for any \( e_1, e_2 \) such that \( e_1 \leq y \leq e_2 \) is in \( * \). Actually, the original set of linear inequalities over \( y \) is equivalent to \( z \leq y \leq \min\{z - x, z + x\} \) plus such bounds. The new inequality set is reported at (b) in the figure above. By replacing backward \( x = 0 \) and \( z = a \), we end up with the final system at (c) in the figure above, where no further elimination is possible. The final system is feasible when the condition \( 0 \leq a \) holds. In this system, we have \( x = 0, y = a, z = a \). Moreover, \( w \leq a \) can be rewritten as: \( w = -\lambda_1 + a \) for some \( \lambda_1 \geq 0 \). Put in a geometrical form, the solution set of \( \phi(z :!) \land c \) is:

\[
\left\{ (x, y, z, w) \mid \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} \left( \begin{pmatrix} \lambda_1 \\ a \end{pmatrix} + \begin{pmatrix} 0 \\ a \end{pmatrix} \right) \right\}
\]

for every \( \lambda_1 \geq 0, \) when \( a \geq 0 \).

Summarizing, the values of \( x, y \) and \( z \) are univocally determined once the parameter \( a \) has been fixed and the system is feasible. Under the same hypotheses, the value of \( w \) is bounded from above (by \( a \) ), but it is not definite. With our notation, \( z :! \vdash c \rightarrow x :!, y :!, z :!, w : \Box \) is valid.

The final form reached in the example resembles the Minkowski’s form for polyhedra, but with a parameterized vector appearing in the vertex matrix. The generalization of the Minkowski’s theorem to parameterized polyhedra is provided in [Loechner and Wilde 1997] and implemented in the polylib library [Loechner 2007].
Define Else let For the parametric polyhedron For every We say that Let Ax over domain includes u instance mating matrix representation of Step 4. . . M ming problems. Called for every parameter instance. First, we introduce a notion to model the variability parameterized vertex and the second one is its Sol Theorem 3.24 (Minkowski's theorem for parameterized polyhedra). Every parameterized polyhedron can be expressed by a generating matrix R and finitely many pairs (v^a(1), C_1 a ≤ c_1), . . . , (v^a(k), C_k a ≤ c_k) where, for i = 1..k, v^a(i) is a vector parameteric in a and Sol(C_i a ≤ c_i) ≠ ∅, as follows:

\[
Sol(Ax ≤ b + B a, u) = \{ x | x = R \lambda, \lambda ≥ 0 \} + ConvexHull(\{v^a(i) | i = 1..k, C_i u ≤ c_i \}),
\]

and \(Sol(Ax ≤ 0) = \{ x | x = R \lambda, \lambda ≥ 0 \}\).

Input: a type declaration d, a linear constraint c, and a set of variables v

Step 0. Define \(v_c = \text{vars}(c)\), n = n(d_1), d = n|B, where \(B = \{ a_r | r ∈ R, r ≥ 0 \}\).

Step 1. Let \(A_c, v_c ≤ b\) be the geometric representation of c, and \(A_d v_c ≤ B_d a d\) the geometric representation of φ(d).

Step 2. For the parametric polyhedron \(\begin{pmatrix} A_c & A_d \end{pmatrix} v_c ≤ \begin{pmatrix} b_c & 0 \end{pmatrix} + \begin{pmatrix} 0 & B_d \end{pmatrix} a\), build the generating matrix \(R\) and the sequence \((v^a(1), C_1 a ≤ c_1), . . . , (v^a(k), C_k a ≤ c_k)\).

Step 3. For every \(x : τ\) in output from LPInfer(d_1, c, v)

Step 4. . . If \(τ ≠ ∅\) or \(x ∉ v_c\). Then output “\(x : τ\)”;

Step 5. . . Else let i such that \(x = v_{c_i}\):

(a). Output “\(x : ∅\)” if row(R, i) = 0 and for 1 ≤ m < n ≤ k, either \(P_{m,n} = 0\) or there exists s such that \(v^a(m_s) ≥ v^a(n_s)\) over \(P_{m,n}\), and

\[r = \text{max}(|0| \cup \{ s | 1 ≤ m < n ≤ k, Sol(P_{m,n}) ≠ ∅, v^a(m_s) ≥ v^a(n_s) \text{ over } P_{m,n}\})\]

where \(P_{m,n} = C_m a ≤ c_m, C_n a ≤ c_n;\)

(b). Output “\(x : ∅\)” otherwise.

The vertex matrix is now replaced by a set of pairs where the first element is a parameterized vertex and the second one is its validity domain. For a parameter instance u, the vertex matrix is built from the (instantiated) vertices whose validity domain includes u. The special case \(k = 0\) models empty parameterized polyhedra, which are empty for any instance of the parameters.

Provided with an explicit form for parameterized polyhedra, we proceed by designing a procedure to check whether a variable has a bounded variability range for every parameter instance. First, we introduce a notion to model the variability range \(r\) of two expressions over the solutions of a polyhedron.

Definition 3.25. We say that \(c_1^T x + \alpha_1 ≃_r c_2^T x + \alpha_2\) over \(Ax ≤ b\) if for every \(x_0 ∈ Sol(Ax ≤ b)\), \(abs(c_1^T x_0 + \alpha_1 - c_2^T x_0 - \alpha_2) ≤ r\), and for some \(x_0 ∈ Sol(Ax ≤ b)\), \(abs(c_1^T x_0 + \alpha_1 - c_2^T x_0 - \alpha_2) = r\).

A direct implementation of the \(≃_r\) operator relies on standard linear programming problems. Called \(M = \text{max}\{c_1 - c_2|^T x + (\alpha_1 - \alpha_2) | Ax ≤ b\}\) and \(m = \text{min}\{(c_1 - c_2|^T x + (\alpha_1 - \alpha_2) | Ax ≤ b)\}\), we have that \(c_1^T x + \alpha_1 ≃_r c_2^T x + \alpha_2\) over \(Ax ≤ b\) iff \(M ∈ R, m ∈ R\) and \(\text{max}\{M, -m\} = r\).

Under the hypothesis that we are provided with the Minkowski's form of \(Ax ≤ b\), an alternative implementation is stated by the next result.
Definition 3.26. The maximum norm of a vector \( \mathbf{v} = (v_1, \ldots, v_n) \) is defined as: \( \| \mathbf{v} \|_\infty = \max \{|v_i| \mid i = 1, \ldots, n\} \).

Lemma 3.27. Assume \( \text{Sol}(\mathbf{Ax} \leq \mathbf{b}) \neq \emptyset \). We have that \( \mathbf{c}_1^T \mathbf{x} + \alpha_1 \succeq_r \mathbf{c}_2^T \mathbf{x} + \alpha_2 \) over \( \mathbf{Ax} \leq \mathbf{b} \) iff called \( \mathbf{R} \) and \( \mathbf{V} \) the generating and vertex matrices of \( \mathbf{Ax} \leq \mathbf{b} \), \( (\mathbf{c}_1 - \mathbf{c}_2)^T \mathbf{R} = \mathbf{0} \) and \( \| (\mathbf{c}_1 - \mathbf{c}_2)^T \mathbf{V} + (\alpha_1 - \alpha_2) \mathbf{1}^T \|_\infty = r \).

We point out that Lemma 3.20 is a special case of this result, obtained for \( r = 0 \).

Also, notice that checking \( \mathbf{c}_1 = \mathbf{c}_2 \) and \( \alpha_1 - \alpha_2 = r \) is a computationally fast calculation, yet being a sufficient condition. In the actual implementation of the \( \succeq_r \) operator, we first check such a sufficient condition and, if not satisfied, will check the more elaborated condition of Lemma 3.27.

Consider now the problem of computing the range of variability of a variable \( \mathbf{x} \) over a parameterized polyhedron. The next result states that it can be computed by considering the pairs of parameterized vertices of the polyhedron. For each pair, we calculate the range of variability of \( \mathbf{x} \) over the intersection of the domains of the two vertices, if not empty. The maximum variability over all the pairs of vertices is then the maximum range of variability of \( \mathbf{x} \) over any instance of the parameterized polyhedron.

Lemma 3.28. Consider the Minkowski’s form of a non-empty parameterized polyhedron as in Theorem 2.24, and let us denote \( P_{m,n} = \mathbf{C}_m \mathbf{a} \leq \mathbf{c}_m, \mathbf{C}_n \mathbf{a} \leq \mathbf{c}_n \).

Let \( S_\mathbf{u} = \{ \mathbf{c}^T \mathbf{x} \mid \mathbf{x} \in \text{Sol}(\mathbf{Ax} \leq \mathbf{b} + \mathbf{Ba}, \mathbf{u}) \} \). For \( \mathbf{r} \in \mathbb{R} \), the following are equivalent:

— for every \( \mathbf{u} \), for every \( x, y \in S_\mathbf{u} \), \( \text{abs}(x - y) \leq r \); and there exist \( \mathbf{u} \), and \( x, y \in S_\mathbf{u} \), such that \( \text{abs}(x - y) = r \)

\[ -\mathbf{c}^T \mathbf{R} = \mathbf{0}; \text{ and, for } 1 \leq m < n \leq k, \text{Sol}(P_{m,n}) = \emptyset \text{ or } \mathbf{c}^T \mathbf{v}^n(m) \succeq_s \mathbf{c}^T \mathbf{v}^n(n) \text{ over } P_{m,n} \text{ for some } s \leq r \text{; and,} \]

\[ r = \max \{\{0\} \cup \{s \mid 1 \leq m < n \leq k, \text{Sol}(P_{m,n}) \neq \emptyset, \mathbf{c}^T \mathbf{v}^n(m) \succeq_s \mathbf{c}^T \mathbf{v}^n(n) \text{ over } P_{m,n}\} \} \).

Summarizing, Lemma 3.28 provides us with a necessary and sufficient condition for inferring \( r \) in a type assertion \( \mathbf{d} \vdash \mathbf{c} \to \mathbf{x} : \Box_r \), by directly reasoning on the Minkowski’s form of the parameterized system in (2). We point out that the polylib library provides the validity domains \( \mathbf{C}_i \mathbf{a} \leq \mathbf{c}_i \) in their Minkowski’s form as well as basic operators on polyhedra, including intersection. Therefore, checking \( \mathbf{c}^T \mathbf{v}^n(m) \succeq_s \mathbf{c}^T \mathbf{v}^n(n) \) can be implemented using Lemma 3.27.

Example 3.29. The type declaration \( z :! \), \( w :! \) and the linear constraint \( z + w \geq y, y \geq z, y \geq w, x = z + 1 \) give rise to a parameterized polyhedron over parameters \( (a, b) \) and variables \( (x, y) \) (we omit \( z \) and \( w \) for space) with generating matrix \( \mathbf{0} \), and with pairs of vertices and domains:

\[ \left(\begin{array}{c} a + 1 \\ b \end{array}\right), a \geq b \geq 0 \quad \left(\begin{array}{c} a + 1 \\ a \end{array}\right), a \geq b \geq 0 \quad \left(\begin{array}{c} a + 1 \\ a + b \end{array}\right), a \geq 0 \land b \geq 0 \).

Let us reason about definiteness of variables \( x \) and \( y \) by using Lemma 3.28. \( x \) is definite, since \( a + 1 \succeq_0 a + 1 \) over any polyhedron. In fact, \( a + 1 \) expressed as a
linear function of the parameters is \((1\ 0)\begin{pmatrix} a \\ b \end{pmatrix} + 1\). The conclusion then holds by Lemma 3.27 using the sufficient condition \(c_1 = (1\ 0) = c_2\) and \(\text{abs}(\alpha_1 - \alpha_2) = 0\).

Consider now \(y\). For the first two vertices, we have \(b \neq a\), thus we cannot use the sufficient condition as in the last case. Since the intersection of the domains of the vertices, namely \(a = b \geq 0\), is not empty, by Lemma 3.27 we proceed by computing its generating and vertex matrices. The vertex matrix is \(0\), so we simply have:

\[
\text{Sol}(a = b \geq 0) = \{(a, b) | \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \lambda_1, \lambda_2 \geq 0\},
\]

and we calculate:

\[
\|((0 1) - (1 0))\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 0 1^T\|_\infty = 0.
\]

Therefore, \(a \approx_0 b\) over \(a = b \geq 0\). Consider now the first and the third vertex. Since \(b \neq a + b\), we compute, as before, the Minkowski’s form of the intersection of their domains:

\[
\text{Sol}(b \geq a \geq 0) = \{(a, b) | \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \lambda_1 + \lambda_2 \geq 0\}.
\]

The vertex matrix \(V\) clearly satisfies \(\|c_1 - c_2\| V + (\alpha_1 - \alpha_2) 1^T\|_\infty = 0\). However, for the generating matrix \(R\), the condition \((c_1 - c_2)^T R = 0\) does not hold:

\[
((0 1) - (1 1))\begin{pmatrix} 1 \\ 0 \end{pmatrix} = (-1 \ 0).
\]

Summarizing, by Lemma 3.27, \(b \not\approx_r a + b\) for any \(r\), and then, by Lemma 3.28, \(y\) cannot be typed as \(2^r\) for any \(r\).

The overall procedure, called POLYInfer, is shown in Fig. 6. POLYInfer terminates and it is sound and complete for inferring validity of type assertions.

**Theorem 3.30 (POLYInfer - soundness and completeness).** POLYInfer is sound for the type inference problem, and it is complete for \(\text{BT}\).

### 3.5 Extension to Strict Inequalities and to Disequalities

So far, we considered equality and non-strict inequality primitive constraints. A generalized linear constraint admits also primitive constraint over the operators \(<,\ >\) (strict inequalities) and \(\neq\) (disequalities). Without any loss of generality, we write a generalized constraint as \(c \land \bigwedge_{i=1}^m e_i \neq \alpha_i\), where \(c\) is a linear constraint and for \(i = 1..m, e_i \neq \alpha_i\) is a disequality. We now extend type assertions to admit generalized constraints. The next result shows that validity of type assertions for a satisfiable generalized linear constraint can be reduced to validity of the type assertions over the linear constraint obtained by removing the disequalities in it.

**Theorem 3.31.** Let \(g = c \land \bigwedge_{i=1}^m e_i \neq \alpha_i\) be a satisfiable generalized linear constraint. \(d_1 \vdash g \rightarrow d_2\) is valid iff \(d_1 \vdash c \rightarrow d_2\) is valid.

The conclusion does not hold for unsatisfiable constraints.
Example 3.32. The type assertion \( \vdash x < 3, x \geq 3, y \leq 0 \rightarrow y :! \) is valid, since the generalized constraint is unsatisfiable. When removing the disequalities (namely, turning \( x < 3 \) into \( x \leq 3 \)), only validity of \( \vdash x \leq 3, x \geq 3, y \leq 0 \rightarrow y :! \) follows.

Checking satisfiability of \( g \) can be easily accomplished in the framework of computing the explicit form of polyhedra. In fact, by independence of negative constraints [Lassez and McAllon 1992], it reduces to show that \( \text{Sol}(\text{Acv} \leq \text{bc}) \neq \emptyset \), and that for every disequality \( e \neq \alpha \) in \( g \), \( e \simeq \alpha \) holds over \( \text{Acv} \leq \text{bc} \).

Lemma 3.27 provides us with a procedure to show that. The same result can be obtained by linear programming. Called \( M = \max\{e - \alpha \mid \text{Acv} \leq \text{bc}\} \) and \( m = \min\{e - \alpha \mid \text{Acv} \leq \text{bc}\} \), we have already noted that \( e \simeq \alpha \) holds iff \( M, m \in \mathbb{R} \) and \( M = m \).

When the generalized constraint contains strict inequalities but no disequality, a single linear programming problem is sufficient for all strict inequalities (see [Greenberg 1996, Theorem 4]).

3.6 Extension to Parametric Types \( \mathcal{PT} \)

The next example introduces type variables in type assertions.

Example 3.33. When writing the following (valid) type assertion:

\[
x : \Box_1 \vdash y = x + 2, x \leq z \leq x + 2 \rightarrow y : \Box_1, z : \Box_3
\]

it is natural to observe that it holds in more general terms: the range of variability for \( y \) is the same as \( x \), and the range of variability for \( z \) is the one for \( x \) plus 2. By introducing a "type variable" \( s \), and assuming \( s \geq 0 \), we can write the general statement as:

\[
x : \Box_s \vdash y = x + 2, x \leq z \leq x + 2 \rightarrow y : \Box_s, z : \Box_{s+2}.
\]

Notice that the kind of involved expressions can be linear combinations of type variables, as in \( x : \Box_s, y : \Box_1 \vdash z = 2x + y \rightarrow z : \Box_{2s+1} \).

Let us formally introduce type variables in the syntax and semantics of type assertions.

Definition 3.34. Let \( V \) be a set of variables distinct from constraint variables, called type variables. A parametric type is \( \square_e \), where \( e = c^T a + r \) is a linear expression over variables \( a \) from \( V \). We define the set of types \( \mathcal{PT} \) as \( \mathcal{BT} \) augmented with parametric types.

The syntax of type declarations and type assertions readily extend to \( \mathcal{PT} \).

The semantics of type assertions over \( \mathcal{PT} \) is defined by extending the \( \phi \) and \( \nu \) functions in Definition 2.4 as follows:

\[
\phi(x : \square_e) = a \leq x \land x \leq a + e \land 0 \leq e \quad \nu(x : \square_e) = \{a\} \cup \text{vars}(e).
\]

For a type assertion \( d_1 \vdash c \rightarrow d_2 \), a type variable that appears (with non-zero coefficient) in \( d_2 \) but that does not appear in \( d_1 \) is called a local type variable.

Intuitively, type variables in parametric\(^3\) types are compiled into parameters of the parameterized linear system underlying the type assertion. Notice that the conjunct \( 0 \leq e \) is syntactically satisfied for \( \square_e \) in the \( \mathcal{BT} \) type system, since we

\(^3\)Notice that we will use the word ‘parametric’ for types and ‘parameterized’ for linear systems in order to resolve possible linguistic ambiguities in the text.
assumed \( r \in \mathcal{R} \) and \( r \geq 0 \). Therefore, the above \( \phi() \) and \( v() \) definitions are conservative extensions of the ones for \( \square_r \). Also, notice that, for a type assertion \( d_1 \vdash c \rightarrow d_2 \), the restriction \( v(d_2) \setminus v(d_1) \) in Definition 2.4 (1) is now fundamental, since a same type variable can appear both in \( d_1 \) and \( d_2 \).

**Example 3.35.** The type assertion \( x : \square_s \vdash y = x + 2 \rightarrow y : \square_p \) states that if \( x \) has a range of variability of \( s \) then \( y \) has some range of variability \( p \), i.e., there exists \( a \) and \( p \geq 0 \) such that \( a \leq y \leq a + p \). But this is equivalent to require \( a \) and \( b \) such that \( a \leq y \leq b \), i.e., to state that \( x : \square_s \vdash y = x + 2 \rightarrow y : \square \) is valid.

Local type variables in a type assertion are existentially quantified. Therefore, there is no loss of generality in assuming no local type variable in a type assertion \( d_1 \vdash c \rightarrow x : \square_c \).

**Lemma 3.36.** A type assertion \( d_1 \vdash c \rightarrow x : \square_c \) with local type variables is valid iff \( d_1 \vdash c \rightarrow x : \square \) is valid.

However, admitting local type variables makes type assertions non-compositional, in the sense highlighted by the following example.

**Example 3.37.** The type assertion \( \vdash 0 \leq x \leq 2, 0 \leq y \leq 1 \rightarrow x : \square_s, y : \square_{1-s} \) is invalid. In fact, \( x \) has a variability range of 2 in the solutions of the linear constraint, whilst the existentially quantified \( s \) is required \( s \geq 0 \) and \( 1 \geq s \). On the contrary, by Lemma 3.36, both \( \vdash 0 \leq x \leq 2, 0 \leq y \leq 1 \rightarrow x : \square_s \) and \( \vdash 0 \leq x \leq 2, 0 \leq y \leq 1 \rightarrow y : \square_{1-s} \) are valid.

The property of compositionality holds for type assertions without local type variables. For the rest of this section we restrict to type assertions of the form \( d_1 \vdash c \rightarrow x : \square_c \).

**Lemma 3.38.** A type assertion \( d_1 \vdash c \rightarrow (d_2, d_3) \) with no local type variable is valid iff \( d_1 \vdash c \rightarrow d_2 \) and \( d_1 \vdash c \rightarrow d_2 \) are valid.

The introduction of parametric types breaks also the basic Lemma 3.4, i.e., satisfiability of the parameterized linear system \( \mathcal{P} = \phi(d) \land c \) in (2) does not depends only on satisfiability of \( c \).

**Example 3.39.** For the type assertion \( x : \square_s, y : \square_{-s-1} \vdash \text{true} \rightarrow x : !, \) we have that \( \phi(x : \square_s, y : \square_{-s-1}) \land \text{true} \) is \( a \leq x \leq a + s, b \leq y \leq b - s - 1, 0 \leq s, 0 \leq -s - 1 \). In particular, the last two primitive constraints can be rewritten as: \( s + 1 \leq 0 \leq s \), which is unsatisfiable.

Lemma 3.4 conservatively generalizes to parametric types by adding the condition that both \( c \) and \( \phi(d) \) are satisfiable. For a type declaration \( d \) in \( \mathcal{E}T \), \( \phi(d) \) is always satisfiable.

**Lemma 3.40.** Consider the parameterized linear system \( \mathcal{P} = \phi(d) \land c \) in (2), with \( d \) type declaration in \( \mathcal{P}T \). There exists a parameter instance \( u \) such that \( \text{Sol}(\mathcal{P}, u) \neq \emptyset \) iff \( \text{Sol}(\mathcal{A}_s v \leq b_c) \neq \emptyset \) and \( \text{Sol}(\mathcal{A}_d v \leq B_d a) \neq \emptyset \).

Concerning type inference, the following example shows that complete procedures have to lift to piecewise linear expressions, which is beyond our syntax of type declarations.
Example 3.41. What parametric type $\tau$ can be inferred for the type assertion $x : \square_s \vdash 0 \leq x \leq 2 \rightarrow x : \tau$?

When $0 \leq s \leq 2$, the range width $s$ for $x$ is stricter than the range width 2 imposed by the constraint $0 \leq x \leq 2$. Hence, $\tau = \square_s$ should be inferred. Nevertheless, inferring $\tau = \square_2$ is sound, i.e., it yields a valid type assertion.

When $s \geq 2$, the range width 2 imposed by $0 \leq x \leq 2$ is stricter than $s$. Hence, $\tau = \square_2$ should be inferred. Nevertheless, inferring $\tau = \square_s$ is sound.

Summarizing, sound procedures for type inference could return either $\tau = \square_s$ or $\tau = \square_2$. A complete procedure, however, should return:

$$\tau = \begin{cases} \square_2 & \text{if } s \geq 2 \\ \square_s & \text{otherwise} \end{cases}$$

i.e., a piecewise linear expression, which is beyond our syntax.

The notions of $\text{lub}()$ and normal form of type declarations are also affected by the same problem.

Example 3.42. By reasoning as in the last example, $\text{lub}(\{\square_s, \square_2\})$ cannot be expressed as $\square_e$ for some linear expression $e$. Rather, we should generalize the syntax of parametric types to allow $\text{lub}_{\min}(x, 2)$. Similarly, one concludes that $\text{nf}(x : \square_s, x : \square_2)$ cannot be expressed with the syntax of linear expressions.

Concerning type checking, however, we are in the position to design a decision procedure. First, an extension of the Theorem 3.22 (Definiteness II) holds.

Theorem 3.43 (Definiteness III). Let $B = PT \setminus BT_\square$. A type assertion $d \vdash c \rightarrow x : \square_e$ with no local type variable is valid iff $d|_s \vdash c \rightarrow x : \square_e$ is valid.

On the contrary, Theorem 3.12 (Definiteness I) does not extend to parametric types, i.e., definiteness cannot be dealt with separately.

Example 3.44. The type assertion $x : \square_s, x : \square_{-s} \vdash \text{true} \rightarrow x : !$ is valid. In fact, the corresponding formula to be shown is:

$$\forall a, b, s \exists e \forall x (a \leq x \leq a + s \land b \leq x \leq b - s \land s \geq 0 \land -s \geq 0 \rightarrow x = c).$$

From the left hand side of the implication, we obtain: $s = 0 \land a + b \leq 2x \leq a + b$, and then $x = (a + b)/2$ is the only value that $s$ can assume.

However, the type assertion $\vdash \text{true} \rightarrow x : !$ is not valid. This shows that Theorem 3.12 does not extend to parametric types. As a consequence, completeness of the Check(IEInfer) procedure is lost for type assertions $d \vdash c \rightarrow x : !$ such that parametric types appear in $d$.

Next, we revisit Lemma 3.27 and Lemma 3.28.

Definition 3.45. Let $e = c_1^T x + r$. We say that $c_1^T x + \alpha_1 \preceq_e c_2^T x + \alpha_2$ over $Ax \leq b$ if for every $x_0 \in \text{Sol}(Ax \leq b)$, $\text{abs}(c_1^T x_0 + \alpha_1 - c_2^T x_0 - \alpha_2) \leq c_2^T x_0 + r$.

$\preceq_e$ is a weakening of the relation $\preceq_r$ from Definition 3.25 in two ways. First, any upper bound is now required, whilst $\preceq_r$ holds for the least upper bound. Second, the upper bound may be expressed using variables of the system, as in $x \preceq_e x + z$. As in the case of $\preceq_r$, the $\preceq_e$ relation can be checked by means of linear programming problems or, if we are provided with the Minkowski’s form of $Ax \leq b$, by means of the following result.
Input: a type assertion $d_1 :: x : \square_e$ with no local type variable

Step 0. Define $d = d_1 | x$, where $B = P \cap BT \cap d \cap \{x\}$, and let $i$ s.t. $x = v_i$.

Step 1. Let $A_v \leq b_v$ be the geometric representation of $e$.

. and $A_d v \leq B_d a_d$ the geometric representation of $\phi(d)$.

Step 2. If $Sol(A_v \leq b_v) = \emptyset$ or $Sol(A_d v \leq B_d a_d) = \emptyset$ Then Output “Valid”

Else

Step 3. For the parametric polyhedron $\left( \begin{array}{c} A_v \\ A_d \end{array} \right) v \leq \left( \begin{array}{c} b_v \\ 0 \end{array} \right) + \left( \begin{array}{c} 0 \\ B_d \end{array} \right) a$, build

. the generating matrix $R$ and the sequence $(v^a(1), C_1 a \leq c_1), \ldots, (v^a(k), C_k a \leq c_k)$

Step 4. (a). Output “Valid” if

(i). $row(R, i) = 0$;

(ii). and for every $1 \leq m < n \leq k$, $v^a(m) \leq v^a(n)$ over $C_m a \leq c_m$, $C_n a \leq c_n$;

(iii). and for $i = 1..k$, $\min\{e \mid C_i a \leq c_i\} \geq 0$.

(b). Output “Not valid” otherwise.

Fig. 7. ParCheck procedure

Lemma 3.46. Let $e = c^T x + r$, and assume $Sol(\mathbf{Ax} \leq \mathbf{b}) \neq \emptyset$.

We have that $c^T x + \alpha_1 \geq \alpha_2 c^T x + \alpha_2$ over $\mathbf{Ax} \leq \mathbf{b}$ iff called $R$ and $V$ the generating and vertex matrices of $\mathbf{Ax} \leq \mathbf{b}$, $\mathbf{(c_1 - c_2 - c_3)^T R} \leq 0$ and $(\mathbf{c_2 - c_1 - c_3})^T \mathbf{V} \leq 0$ and $(\mathbf{c_1 - c_2 - c_3})^T \mathbf{V} + \mathbf{(\alpha_1 - \alpha_2)} \mathbf{1}_p^T \leq 0$.

Notice that, when $e = r$, then the necessary and sufficient condition can be stated in a simpler form as: $\mathbf{(c_1 - c_2)^T R} = 0$ and $\| (\mathbf{c_1 - c_2)^T V} + \mathbf{(\alpha_1 - \alpha_2)} \mathbf{1}_p^T \|_\infty \leq r$, which closely resembles the one in Lemma 3.27. Lemma 3.28 is extended to the checking problem in the presence of parametric types as follows.

Lemma 3.47. Consider the Minkowski’s form of a non-empty parameterized polyhedron as in Theorem 3.24, and let us denote $P = P_{m,n} = C_m a \leq c_m$, $C_n a \leq c_n$.

Let $S_u = \{ c^T x \mid x \in Sol(\mathbf{Ax} \leq \mathbf{b + Bu}, u) \}$, and $e = c^T a + r$.

We have that for every $u$, for every $x, y \in S_u$, abs$(x - y) \leq c^T u + r$ iff the following hold: (i) $c^T R = 0$; (ii) and, for $1 \leq m < n \leq k$, $c^T v^a(m) \leq c^T v^a(n)$ over $P_{m,n}$; (iii) and, for $i = 1..k$, $\min\{e \mid C_i a \leq c_i\} \geq 0$.

As in the case of Lemma 3.28, we point out that the polylib library provides the validity domains $C_i a \leq c_i$ in their Minkowski’s form as well as basic operators on polyhedra, including intersection. Therefore, condition (ii) can be implemented using Lemma 3.46. Analogously, condition (iii) can be checked by solving linear programming problems or as follows.

Lemma 3.48. Let $e = c^T x + r$, and assume $Sol(\mathbf{Ax} \leq \mathbf{b}) \neq \emptyset$. We have that $\min\{e \mid \mathbf{Ax} \leq \mathbf{b}\} \geq 0$ iff called $R$ and $V$ the generating and vertex matrices of $\mathbf{Ax} \leq \mathbf{b}$, $c^T R \geq 0$ and $c^T V + r \mathbf{1}_p^T \geq 0$.

Condition (iii) is needed to cover contrived type assertions, as shown in the next example. We refer the reader to Appendix A.7 for additional technical details.

Example 3.49. Consider the type assertion $x : \square_1, y : \square_2 \vdash z = 0 \rightarrow z : \square_2$. By instantiating $s = 0, r = 1$, we obtain $x :: y : \square_1 \vdash z = 0 \rightarrow z : \square_2$, which is syntactically malformed as a type assertion in $\mathcal{ST}$ (since the requirement $−1 > 0$ is false). As a type assertion in $\mathcal{PT}$, however, it is well-formed, yet invalid. In fact,
Consider linear constraints of the form:

\[ \phi(x !, y : \Box_1) \land z = 0 \] is a \( a \leq x \leq a + 0 \land 0 \geq 0 \land b \leq y \leq b + 1 \land 1 \geq 0 \land z = 0 \), which admits solutions. However, \( \phi(z : \Box_{-1}) \) is \( c \leq z \leq c - 1 \land -1 \geq 0 \), hence false.

Figure 7 summarizes the checking procedure PARCheck, in the case of input type assertion \( d_1 \vdash c \rightarrow x : \Box_e \). Step 4 (a)(iii) is implemented using Lemma 3.46, and Step 4 (a)(iii) is implemented using Lemma 3.48.

**Theorem 3.50** (PARCheck - soundness and completeness).

**PARCheck** is a decision procedure for the type checking problem of type assertions \( d_1 \vdash c \rightarrow x : \Box_e \) with no local type variable.

When local type variables exist, \( d_1 \vdash c \rightarrow x : \Box_e \) is equivalent to \( d_1 \vdash c \rightarrow x : \Box \), as shown in Lemma 3.36. In general, for a type assertion \( d_1 \vdash c \rightarrow x : \tau \) with \( \tau \in BT\), the checking procedure can resort to CHECK(LPInfer) by reducing the problem to type assertions for which CHECK(LPInfer) is sound and complete.

**Lemma 3.51.** Let \( d_1 \) be a type declaration (in \( PT \)) such that \( \phi(d_1) \) is satisfiable. Let \( d_1' \) be the type declaration (in \( BT\)) obtained from \( d_1 \) by replacing every \( y : \Box_e \) with \( y : \Box \). For \( \tau \in BT\), \( d_1 \vdash c \rightarrow x : \tau \) is valid iff \( d_1' \vdash c \rightarrow x : \tau \) is valid.

### 3.7 Computational Complexity Issues

Let us consider the issue of computational complexity of the type checking problem.

First, we observe that the LPInfer procedure has a polynomial time complexity when a polynomial time algorithm (such as the one in [Khachiyan 1979]) is adopted for solving the required linear programming problems. By Theorem 3.7, CHECK(LPInfer) is then a polynomial time decision procedure for type assertions \( d_1 \vdash c \rightarrow d_2 \) such that types in \( d_1 \) belong to \( BT \), and types in \( d_2 \) belong to \( BT\).

The conclusion can be strengthened to \( BT \) since the additional computations of IEinfer consist of extracting implicit equalities and of performing Gaussian elimination, both of which have polynomial time complexity.

**Theorem 3.52.** The problem of checking whether \( d_1 \vdash c \rightarrow d_2 \) is valid is polynomial time decidable, if all types in \( d_2 \) belong to \( BT \).

Notice that the instantiation of LPInfer using the Minkowski’s form of polyhedra has an exponential time complexity, since the number of vertices of a (not even parameterized) polyhedron can be exponential in the worst case. A fortiori, neither CHECK(LPInfer) nor CHECK(IEinfer), can have polynomial time complexity when they rely on computing the Minkowski’s form of polyhedra.

**Example 3.53.** Consider linear constraints of the form: \( cw_n = 0 \leq x_1 \leq 1, \ldots, 0 \leq x_n \leq 1 \), for \( n \geq 0 \). The polyhedra \( Sol(cw_n) \) is called a hypercube, since it consists of \( 2^n \) vertices of the form \( (a_1, \ldots, a_n) \) such that for \( i = 1 \ldots n \), \( a_i \in \{0, 1\} \). Computing the vertex matrix of \( Sol(cw_n) \) has then an exponential time computational complexity.

When restricting to homogeneous systems, as required in LPInfer, the worst case is still exponential.

**Example 3.54.** The transformation of [Goldman 1956; Wilde 1993] over \( cw_n \) yields the homogeneous constraint \( hw_n = 0 \leq x_1, 0 \leq \xi - x_1, \ldots, 0 \leq x_n, 0 \leq \xi )
Typing Linear Constraints

\[ \xi - x_n, 0 \leq \xi. \]

For each vertex \( v \) of \( \text{Sol}(cw_n) \), \( \text{Sol}(hw_n) \) has a ray of the form \((\lambda v, \lambda)\), for some \( \lambda > 0 \). Therefore, complexity of computing the vertex matrix of \( \text{Sol}(cw_n) \) has been shifted to the computation of the generating matrix of \( \text{Sol}(hw_n) \).

We refer the reader to [Borgwardt 2007] for a discussion of the average-case complexity of the double description method, the core algorithm for the extraction of the Minkowski’s form of (parameterized) polyhedra. Under a reasonable stochastic model over the input linear system of inequalities, the average computational complexity of the method is shown to be polynomial in the number of inequalities (but not in the number of variables, which is the case of the last two examples).

Let us consider now \( \text{Check(POLYInfer)} \) and \( \text{ParCheck} \). They require to compute the pairs \((v^a(1), C_1 a \leq c_1), \ldots, (v^a(k), C_k a \leq c_k)\) as from Theorem 3.24. The algorithm proposed in [Loechner and Wilde 1997] for the purpose consists of first computing the vertices of \( \text{Sol}(P) \) where \( P \) is the linear system (2) in the space of variables plus parameters; then to project each vertex over the parameter space. The first step leads to conclude non-polynomial time complexity, as for \( \text{LPInfer} \). The second step leads to conclude that the number of parameterized vertices \( k \) is bounded by the number of vertices in \( \text{Sol}(P) \). Since the projection at the second step requires polynomial time [Loechner and Wilde 1997, Section 5.3], under the assumptions of [Borgwardt 2007], the computation of the parameterized vertices has a polynomial time average complexity in the number of inequalities.

Finally, in Section 5 we report an experimental evaluation of the execution times of the various decision procedures over several testbeds.

4. MODING CLP(\( \mathbb{R} \)) PROGRAMS

Constraint logic programming provides an elegant scheme for dynamically building complex constraints, exploiting recursion, non-determinism and intertwined constraint generation & solving. It is a natural candidate for exploiting the type system theory developed in the previous sections. Here, we propose an extension of the notion of well-moding from pure logic programming.

4.1 Well-moding

Modes for pure logic programs assign to every predicate argument an input-output behavior. Input means that the predicate argument is ground on calls. Output means that is it ground on answers. As discussed in the introduction, groundness (i.e., definiteness) is restrictive in the CLP context. Based on types, we can extend the notion of moding to upper and/or lower bounds as well.

**Definition 4.1 (moding).** A mode for an \( n \)-ary predicate \( p \) is a function \( d_p \) from \( \{1, \ldots, n\} \) to \( BT \times BT \). We write \( d_p \) as \( p(\tau_1 \times \mu_1, \ldots, \tau_n \times \mu_n) \), where \( d_p(i) = (\tau_i, \mu_i) \) for \( i = 1..n \).

A mode for a CLP(\( \mathbb{R} \)) program \( P \) is a set of modes, one for each predicate in \( P \). For an atom \( p(x) \), we write \( p(x : \tau \times \mu) \) to denote that \( x \) is the collection of variables occurring in the atom, and \( p(\tau \times \mu) \) is the mode of \( p \). Types in \( \tau \) are called input modes while types in \( \mu \) are called output modes.

By fixing a predicate argument mode to \(!x!\) or to \(*x!*\) we get back to the logic programming input-output behavior, respectively denoted by \(+\) and \(−\). Also the
mode $\star \times \star$ means that the predicate argument is of no relevance in the analysis, a case denoted by ? in logic programming approaches. We recall that programs are assumed in flat form. Several notions of moding have been proposed (see [Apt 1997] for a review) for logic programs. We consider here well-moding by extending it to CLP(R).

**Definition 4.2 (well-moding).** A CLP(R) clause:

$$p_0(x_0 : \tau_0 \times \mu_0) \leftarrow c, p_1(x_1 : \tau_1 \times \mu_1), \ldots, p_n(x_n : \tau_n \times \mu_n)$$

is well-moded if the following type assertions are valid:

For $i = 1..n$, $x_0 : \tau_0, x_1 : \mu_1, \ldots, x_{i-1} : \mu_{i-1} \vdash c \rightarrow x_i : \tau_i,$

$$x_0 : \tau_0, x_1 : \mu_1, \ldots, x_n : \mu_n \vdash c \rightarrow x_0 : \mu_0.$$ 

A CLP(R) program $P$ is well-moded if every clause in it is well-moded.

The definition of a well-moded clause constrains the “flow of data” through the atoms to follow a left-to-right sequence, starting with input position in the head of the clauses and ending with output positions in the head.

**Example 4.3.** The **MORTGAGE** program is well-moded with moding mortgage($!x, \Box \times !, !x, \star \times !$), which is intended for covering the first two queries in the introduction. For clause (m1) we have to show that:

$$P :!, T : \Box, R :!, B : \star \vdash T = 0, B = P \rightarrow P :!, T :!, R :!, B :!$$

is valid, which is immediate. For clause (m2), called $c$ the constraint $T >= 1, NP = P + P * 0.05 - R, NT = T - 1$, we have to show validity of:

$$P :!, T : \Box, R :!, B : \star \vdash c \rightarrow NP :!, NT : \Box, R :!, B : \star,$$

$$P :!, T : \Box, R :!, B : \star, NP :!, NT : \Box, R :!, B : \star \vdash c \rightarrow P :!, T :!, R :!, B :!$$

which are both readily checked. Analogously, **MORTGAGE** is well-moded with the moding mortgage($\star \times \Box, \Box \times !, \Box \times \Box, \Box \times \Box$), which is intended for covering the third query in the introduction.

We recall that the operational semantics of CLP consists of a transition system from states to states (see Appendix A.9 for details). A state is a pair $\langle Q \| c \rangle$ where $Q$ is a query and $c$ is a constraint, called the constraint store. Initial states are of the form $\langle Q \| \text{true} \rangle$. Final states (if any) are of the form $\langle \varepsilon \| c \rangle$, where $\varepsilon$ is the empty query. The definition of well-moding extends to states.

**Definition 4.4.** A state $\langle c, p_1(x_1 : \tau_1 \times \mu_1), \ldots, p_n(x_n : \tau_n \times \mu_n) \rangle$, with $n \geq 0$, is well-moded if for $i = 1..n$ the type assertion:

$$x_1 : \mu_1, \ldots, x_{i-1} : \mu_{i-1} \vdash (c \land c') \rightarrow x_i : \tau_i$$

is valid. A query $Q$ is well-moded if the state $\langle Q \| \text{true} \rangle$ is well-moded.

### 4.2 Well-moding for Static Analysis

Widely studied properties of well-moding in logic programming include persistency along derivations, call pattern characterization and computed answer characterization. They are at the basis of several techniques for program analysis, such as
Typing Linear Constraints

Theorem 4.5. Let \( P \) be a well-moded \( \mathsf{CLP}(R) \) program and \( Q = \leftarrow c, p_1(x_1: \tau_1 \times \mu_1), \ldots, p_n(x_n: \tau_n \times \mu_n) \) a well-moded query.

[**persistence**]. Every state selected in a left-derivation of \( P \) and \( Q \) is well-moded.

[**call patterns**]. For every state of the form \( \langle \leftarrow q(x: \tau \times \mu), R \parallel c' \rangle \) selected in a left-derivation of \( P \) and \( Q \), \( \vdash c' \rightarrow x : \tau \) is valid.

[**answers**]. For every final state \( \langle \leftarrow \epsilon \parallel c' \rangle \), \( \vdash c' \rightarrow x_1 : \mu_1, \ldots, x_n : \mu_n \) is valid.

The next two examples provide hints on the kind of analyses that well-moding allows for by exploiting the properties above.

Example 4.6. The two queries from the introduction \( \leftarrow \text{mortgage}(100, 5, 20, B) \) and \( \leftarrow 3 \leq T, T \leq 5, \text{mortgage}(100, T, 20, B) \) are well-moded with the moding \( \text{mortgage}(\star \times \star, \star \times \star, \star \times \star) \). Since the \textsc{Mortgage} program was shown to be well-moded in Example 4.3, by Theorem 4.5 we conclude definiteness of balance in every answer constraint store.

The third query from the introduction \( \leftarrow 0 \leq B, B \leq 10, 15 \leq R, R \leq 20, \text{mortgage}(P, 5, R, B) \) is well-moded with the moding \( \text{mortgage}(\star \times \star, \star \times \star, \star \times \star) \). Again, by Theorem 4.5 we conclude boundedness of principal in every answer constraint store.

Example 4.7. The full version of the \textsc{Mortgage} program takes the interest rate as a further predicate argument.

\[
\begin{align*}
(n1) \quad \text{mortgage}(P, T, I, R, B) & \leftarrow \\
& T = 0, \\
& B = P.
\end{align*}
\]

\[
\begin{align*}
(n2) \quad \text{mortgage}(P, T, I, R, B) & \leftarrow \\
& T > 1, \\
& NP = P + P \times I - R, \\
& NT = T - 1, \\
& \text{mortgage}(NP, NT, I, R, B).
\end{align*}
\]

However, this leads to a non-linear constraint appearing in clause \( (n2) \). How can we reason on it? Our framework is heavily based on linearity. Also, many constraint solvers are incomplete with respect to non-linear constraints and delay their evaluation until they become linear: even an approach explicitly dealing with non-linear constraints fails with such constraint solvers. We exploit the call pattern characterization property of well-moding by factoring out the \( P \times I \) term.

\[
\begin{align*}
(n2') \quad \text{mortgage}(P, T, I, R, B) & \leftarrow \\
& T > 1, \\
& NP = P + M - R, \\
& NT = T - 1, \\
& \text{mult}(P, I, M), \\
& \text{mortgage}(NP, NT, I, R, B).
\end{align*}
\]

Consider now as if the predicate \text{mult} is a built-in of the system, and the input-output properties of Theorem 4.5 are guaranteed for the mode \text{mult}(\star \times \star, \star \times \star, \star \times \star). The rest of the program, namely clauses \( (n1) \) and \( (n2') \), is readily checked.
to be well-moded with moding mortgage(!×!, !×!, !×!, !×!, !×!, !×!, !×!). Therefore, for every call to mult the first and the second arguments are definite, and then the non-linear constraint \( P \times I = M \) becomes linear at run-time. Finally, notice that we can fold back the mult predicate in (n2') to conclude that the non-linear constraint in (n2) becomes linear at run-time.

Notice, however, that the approach requires an appropriate choice of the mode of mult and of the position of the call to mult. For instance, consider the mode mortgage(\( \star \times \square \), \( \square \times \!', \) !×!, !×!, \( \square \times \square \), \( \square \times \square \)), where we intend to calculate the principal given the other arguments. The non-linear constraint \( NP = P + P \times I - R \) in (n2) can be rewritten as \( P \times (I+1) = NP + R \). This suggests that mult(P, I+1, NP+R) can be called to compute P from I+1 and NP+R. There are two issues, however. First, the mode of mult must be mult(\( \star \times \square \), !×!, \( \square \times \square \)), since the input arguments are now the second and the third ones. Second, the argument NP+R has type \( \square \) only after the recursive call. As a consequence, the call to mult must occur after the recursive call to mortgage. Summarizing, we have the following version of the (n2) clause:

(n2'') mortgage(P,T,I,R,B) ←
\[
T \geq 1,
I_1 = I + 1,
M = NP + R,
NT = T - 1,
mortgage(NP,NT,I,R,B),
mult(P, I_1, M).
\]

With the provided modes, the non-linear constraint in mult becomes linear at run-time. It is worth noting that the position of the call to mult in (n2'') can be interpreted as the point in the left-to-right flow of the calls when the non-linear constraint in (n2) becomes linear. Finally, notice that the approach of rearranging body atoms to fit the declared modes is a well-studied technique [Somogyi et al. 1996].

4.3 Extension to CLP(\( \mathbb{R} \)) with Terms

Several existing CLP(\( \mathbb{R} \)) languages and systems actually adopt a multisorted constraint domain, usually including at least the domain Term of pure logic programming terms. In those languages, terms are trees whose leaves are either symbolic constants from Term or linear expressions over \( \mathbb{R} \). Definitions and results for multisorted domain extends naturally from the single-sorted ones. More specifically, an atom is now of the form \( p(t_1, \ldots, t_n) \) where \( p \) is a predicate of arity \( n \) and \( t_1, \ldots, t_n \) are terms.

**Example 4.8.** The following LISTSUM program calculates the sum of the elements in a list.

\( ^4 \)Predicate occurring in two or more sorts need to be renamed apart or to be syntactically distinguished. As an example, the unification predicate \( = \) (defined by the clause \( X = X \). which is implicitly part of any program with term) is used also as the linear equality predicate. Other examples are concerned with Prolog arithmetic built-in’s. In order to distinguish the two sorts, concrete programming languages, such as SWI-Prolog, write constraints between curly brackets.
Let every element in the list be definite. If the latter states that the sum is definite if every element of the list is upper and lower bounded. The latter states that the sum is definite if every element in the list is definite.

The definition of well-moding readily extends to programs with terms, provided that the notion of type declaration is lifted to type terms, in addition to variables over reals. Let us denote by $\text{vars}(t)$ the set of variables of a term $t$. The following definition extends atd’s (and a fortiori, type assertions) to terms.

**Definition 4.9.** Let $t$ be a term and $\text{vars}(t) = \{x_1, \ldots, x_n\}$. We write $t : \tau$ as a shorthand for $x_1 : \tau, \ldots, x_n : \tau$.

The definition is conservative: when $t$ is a variable, the shorthand reduces to an atd. The semantics underlying the definition is that a term is typed as $\tau$ if every variable over the reals in it is typed as $\tau$, and every variable over terms in it is typed as $\tau$.

**Example 4.10.** The LISTSUM program is well-moded w.r.t. listsum($[\times! , \times! ]$). For instance, for clause (s2), the type assertions:

\[
X \times!, Xs : \times! \vdash S = X + S1 \rightarrow S : ! \quad X \times!, Xs : ! \vdash S = X + S1 \rightarrow S : !
\]

are readily checked to be valid.

Using this notation, it is readily checked that the proposed notion is a conservative extension of well-moding for pure logic programs [Apt 1997]. In fact, consider a clause with constraint $\text{true}$:

\[
p_0(t_0 : \tau_0 \times \mu_0) \leftarrow p_1(t_1 : \tau_1 \times \mu_1), \ldots, p_n(t_n : \tau_n \times \mu_n).
\]

By Lemma 2.11, the type assertions to be checked by Definition 4.2 are valid iff:

\[
\text{for } i = 1..n, \quad \text{nf}(t_0 : \tau_0, t_1 : \mu_1, \ldots, t_{i-1} : \mu_{i-1}) \geq \text{nf}(t_i : \tau_i), \\
\quad \text{nf}(t_0 : \tau_0, t_1 : \mu_1, \ldots, t_n : \mu_n) \geq \text{nf}(t_0 : \mu_0).
\]

Assume now that only pairs $\times!$ (input) and $\times\times!$ (output) are used. By expanding the definition of typed terms, the conditions can be rewritten as follows:

\[
\text{for } i = 1..n, \quad \bigcup_{j=1}^{i-1} \text{vars}(\text{outp}(t_j)) \cup \text{vars}(\text{inp}(t_0)) \supseteq \text{vars}(\text{inp}(t_i)), \\
\quad \bigcup_{j=1}^{n} \text{vars}(\text{outp}(t_j)) \cup \text{vars}(\text{inp}(t_0)) \supseteq \text{vars}(\text{outp}(t_0)).
\]

where, for $i = 0..n$, $\text{inp}(t_i)$ is the subset of $t_i$ typed as $\times!$ in $p_i(t_i : \tau_i \times \mu_i)$, and $\text{outp}(t_i)$ is the subset typed as $\times\times!$. This is exactly the formulation of well-moding for pure logic programs.

Theorem 4.5 readily extends to programs and queries with terms by the following fact, which states that validity of type assertion is monotonic w.r.t. term instantiation. For a term substitution $\theta$ and an atd $d = t : \tau$, we write $d\theta$ to denote $t\theta : \tau$. This naturally extends to type declarations.
Lemma 4.11. Assume that \( d_1 \vdash c \rightarrow d_2 \) is valid, and let \( \theta \) be a term substitution such that \( c\theta = c \). Then \( d_1\theta \vdash c \rightarrow d_2\theta \) is valid.

Intuitively, the lemma allows for the conclusion that an instance of a well-moded clause is well-moded, provided that variables over the reals in the constraint from the clause are not instantiated. A run-time error is typically raised if an incorrectly instantiated constraint is introduced in the current state during program execution.

4.4 Extension to Parametric Types

In this section, the definition of well-moding is extended to admit parametric types. The resulting moding system allows for more expressive analyses.

Definition 4.12 (parametric moding). A parametric mode for an \( n \)-ary predicate \( p \) is a function \( d_p \) from \( \{1, \ldots, n\} \rightarrow PT \times PT \). We extend the notation used for modes to parametric modes.

We assume that, for a parametric mode \( p(\tau \times \mu) \), if \( s \) is a type variable in \( \mu \), then \( s \) is a type variable in \( \tau \), i.e., no “local to output” type variable exist.

In the following, we omit the adjective parametric when it is clear from the context.

Example 4.13. Consider the program ACKERMANN for computing the Ackerman function.

(a1) \( \text{ack}(M, N, R) \leftarrow \ M = 0, R = N+1. \)

(a2) \( \text{ack}(M, N, R) \leftarrow \ M >= 1, N = 0, \ N1 = N-1, \ M1 = M - 1, \text{ack}(M1, N1, R). \)

(a3) \( \text{ack}(M, N, R) \leftarrow \ M >= 1, N >= 1, \ N1 = N-1, \ M1 = M - 1, \text{ack}(M, N1, T), \text{ack}(M1, T, R). \)

The mode \( \text{ack}(\star \times! \ldots, \square_b \times \square_b, \star \times \square_b) \) states that if \( \text{ack}(M, N, R) \) is called with a variability range of \( b \) for \( N \) then, after it is completely resolved, \( M \) is definite and \( R \) has a variability range of \( b \). Consider clause (a2), and let \( c \) be \( M >= 1, N >= 1, N1 = N-1, M1 = M - 1. \) In order to show that the program is well-moded, the following parametric type assertions must be valid: \( M : \star, N : \square_b, R : \star \vdash c \rightarrow M1 : \star, N1 : \square_b, R : \star, M : \square_b, N : \square_b \). They are both readily checked.

The query \( \leftarrow 0 <= M, M <= 5, 1 <= N, N <= 2, \text{ack}(M, N, R) \) is well-moded, with \( b = 1 \). In automated proofs, this can be established by running the POLY-INFER procedure to conclude \( \vdash 0 <= M, M <= 5, 1 <= N, N <= 2 \rightarrow N : \square. \) For every constraint store \( c \) in a final state, the type assertion \( \vdash c \rightarrow R : \square \) is then valid. Stated in other words, if we start with a value for \( N \) known with an approximation of at most \( \pm 0.5 \), in each computed answer the result \( R \) is stated with at most the same approximation. Notice that this does not mean that two results \( R \) from two different computed answers differ by at most \( \pm 0.5 \).

As outlined at the end of the example, however, when calling a predicate with parametric mode, it may be necessary to instantiate type variables. In the general case, this requires an extended formulation of Definition 4.2.

Example 4.14. Consider the program ACKERMANN with the additional clause:
(a4) \( \text{ack2}(N, R) \leftarrow \text{ack}(N, N, R). \)

and mode \( \text{ack2}(\Box_a \times \Box_a, \star \times \Box_a). \) Is (a4) well-moded? Using Definition 4.2, we should show validity of the type assertions \( N: \Box_a, R: \star \vdash \text{true} \rightarrow N: \Box_a, R: \star \) and \( M: \Box_a, R: \star, N: \Box_b, R: \Box_b \vdash \text{true} \rightarrow N: \Box_a, R: \Box_a, \) which is not the case since \( b \) and \( a \) are distinct type variables. Intuitively, however, \( b \) has a meta-level usage, in the sense that when calling the predicate \( \text{ack} \) the type variable \( b \) can be instantiated to satisfy the call context. By instantiating \( b = a \), the type assertions above become \( N: \Box_a, R: \star \vdash \text{true} \rightarrow N: \Box_a, R: \star \) and \( M: \Box_a, R: \star, N: \Box_b, R: \Box_b \vdash N=1 \rightarrow \text{true} \rightarrow N: \Box_a, R: \Box_a, \) which are both valid. Similarly, consider:

(a5) \( \text{ack3}(M, R) \leftarrow N=1, \text{ack}(M, N, R). \)

with mode \( \text{ack3}(\star! \times \star!, \star! \times \star!). \) The type assertions to be shown are \( M: \star, R: \star \vdash N=1 \rightarrow M: *, N: \Box_b, R: \star \) and \( N: \Box_b, R: \Box_b \vdash N=1 \rightarrow R: !, M: !, \) where the former is valid and the latter is not. By instantiating \( b = 0 \), the type assertions become \( M: \star, R: \star \vdash N=1 \rightarrow M: *, N: !, R: \star \) and \( N: !, R: !, N: ! \vdash N=1 \rightarrow R: !, M: !, \) which are both valid.

Let us formalize type variable instantiation by means of type substitutions.

**Definition 4.15.** A type substitution \( \vartheta \) is a function mapping type variables into linear expressions over type variables.

Type substitutions readily lift to parametric types, by setting \( \vartheta(\Box_a) = \Box_{\vartheta(e)} \) and \( \vartheta(\tau) = \tau \) for \( \tau \in BT_\mathcal{C}. \) We are now in the position to (conservatively) extend the notion of well-moding.

**Definition 4.16 (Well-moding with Parametric Types).** A CLP(\( \mathcal{R} \)) clause with parametric modes:

\[
p_0(x_0: \tau_0 \times \mu_0) \leftarrow \ldots p_n(x_n: \tau_n \times \mu_n)
\]

is well-moded if there exist type substitutions \( \vartheta_1, \ldots, \vartheta_n \) such that the following type assertions have no local type variable and are valid:

\[
\begin{align*}
\text{for } i = 1, \ldots, n, \quad & x_0: \tau_0, x_1: \vartheta_i(\mu_1), \ldots, x_{i-1}: \vartheta_{i-1}(\mu_{i-1}) \vdash c \rightarrow x_i: \vartheta_i(\tau_i), \\
\quad & x_0: \tau_0, x_1: \vartheta_i(\mu_1), \ldots, x_n: \vartheta_n(\mu_n) \vdash c \rightarrow x_0: \mu_0.
\end{align*}
\]

A CLP(\( \mathcal{R} \)) program \( P \) is well-moded if every clause in it is well-moded.

The requirement that there is no local type variable formalizes the intuitions of Example 4.14 that type variables in the modes of atoms in in the body of the clause must be instances of type variables occurring in the mode of the head.

Similarly, well-moding for a state (and then for a query) \( \langle c, p_1(x_1: \tau_1 \times \mu_1), \ldots, p_n(x_n: \tau_n \times \mu_n) \rangle \) is extended by assuming type substitutions \( \vartheta_1, \ldots, \vartheta_n \) such that the following type assertions have no local type variable and are valid:

\[
\begin{align*}
\text{for } i = 1, \ldots, n, \quad & x_1: \vartheta_i(\mu_1), \ldots, x_{i-1}: \vartheta_{i-1}(\mu_{i-1}) \vdash c \land c' \rightarrow x_i: \vartheta_i(\tau_i).
\end{align*}
\]

The basic property of persistency holds for the revised notions of well-moded programs and queries. Call pattern and computed answer characterizations readily follow from persistency.
**Theorem 4.17.** (Persistency) Let $P$ be a $\text{CLP}(\mathcal{R})$ program and $Q$ a query, both well-moded with parametric modes. Every state selected in a left-derivation of $P$ and $Q$ is well-moded.

The natural question with the revised definition of well-moding is: how to derive the type substitutions $\vartheta$?

For a well-moded atomic query $\leftarrow c, p(x_1 : \tau \times \mu)$, we have that $\vartheta_1(\tau)$ must be types in $BT$. Thus, the POLYInfer procedure can be adopted for the purpose, as it was suggested in Example 4.13. Also, notice that, since we assume no local to output type variables, $\vartheta_1(\mu)$ is in $BT$ as well, i.e., the output types are completely determined by $\vartheta_1$.

Concerning program clauses, in paper & pencil proofs, $\vartheta$ comes out from the specific instance of the call to the clause body predicate. In automated proofs, an inference procedure is needed, as done in Example 4.13, to infer the type of $x$ at the point of a call to $p(..., x : \bowtie_b \times \tau...)$. If the type inferred is $\bowtie_c$, we can set $b = c$. As discussed in Section 3.6, however, a complete inference procedure may not exist if we restrict to (non-piecewise) linear expressions.

5. EXPERIMENTAL RESULTS

5.1 The clpt system

Proof obligations of well-moding consist of repeatedly calling a decision procedure for checking validity of type assertions syntactically built from the clauses of the program under consideration. Checking well-moding is then a representative testbed for testing the efficiency in practice of the procedures designed in the paper.

We have implemented, in standard C++, the checking procedure for well-moding, including the extension to $\text{CLP}(\mathcal{R})$ with terms, and the following decision and inference procedures: $\text{Check}(\chi)$, LPInfer, IEInfer, POLYInfer, ParCheck. Validity of a type assertion $d_1 \vdash c \rightarrow d_2$ is checked on the basis of the type system $d_1$ and $d_2$ defined in: by $\text{Check}(\text{IEInfer})$ for $BT$; by $\text{Check}(\text{POLYInfer})$ for $BT$; and by the ParCheck procedure for $PT$. For non-valid type assertions, the POLYInfer procedure is called to provide the most general type that can be inferred for variables in $d_2$. The implementation relies on the polylib library [Loechner 2007] for the calculation of the Minkowski’s form of (parameterized) polyhedra. The overall system, called clpt, is available as open source from http://www.di.unipi.it/~ruggieri/software.

5.2 Testbeds and Experiments

Tables I-IV report the execution times over several testbeds of programs. Tests were run on a PC Xeon 2.8GHz with Linux 2.6.17. In brief, those results provide us with confidence on the efficiency of the proposed approach in practice. More in depth, let us consider the various testbeds.

The folk testbed. This testbed consists of small-size programs with linear constraints only, from the $\text{CLP}(\mathcal{R})$ folklore. For a same program, we considered different modes of usage, as done for the MORTGAGE example throughout the paper. Also, we consider different type systems for specifying modes. For instance, the ack program is tested for modes whose types belong to $BT_0$, $BT_1$, $BT$, and $PT$. 


Typing Linear Constraints · 33

<table>
<thead>
<tr>
<th>program</th>
<th>C</th>
<th>A</th>
<th>mode of the main predicates</th>
<th>time</th>
</tr>
</thead>
<tbody>
<tr>
<td>ack</td>
<td>3</td>
<td>6</td>
<td>ack(+ × , × )</td>
<td>0.0008</td>
</tr>
<tr>
<td>ack</td>
<td>3</td>
<td>6</td>
<td>ack(+ × !, ! × )</td>
<td>0.0010</td>
</tr>
<tr>
<td>ack</td>
<td>3</td>
<td>6</td>
<td>ack(+ × !, ! × )</td>
<td>0.0025</td>
</tr>
<tr>
<td>ack</td>
<td>3</td>
<td>6</td>
<td>ack(+ × !, ! × )</td>
<td>0.0030</td>
</tr>
<tr>
<td>fib</td>
<td>2</td>
<td>4</td>
<td>fib(! × , ! × )</td>
<td>0.0007</td>
</tr>
<tr>
<td>fib</td>
<td>2</td>
<td>4</td>
<td>fib(! × , ! × )</td>
<td>0.0019</td>
</tr>
<tr>
<td>mc91</td>
<td>2</td>
<td>4</td>
<td>mc(! × ! , × )</td>
<td>0.0005</td>
</tr>
<tr>
<td>mortgage</td>
<td>2</td>
<td>4</td>
<td>mortgage( × , ! × )</td>
<td>0.0007</td>
</tr>
<tr>
<td>mortgage</td>
<td>2</td>
<td>4</td>
<td>mortgage( × , ! × )</td>
<td>0.0009</td>
</tr>
<tr>
<td>schedule</td>
<td>10</td>
<td>21</td>
<td>schedule( ! × , × × )</td>
<td>0.0020</td>
</tr>
<tr>
<td>schedule</td>
<td>10</td>
<td>21</td>
<td>schedule( ! × , × × )</td>
<td>0.0080</td>
</tr>
<tr>
<td>send+more</td>
<td>4</td>
<td>9</td>
<td>send+more( × , ! × )</td>
<td>0.0633</td>
</tr>
<tr>
<td>send+more</td>
<td>4</td>
<td>9</td>
<td>send+more( × , ! × )</td>
<td>0.4846</td>
</tr>
<tr>
<td>send+more</td>
<td>4</td>
<td>9</td>
<td>send+more( × , ! × )</td>
<td>1.1526</td>
</tr>
<tr>
<td>tak</td>
<td>3</td>
<td>8</td>
<td>tak(! × , ! × )</td>
<td>0.0010</td>
</tr>
</tbody>
</table>

Table I. The folk testbed. Elapsed time in seconds, C = no. of clauses, A = no. of atoms.

<table>
<thead>
<tr>
<th>program</th>
<th>C</th>
<th>A</th>
<th>mode of the main predicates</th>
<th>time</th>
</tr>
</thead>
<tbody>
<tr>
<td>listsun</td>
<td>2</td>
<td>3</td>
<td>listsun( × , × )</td>
<td>0.0005</td>
</tr>
<tr>
<td>listsun</td>
<td>2</td>
<td>3</td>
<td>listsun( × , × )</td>
<td>0.0005</td>
</tr>
<tr>
<td>nqueens</td>
<td>13</td>
<td>25</td>
<td>nqueens( ! × , ! × )</td>
<td>0.0016</td>
</tr>
<tr>
<td>plate</td>
<td>6</td>
<td>11</td>
<td>plate( × )</td>
<td>0.0024</td>
</tr>
<tr>
<td>quicksort</td>
<td>7</td>
<td>14</td>
<td>quicksort( ! × )</td>
<td>0.0010</td>
</tr>
<tr>
<td>quicksort</td>
<td>7</td>
<td>14</td>
<td>quicksort( × , × )</td>
<td>0.0015</td>
</tr>
<tr>
<td>quicksort</td>
<td>7</td>
<td>14</td>
<td>quicksort( × , × )</td>
<td>0.0017</td>
</tr>
<tr>
<td>sequence</td>
<td>15</td>
<td>32</td>
<td>sequence( ! × )</td>
<td>0.0017</td>
</tr>
<tr>
<td>tree_layout</td>
<td>35</td>
<td>82</td>
<td>tree_layout( ! × , × )</td>
<td>0.0053</td>
</tr>
</tbody>
</table>

Table II. The term testbed. Elapsed time in seconds, C = no. of clauses, A = no. of atoms.

Notice how the execution times in Table I increase with the expressiveness of the type system and, a fortiori, with the complexity of the checking procedure adopted. Running times are very low on average. The send+more=money program is the most demanding one, due to the presence of a worst-case constraint in it (see later on the worst testbed).

The term testbed. It includes programs mixing linear constraints and terms, mainly from CLP textbooks [Marriott and Stuckey 1998]. As a special case, when clauses contain no constraint, pure Prolog programs belong to this testbed. Table II shows very low execution times, better than in the folk testbed. In fact, we observe that a variable x representing a term (e.g., a list) does not typically occur in a clause constraint, hence proving validity of a type assertion d ⊢ c ← x : τ reduces to checking d ⊢ true ← x : τ which, by Lemma 2.11, reduces to checking whether
Table III. The *mlsize* testbed. Elapsed time in seconds, C = no. of clauses, A = no. of atoms.

<table>
<thead>
<tr>
<th>program</th>
<th>C</th>
<th>A</th>
<th>mode of the main predicates</th>
<th>time</th>
</tr>
</thead>
<tbody>
<tr>
<td>assoc</td>
<td>13</td>
<td>23</td>
<td>assoc_to_list( !\times!, \times \circ)</td>
<td>0.0039</td>
</tr>
<tr>
<td>credit</td>
<td>33</td>
<td>63</td>
<td>credit( !\times!, \times \circ)</td>
<td>0.0094</td>
</tr>
<tr>
<td>sequence</td>
<td>5</td>
<td>18</td>
<td>q(\times!)</td>
<td>0.0041</td>
</tr>
<tr>
<td>ttt</td>
<td>36</td>
<td>75</td>
<td>play(\times!)</td>
<td>0.0136</td>
</tr>
<tr>
<td>Ackermann</td>
<td>7</td>
<td>14</td>
<td>all arguments moded as !\times!</td>
<td>0.0015</td>
</tr>
<tr>
<td>CaffeineMark</td>
<td>4944</td>
<td>9888</td>
<td>all arguments moded as !\times!</td>
<td>3.1226</td>
</tr>
<tr>
<td>JLex</td>
<td>850</td>
<td>1700</td>
<td>all arguments moded as !\times!</td>
<td>0.3400</td>
</tr>
<tr>
<td>Kitten</td>
<td>2320</td>
<td>4640</td>
<td>all arguments moded as !\times!</td>
<td>0.6115</td>
</tr>
<tr>
<td>NQueens</td>
<td>290</td>
<td>580</td>
<td>all arguments moded as !\times!</td>
<td>0.0970</td>
</tr>
<tr>
<td>RayTracer</td>
<td>75</td>
<td>150</td>
<td>all arguments moded as !\times!</td>
<td>0.0300</td>
</tr>
</tbody>
</table>

Table IV. The *worst* testbed. Elapsed time in seconds, C = no. of clauses, A = no. of atoms.

<table>
<thead>
<tr>
<th>program</th>
<th>C</th>
<th>A</th>
<th>mode of the main predicates</th>
<th>time</th>
</tr>
</thead>
<tbody>
<tr>
<td>worst_10</td>
<td>1</td>
<td>1</td>
<td>worst(\times \circ)</td>
<td>0.07</td>
</tr>
<tr>
<td>worst_11</td>
<td>1</td>
<td>1</td>
<td>worst(\times \circ)</td>
<td>0.32</td>
</tr>
<tr>
<td>worst_12</td>
<td>1</td>
<td>1</td>
<td>worst(\times \circ)</td>
<td>1.23</td>
</tr>
<tr>
<td>worst_13</td>
<td>1</td>
<td>1</td>
<td>worst(\times \circ)</td>
<td>7.00</td>
</tr>
<tr>
<td>worst_14</td>
<td>1</td>
<td>1</td>
<td>worst(\times \circ)</td>
<td>33.50</td>
</tr>
<tr>
<td>worst_15</td>
<td>1</td>
<td>1</td>
<td>worst(\times \circ)</td>
<td>135.52</td>
</tr>
</tbody>
</table>

\(\tau = \star\), or \(x : \tau\) belongs to \(nf(d)\).

The *mlsize* testbed. This set of medium-to-large size programs is automatically generated from Prolog and Java programs. Prolog programs are transformed into CLP(\(R\)) programs by applying the term-size or the list-size norm to predicate arguments, a basic abstract interpretation technique used by some termination analysers [Lagoon et al. 2003; Mesnard and Ruggieri 2003]. Java bytecode programs are transformed into CLP(\(R\)) programs by applying the *Julia+BinTerm* system [Spoto et al. 2009]. This analyzer combines information from sharing, cyclicity, and path-length analysis to generate binary CLP programs, the termination of which ensures termination of the original Java programs.

The execution times reported in Table III show that the *clpt* system scales up to large-size programs. This is theoretically justified by noting that proof obligations of well-moding consider each clause in isolation.

The *worst* testbed. This testbed consists of programs *worst_n* with only one rule of the form \(\text{worst}(y) \leftarrow cw_n, y = 1\), where \(cw_n\) is the worst-case linear constraint of Example 3.53. We recall that \(cw_n\) yields and exponential number of vertices, even for a non-parameterized polyhedra. Such an exponential grow is reflected by the *worst_n* program execution times shown in Table IV. Note that the mode *worst(\times \circ \approx)* is such that the CHECK(LPInfer) procedure is actually called by the *clpt* system.

5.3 Polyhedra-based vs Simplex-based Implementations

We are interested in comparing performances of the polyhedra-based implementation of the checking procedures with a Simplex-based one. This can be done
Typing Linear Constraints

Table V. Elapsed times for the Check(LP2Infer) procedure, and ratio over Check(LPInfer). Elapsed time in seconds, $C =$ no. of clauses, $A =$ no. of atoms. $T_2 =$ time of Check(LP2Infer), ratio = $T_2$ / (time of Check(LPInfer)).

<table>
<thead>
<tr>
<th>Program</th>
<th>$C$</th>
<th>$A$</th>
<th>Modes</th>
<th>Time</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>ack</td>
<td>3</td>
<td>6</td>
<td>$\text{ack}(\star \times \Box, \Box \times \Box, \star \times \Box)$</td>
<td>0.0049</td>
<td>6.33</td>
</tr>
<tr>
<td>listsun</td>
<td>2</td>
<td>3</td>
<td>$\text{listsun}(\Box \times \Box, \star \times \Box)$</td>
<td>0.0027</td>
<td>6.13</td>
</tr>
<tr>
<td>mc91</td>
<td>2</td>
<td>4</td>
<td>$\text{mc}(\Box \times \Box, \star \times \Box)$</td>
<td>0.0023</td>
<td>5.58</td>
</tr>
<tr>
<td>nqueens</td>
<td>13</td>
<td>25</td>
<td>$\text{nqueens}(\Box \times \Box, \star \times \Box)$</td>
<td>0.0089</td>
<td>5.94</td>
</tr>
<tr>
<td>schedule</td>
<td>10</td>
<td>21</td>
<td>$\text{schedule}(\Box \times \Box, \star \times \Box, \star \times \Box)$</td>
<td>0.0139</td>
<td>6.95</td>
</tr>
<tr>
<td>send+more</td>
<td>4</td>
<td>9</td>
<td>$\text{solve}(\star \times \Box, \star \times \Box, \star \times \Box, \star \times \Box)$</td>
<td>0.0102</td>
<td>0.16</td>
</tr>
<tr>
<td>worst_13</td>
<td>1</td>
<td>1</td>
<td>$\text{worst}(\star \times \Box)$</td>
<td>0.0020</td>
<td>0.0028</td>
</tr>
<tr>
<td>worst_14</td>
<td>1</td>
<td>1</td>
<td>$\text{worst}(\star \times \Box)$</td>
<td>0.0021</td>
<td>0.0009</td>
</tr>
<tr>
<td>Ackermann</td>
<td>7</td>
<td>14</td>
<td>all arguments moded as $\Box \times \Box$</td>
<td>0.0115</td>
<td>7.67</td>
</tr>
<tr>
<td>CaffeineMark</td>
<td>4944</td>
<td>9888</td>
<td>all arguments moded as $\Box \times \Box$</td>
<td>19.013</td>
<td>6.45</td>
</tr>
<tr>
<td>JLex</td>
<td>850</td>
<td>1700</td>
<td>all arguments moded as $\Box \times \Box$</td>
<td>3.088</td>
<td>6.01</td>
</tr>
<tr>
<td>Kitten</td>
<td>2320</td>
<td>4640</td>
<td>all arguments moded as $\Box \times \Box$</td>
<td>4.241</td>
<td>7.30</td>
</tr>
<tr>
<td>NQueens</td>
<td>290</td>
<td>580</td>
<td>all arguments moded as $\Box \times \Box$</td>
<td>0.632</td>
<td>6.87</td>
</tr>
<tr>
<td>RayTracer</td>
<td>75</td>
<td>150</td>
<td>all arguments moded as $\Box \times \Box$</td>
<td>0.172</td>
<td>6.02</td>
</tr>
</tbody>
</table>

for the $BT$ type system by comparing the Check(LPInfer) procedure (using Minkowski’s form of polyhedra) with an instance using a Simplex-based linear programming solver, which we call Check(LP2Infer).

We have implemented the Check(LP2Infer) procedure as part of the clpt system by relying on the lpsolve library [Berkelaar et al. 2008]. Table V reports the execution times for some programs from the previously introduced testbeds moded with types in $BT$. Apart from the worst-case scenario (we recall that send+more=money contains a worst case constraint), on average the polyhedra-based approach is 5 to 7 times faster than a Simplex-based one.

Although the results may be biased by (in)efficiencies of the adopted libraries, we observe that, on average, the constraints appearing in a program are quite small-sized, as per number of variables and number of inequalities. Moreover, a same constraint is involved in $n + 1$ type assertions, where $n$ is the number of atoms in the clause body. Therefore, an approach computing the set of (parameterized) vertices once for all $n + 1$ type assertions can reasonably outperform a Simplex-based one with relatively large setup times.

Finally, a polyhedra-based implementation has the additional benefit of sharing some computations with Check(POLYInfer) and ParCheck. Adopting a polyhedra-based implementation for Check(LPInfer) can then speed up those procedures, which do not have a linear programming equivalent.

---

5Since the double description method adopted by the polylib library to compute the Minkowski’s form of (parameterized) polyhedra is incremental, the Check(POLYInfer) and ParCheck procedures benefit from the availability of the Minkowski’s form of $A_v \leq b$, computed by Check(LPInfer) to test the satisfiability of the input constraint $c$. 
6. RELATED WORK

A class of formulas, called parametric queries, is investigated in [Huynh et al. 1991]. It includes formulas $\exists a \forall v \ c \rightarrow x \sim a$, where $\sim \in \{\leq, =, \geq\}$, or, with our notation, type assertions of the form $\vdash c \rightarrow x : \tau$ with $\tau \in \mathcal{BT}_\emptyset$. The approach switches from the problem of checking $\max \{c^T x \mid A_c v \leq b_c\} \leq a$ to its dual form $\max \{0 \mid y^T A_c = c, a = y^T b_c + q, y \geq 0, q \geq 0\} = 0$, namely on checking feasibility of $y^T A_c = c, a = y^T b_c + q, y \geq 0, q \geq 0$. This requires to solve a distinct linear programming problem for each variable to be typed. By switching to the homogeneous problem, our approach allows for concentrating on maximizing several linear functions on a single polyhedron, for which we need to extract its Minkowski’s form only once. More importantly, as soon as general type assertions $d_1 \vdash c \rightarrow d_2$ are considered, switching to the dual form yields a non-linear problem.

The problem of maximizing a linear function over a parameterized system of linear inequalities is the subject of (multi)parameterized linear programming. The solution of the problem can be expressed as a piecewise linear function of the parameters [Gal and Nedoma 1972; Gal 1995], or as the maximum of a finite set of linear functions [Borrelli et al. 2003; Keerthi and Sridharan 1990; Schechter 1987]. Therefore, an approach alternative to procedure $\text{Check(POLYInfer)}$ is to compute (for each variable to be typed) the $\max$ and $\min$ functions of a parameterized linear programming problem and then to compare the resulting piecewise linear functions on each pair of breaks they are defined on. It is worth mentioning that, even for a single parameter, the number of breaks can be exponential [Murty 1983].

Definiteness analysis for CLP($\mathcal{R}$), also called groundness analysis in logic programming, has been investigated in several works [Baker and Søndegaard 1993; Codish et al. 2001; de la Banda et al. 1996; Howe and King 2000], and it is used as a basic tool in CLP($\mathcal{R}$) compiler optimizations [Kelly et al. 1998]. Here we have extended the concept from definite values, namely the $!$ type, to ranges, namely the $\sqcap r$ type. The cited papers adopt abstract interpretation techniques to infer boolean expressions relating definiteness of predicate arguments. E.g., an inferred $p(x, y) = x \rightarrow y$ states that if $x$ is definite when $p(x, y)$ is called then $y$ is definite when it is completely resolved. Compared to the notion of well-moding, inference does not need modes to be specified. However, the mentioned approaches restrict to consider equality constraints only. Also, it is worth noting that groundness inference for logic programs is shown to be exponential in the worst case [Genaim et al. 2001]. Finally, we include in this stream of research also the work of [Hanus 1995] which adopts abstract interpretation to detect non-linear constraints that become linear at run-time.

A survey of applications of polyhedra and their Minkowski’s form to the analysis and verification of hardware and software systems is reported in [Bagnara et al. 2007]. The definition of the Minkowski’s form has been extended in [Bagnara et al. 2005] to explicitly take into account strict inequalities. Finally, we refer the reader to [Bagnara et al. 2008] for an experimental comparison of several libraries, including polylib, for reasoning about polyhedra.
7. CONCLUSION

We have introduced a type system for linear constraints over the reals that is able to reason about definiteness, upper bounds, lower bounds, range width (or approximatively known values) of variables. The problems of inferring and checking validity of type assertions have been investigated and solved by proposing specialized procedures of increasing complexity and expressiveness. Extensions to generalized constraints and parametric types are also presented.

As an application area, types are used for annotating (moding) CLP(\(\mathcal{R}\)) programs, while type assertions represent the basic tool for extending the notion of well-moding from logic programming to CLP(\(\mathcal{R}\)). The extension is conservative, since it can reason on programs mixing linear constraints and logic terms.

We have implemented in standard C++ all the type assertion checking and inference procedures, and the well-moding checking procedure. The system developed, called \texttt{clpt}, is released as open source. We have conducted an experimental evaluation of \texttt{clpt} over several testbeds of programs. The results show the efficiency of the proposed approach in practice.

Acknowledgements. We thank Maurice Bruynooghe and Sven Verdoolaege for suggesting the use of implicit equalities in checking \(\models \vdash c \rightarrow x : !\). We are grateful to Vincent Loechner for his precious support with the \texttt{polylib} library.

REFERENCES


S. Ruggieri and F. Mesnard


Typing Linear Constraints


A. PROOFS
A.1 Section 2.1

The proofs in this section follow a proof-theoretic approach based on first order predicate calculus. We recall that the notation \( \phi[t/x] \) denotes the substitution of free occurrences of the variable \( x \) by term \( t \) within the formula \( \phi \).

To keep the notation simple, we will exploit the fact that, since the \( v() \) function assumes fresh variables, the quantification \( \exists v(d_2) \setminus v(d_1) \) in (1) in Definition 2.4 is equivalent to \( \exists v(d_2) \) as far as the type system \( BT \) is considered. Also, when writing instances of (1), \( v \) will denote \( vars(c) \cup vars(d_1) \cup vars(d_2) \) where \( c \) is the constraint and \( d_1, d_2 \) the type declarations involved in the instance.

**Lemma 2.8.**

**Proof.** Assume that \( \vdash d_1 \vdash c \rightarrow d_2 \) is valid, i.e. \( \forall v(d_1) \exists v(d_2) \forall v. \phi(d_1) \land c \rightarrow \phi(d_2) \) holds. The result follows from the parts (a), (b) and (c) below.

(a) Since \( d_i' \geq_i d_i \) implies \( \phi(d_i') \rightarrow \phi(d_i) \) (modulo renaming of parameters), we have that \( \phi(d_i') \land c \rightarrow \phi(d_i) \land c \). By validity of \( \vdash d_1 \vdash c \rightarrow d_2 \), we conclude that \( \forall v(d_1) \exists v(d_2) \forall v. \phi(d_i') \land c \rightarrow \phi(d_i) \land c \) is true, i.e. \( \vdash d_i' \vdash c \rightarrow d_2 \) is valid.

(b) If \( R \models c' \rightarrow c \) holds, then \( \forall v(d_1') \exists v(d_2') \forall v. \phi(d_1') \land c \rightarrow \phi(d_2) \) implies \( \forall v(d_1') \exists v(d_2') \forall v. \phi(d_1') \land c' \rightarrow \phi(d_2') \), i.e. \( \vdash d_i' \vdash c' \rightarrow d_2' \) is valid.

(c) Since \( d_2 \geq_1 d_2' \) implies \( \phi(d_2) \rightarrow \phi(d_2') \) (modulo renaming of parameters), then \( \forall v(d_1') \exists v(d_2') \forall v. \phi(d_1') \land c \rightarrow \phi(d_2) \) implies \( \forall v(d_1') \exists v(d_2') \forall v. \phi(d_1') \land c' \rightarrow \phi(d_2') \), i.e. \( \vdash d_i' \vdash c' \rightarrow d_2' \) is valid. \( \square \)

We state a few intuitive properties of the \( \vdash \) relation.

**Lemma A.1.** Let \( d_2 \) be \( x_1 : \tau_1, \ldots, x_n : \tau_n \). The following are equivalent:

(i) \( \vdash d_1 \vdash c \rightarrow d_2 \) is valid;
(ii) \( \vdash d_1 \vdash c \rightarrow nf(d_2) \) is valid;
(iii) \( \vdash nf(d_1) \vdash c \rightarrow d_2 \) is valid;
(iv) for \( i = 1 \ldots n \), \( \vdash d_1 \vdash c \rightarrow x_i : \tau_i \) is valid.

**Proof.** (i) is equivalent to (ii) and to (iii) since in order to obtain the normal form, we have to go through a sequence of deletions (remove duplicate atd’s or that assign lower types than existing ones) and/or mergings (merge \( x : \sqcap \vdash x : \sqcup \) into \( x : \sqcup \)). Merging has no change on \( \phi() \). Removing duplicates or lower types yields an equivalent formula to be shown. For instance, \( \forall a. \exists x \geq a \forall x. x_1 \leq a \land x_1 = b \rightarrow x_1 \leq c \land x_1 = d \) holds iff \( \forall y \exists x. x_1 = b \rightarrow x_1 = d \). The if part follows by setting \( c = d \). The only-if part follows by setting \( a = b \).

(i) is equivalent to (iv) since, by distributivity laws, we can rewrite \( \forall v(d_1) \exists v(d_2) \forall v. \phi(d_1) \land c \rightarrow \phi(d_2) \) into \( \forall \tau_a \forall v(d_1) \exists v(d_2) \forall v. \phi(d_1) \land c \rightarrow \phi(d_2) \), which is our conclusion. \( \square \)

When showing validity of \( \vdash d_1 \vdash c \rightarrow d_2 \), atd’s in \( d_2 \) typing variables not in \( vars(c) \) can be dealt with by means of the \( \geq_1 \) relation.

**Lemma A.2.** Let \( \vdash d_1 \vdash c \rightarrow d_2 \) be a type assertion with \( c \) satisfiable. Called \( \nu_c = vars(c) \), we have that: \( \vdash d_1 \vdash c \rightarrow d_2 \) is valid iff (i) \( d_1|_{\nu_c} \vdash c \rightarrow d_2|_{\nu_c} \) is valid; and (ii) \( \vdash nf(d_1) \geq_1 nf(d_2|_{\nu(c)} \setminus \nu_c) \).
Proof. By Lemma A.1(i-iii), we can assume that \( d_1 \) and \( d_2 \) are in normal form. Also, by Lemma A.1 (iv), we have to show that for every \( x : \tau \) in \( d_2 \), \( d_1 \vdash c \rightarrow x : \tau \) is valid iff (i) \( d_1|_{\vars(c)} \vdash c \rightarrow x : \tau \) is valid if \( x \in \vars(c) \); and (ii) \( \nf(d_1) \geq t, \nf(x : \tau) \) if \( x \notin \vars(c) \). The result is immediate for \( \tau = \ast \), so we can assume \( \tau \neq \ast \) in the following. Let us distinguish two cases.

(i) \( x \in \vars(c) \).

Let \( v = \vars(d_1) \cup \vars(c) \). By definition, \( d_1 \vdash c \rightarrow x : \tau \) is valid iff:

\[
\forall v(\phi(d_1) \land c) \rightarrow \phi(x : \tau).
\]  

(3)

Called \( d_1^{\ast}_{\vars(c)} = d_1|_{\vars(d_1) \setminus \vars(c)} \), it can be rewritten as:

\[
\forall v(\phi(d_1^{\ast}_{\vars(c)}) \land c) \rightarrow \phi(x : \tau) \lor (\forall v \setminus c. \neg \phi(d_1^{\ast}_{\vars(c)})).
\]

The formula \( \forall v \setminus c. \neg \phi(d_1^{\ast}_{\vars(c)}) \) is false for any instance of \( v(d_1) \). A solution for \( \phi(d_1^{\ast}_{\vars(c)}) \) can be found by setting variables to the values of the parameters they are constrained by. As an example, for \( a \leq x \leq a + 2 \), set \( x = a \). Since \( d_1 \) is in normal form, this definition is well-formed. Therefore, (3) reduces to \( \forall v(d_1) \exists v(x : \tau) \forall v. \phi(d_1^{\ast}_{\vars(c)} \cup c) \rightarrow \phi(x : \tau) \). Since universal quantification over \( v(d_1^{\ast}_{\vars(c)}) \) has no effect, this is equivalent to the desired conclusion: \( d_1|_{\vars(c)} \vdash c \rightarrow x : \tau \) is valid.

(ii) \( x \notin \vars(c) \).

Let \( v = \vars(d_1) \cup \vars(c) \cup \{x\} \). By definition, \( d_1 \vdash c \rightarrow x : \tau \) is valid iff (3) holds. Called \( d_1^{\ast} = d_1|_{\vars(x)} \) and \( d_1^{\ast}_{\setminus x} = d_1|_{\setminus \{x\}} \), (3) can be rewritten as:

\[
\forall v(\phi(d_1^{\ast}_{\setminus x}) \land c) \rightarrow \phi(x : \tau) \lor (\forall v \setminus \{x\}. \neg \phi(d_1^{\ast}_{\setminus x} \land c)).
\]

since \( \vars(d_1^{\ast}) \cup \vars(x : \tau) \subseteq \{x\} \) and \( \vars(d_1^{\ast}_{\setminus x}) \cup \vars(c) \subseteq v \setminus \{x\} \). Moreover, by factoring out universal quantification over \( v(d) \), it can be further rewritten as:

\[
(\forall v(d_1^{\ast}) \exists v(x : \tau) \forall x. \phi(d_1^{\ast}) \rightarrow \phi(x : \tau)) \lor (\forall v(d_1^{\ast}_{\setminus x}) \forall \{x\}. \neg \phi(d_1^{\ast}_{\setminus x} \land c)).
\]

The formula \( \forall v(d_1^{\ast}) \forall \{x\}. \neg \phi(d_1^{\ast}_{\setminus x} \land c) \) is false, since we \( \phi(d_1^{\ast}_{\setminus x} \land c) \) is satisfiable. In fact, a solution of \( c \) can be extended to a solution of \( \phi(d_1^{\ast}_{\setminus x} \land c) \) by setting the value of a parameter in \( v(d_1^{\ast}_{\setminus x}) \) to the value of the variable it constrains, e.g., by setting \( a \) to the value of \( y \) if \( a \leq y \leq a + 2 \) is in \( \phi(d_1^{\ast}_{\setminus x}) \).

Therefore, (3) reduces \( \forall v(d_1^{\ast}) \exists v(x : \tau) \forall x. \phi(d_1^{\ast}) \rightarrow \phi(x : \tau) \), namely to validity of \( d_1^{\ast} \vdash \true \rightarrow x : \tau \). We have then to show that \( d_1^{\ast} \vdash \true \rightarrow x : \tau \) is valid if \( d_1^{\ast} \geq t \), \( x \) holds, where, since \( d_1 \) is in normal form, \( d_1^{\ast} \) is either empty or \( x : \mu \), with \( \mu \neq \ast \). If \( d_1^{\ast} \) is empty, then neither \( d_1^{\ast} \vdash \true \rightarrow x : \tau \) can be valid nor \( d_1^{\ast} \geq t \), \( x \) holds, since \( \tau \neq \ast \). If \( d_1^{\ast} \) is \( x : \mu \), then \( d_1^{\ast} \geq t \), \( x \) iff \( \mu \geq t \tau \) which, by definition of \( \geq t \), holds iff \( x : \mu \vdash \true \rightarrow x : \tau \). \( \square \)

We are now in the position to show the other results of Section 2.1.

Lemma 2.11.

Proof. An immediate instance of Lemma A.2, by noting that \( \vars(c) = \emptyset \). \( \square \)

Lemma 2.12.

Proof. Let \( v = \vars(c) \cup \vars(d_1) \cup \vars(d_2) \cup \vars(d_3) \). Consider an instance \( u_1 \) of parameters in \( v(d_1) \). By validity of \( d_1 \vdash c \rightarrow d_2 \), there exists \( u_2 \) instance of parameters in \( v(d_2) \) such that \( \forall v. \phi(d_1)[v(d_1) / u_1] \land c \rightarrow \phi(d_2)[v(d_2) / u_2] \).
The only-if part is trivial. Concerning the if part, we observe that, given that set, it is included in the set of solutions of $\phi(d_1)[v(d_1)/u_1] \land c \rightarrow \phi(d_2)[v(d_2)/u_2]$ is true. By transitivity of implication, we conclude that:

$$\forall v. \phi(d_1)[v(d_1)/u_1] \land \phi(d_2)[v(d_2)/u_2] \land c \rightarrow \phi(d_3)[v(d_3)/u_3]$$

is true. By eliminating from the quantification over $v$ those variables not in $\text{vars}(c) \cup \text{vars}(d_1) \cup \text{vars}(d_3)$ and by reintroducing existential quantification over $v(d_3)$ and universal quantification over $v(d_1)$, we get the desired conclusion.

A.2 Section 2.2

**Lemma 2.14.**

**Proof.** (One solution) By Lemma A.1 (iv), we can restrict to the case $d_1 \vdash c \rightarrow x : \tau$. Let $V = \{r \in \mathcal{R} | (d_1 \vdash c \rightarrow x) : \text{valid} \}$.

If $V = \emptyset$, then by defining $\tau = \text{lub}\{\mu \in \text{BT}_0 | (d_1 \vdash c \rightarrow x) : \mu \text{ valid} \}$, we have that $d_2 = x : \tau$ satisfies the requirements of Definition 2.13. If $V \neq \emptyset$, we distinguish two cases based on $\tau = \inf V$, the infimum of $V$.

Assume first that $\tau \in V$. Then $d_1 \vdash c \rightarrow x : \tau$ is valid iff $\tau \in \text{BT}_0$ or $\tau = \Box r$, with $r \in V$ iff $\Box r \geq_1 \tau$ iff $x : \Box r \geq_1 x : \tau$. Thus, $d_2 = x : \Box r$ satisfies the conclusion.

Assume now that $\tau \notin V$. Consider an instance $u$ of parameters in $v(d_1)$. Since $(\phi(d_1) \land c)[v(d_1)/u]$ is a linear constraint, the set of its solutions $S_u$ is a closed set (actually, it is a polyhedron). For some $a$, $S_u$ is included in the set of solutions of $\phi(x : \Box r) = a \leq x \leq a + r$ for any $r \in V$, namely for any $r > \tau$. Since $S_u$ is a closed set, it is included in the set of solutions of $\phi(x : \Box r) = a \leq x \leq a + r$ as well. Since this holds for any $u$, we conclude that $d_1 \vdash c \rightarrow x : \Box r$ is valid, namely that $\tau \in V$, which is absurd.

(Only one normal form) Assume there exist two type declarations $d_2$ and $d_2'$ satisfying the property of completeness. Since $d_2 \geq_1 d_2'$, we have that $d_1 \vdash c \rightarrow d_2$ is valid. Thus, $d_2' \geq_1 d_2$ must hold. This implies $\text{nf}(d_2') \geq_1 \text{nf}(d_2)$.

Analogously, one shows $\text{nf}(d_2') \geq_1 \text{nf}(d_2')$. By definition of $\geq_1$, $x : \tau$ is in $\text{nf}(d_2)$ iff it is in $\text{nf}(d_2')$, i.e. $\text{nf}(d_2) = \text{nf}(d_2')$ (modulo reordering of $\text{atd}'s$).

**Lemma 2.17.**

**Proof.** (Output “valid”) By soundness of $\chi$, $d_1 \vdash c \rightarrow d$ is valid. By the monotonicity Lemma 2.8 and the test $d \geq_1 d_2$ at Step $I$, $d_1 \vdash c \rightarrow d_2$ is valid as well.

(Output “not valid”) By completeness of $\chi$ for $\mathcal{B}$, if $d_1 \vdash c \rightarrow d_2$ were valid, then the test $d \geq_1 d_2$ would have passed.

A.3 Section 3.2

**Lemma 3.4.**

**Proof.** The only-if part is trivial. Concerning the if part, we observe that, given $x \in \text{Sol}(A_3, v \leq b_3)$, we can choose parameter values that are exactly equal to the
value of variables they constraint in $\phi(d)$, e.g., when $a \leq x \leq a + 2$ is in $\phi(d)$ we fix $a$ to the value of $x$ in $x$. Since parameters in $v(x : \tau)$ are fresh variables, this definition is well-formed. □

We recall the following result, which relates the solutions of a linear programming problem over $Ax \leq b$ to the ones over its homogeneous version $Ax \leq 0$.

**THEOREM A.3.** Let $Sol(Ax \leq b)$ be a non-empty polyhedron. We have:

$$\max \{c^T x \mid Ax \leq b\} \in \mathcal{R} \iff \max \{c^T x \mid Ax \leq 0\} = 0$$

**PROOF.** See [Murty 1983, Corollary 3.1]. □

**LEMMA 3.5.**

Consider the if part. By Lemma 3.4, there exists some $u$ such that $Sol(P, u) \neq \emptyset$. By hypothesis, we have $max\{c^T v \mid v \in Sol(P, u)\} \in \mathcal{R}$. By Theorem A.3, we conclude $max\{c^T v \mid v \in Sol(H)\} = 0$. Consider now the only-if part. Let $u$ be such that $Sol(P, u) \neq \emptyset$. By Theorem A.3, $max\{c^T v \mid v \in Sol(H)\} = 0$ implies $max\{c^T v \mid v \in Sol(P, u)\} \in \mathcal{R}$. □

**THEOREM 3.7 (LPInfer - soundness and completeness).**

**PROOF.** (Soundness) Working on $n = nf(d_1)$ at Step 0 is correct by Lemma A.1. By the same lemma, each variable $x$ in $v$ can be considered separately.

**Step 2** is justified by Lemma 3.4.

If $x \notin v_c = vars(c)$ then, by Lemma A.2, validity of $n \vdash c \rightarrow x : \tau$ is equivalent to $nf(n) = n \geq, nf(x : \tau)$. This holds if either $x : \tau$ is in $n$, which justifies Step 3 (a), or $\tau = \ast$, which justifies Step 3 (b).

If $x \in v_c$ then validity of $n \vdash c \rightarrow x : \tau$ is equivalent to validity of $n_{v_c} \vdash c \rightarrow x : \tau$, i.e., by the definition of $d$ at Step 0, to validity of $d \vdash c \rightarrow x : \tau$. The outputs at Step 4 are then justified by Lemma 3.5.

(Completeness) By Lemma A.1 we can consider each variable $x$ in $v$ separately. Assume that $d_1 \vdash c \rightarrow x : \tau$ is valid. If $c$ is unsatisfiable, completeness follows by Lemma 3.4 and Step 2. If $x \notin v_c$, completeness follows since Step 3 in the soundness part hold as if and only-if by Lemma A.2. If $x \in v_c$ and $\tau \in BT_{\emptyset}$, completeness follows since Step 4 holds as if and only-if by Lemma 3.5. □

**LEMMA 3.9.**

**PROOF.** If $c^T R \leq 0$ then $c^T RA \leq 0$ for every $\lambda \geq 0$. Moreover, for $\lambda = 0$ we have $c^T RA = 0$. Summarizing, $max\{c^T x \mid Ax \leq 0\} = max\{c^T RA \mid \lambda \geq 0\} = 0$.

Conversely, if $c^T R \not\leq 0$ then there exists a column $R_j$ of $R$ such that $c^T R_j > 0$. By choosing $\lambda'_i = 0$ for $i \neq j$ and $\lambda'_j = 1$, for $x = RA'$ we get $c^T RA' = c^T R_j > 0$. Summarizing, $max\{c^T x \mid Ax \leq 0\} = max\{c^T RA \mid \lambda \geq 0\} \geq c^T RA' > 0$. □

**A.4 Section 3.3**

We recall that a polyhedron is a convex set, namely for $x, y$ in $Sol(Ax \leq b)$ and any $0 \leq \lambda \leq 1$ the vector $\lambda x + (1 - \lambda)y$ belongs $Sol(Ax \leq b)$ as well. Geometrically, this means that the segment from $x$ to $y$ is included in the polyhedron.
Theorem 3.12 [(Definiteness I)].

Proof. The if-part is immediate by Lemma 2.8 since $d \geq_1 d|_1$.

Consider now the only-if part. If $c$ is unsatisfiable, the conclusion readily follows.

Also, we can assume that $d$ is in normal form. In fact, let $d \vdash c \rightarrow x : !$ be valid. By Lemma A.1, $nf(d)|_1 \vdash c \rightarrow x : !$ is valid as well. If we could show the conclusion for normal forms, we get that $nf(d)|_1 \vdash c \rightarrow x : !$ is valid. Finally, since $nf(d)|_1 = d|_1$, we conclude that $d|_1 \vdash c \rightarrow x : !$ is valid.

Let $v = vars(c) \cup vars(d) \cup \{x\}$, and let $i$ such that $v_i = x$. Since the order of atd’s in $d$ is irrelevant for validity, we can assume that $d = d_1, d|_1$, with $d_1$ assigning no variable to the $!$ type. We reason by induction on the length of $d_1$. If $d_1$ is the empty sequence, the conclusion is immediate since $d = d|_1$. Assume now that $d_1 = y : \tau, d_2$, with $\tau \neq !$, and let $j$ such that $v_j = y$. We will show that $d_2, d|_1 \vdash c \rightarrow x : !$ is valid, and then, by inductive hypothesis, we have that $d|_1 \vdash c \rightarrow x : !$ is valid. Let us consider the possible cases for $\tau$.

$(\tau = \ast)$. This cannot occur, since $d$ is in normal form.

$(\tau = \exists)$. Assume, by absurd, that $d_2, d|_1 \vdash c \rightarrow x : !$ is not valid. Called $d = d_2, d|_1$, there exists an instance $u$ of parameters $a = v(d)$ of the parameterized system $\phi (d) \land c$ such that the instantiated system has at least two solution points $w$ and $z$ which differ on the value of $x$, i.e., in formulas:

$$A_c w \leq b_c, A \tau w \leq B \tau u \quad \text{and} \quad A_c z \leq b_c, A \tau z \leq B \tau u,$$

and $w_i \neq z_i$. Let $m = \max \{w_j, z_j\}$. We have: $A_c w \leq b_c, A \tau w \leq B \tau u, w_j \leq m$ and $A_c z \leq b_c, A \tau z \leq B \tau u, z_j \leq m$, where $w_j \leq m$ and $z_j \leq m$ correspond to the instantiation of $\phi (y ; \tau)$. Therefore, there exists a parameter instance $(u, m)$ in the parameter space augmented by one dimension (needed since $y$ is not typed in $d$ as $d$ is in normal form) such that $\phi (d) \land c$ admits two solutions with distinct values for $x$. Summarizing, $d \vdash c \rightarrow x : !$ is not valid, contrarily to the hypothesis.

$(\tau = \forall)$. Analogous to $\tau = \exists$, but considering $m = \min \{w_j, z_j\}$.

$(\tau = \sqcap)$. Analogous to $\tau = \exists$, but now consider both an upper bound $m_1 = \max \{w_j, z_j\}$ and a lower bound $m_2 = \min \{w_j, z_j\}$.

$(\tau = \sqcup, \text{with } s > 0)$. Analogous to $\tau = \sqcap$, under the assumption that $\text{abs}(w_j - z_j) \leq s$. Let us show that such an assumption holds. Let $w$ and $z$ be any two solutions such that $w_i \neq z_i$. If $\text{abs}(w_j - z_j) \leq s$, we are done. Otherwise, since polyhedra are convex sets, $z = \lambda w + (1 - \lambda) z$ is also a solution for every $0 \leq \lambda \leq 1$. In particular, for $\lambda_0 = 1 - s/\text{abs}(w_j - z_j)$, we have:

$$\text{abs}(w_j - z_j) = \text{abs}(1 - \lambda_0)(w_j - z_j) = (1 - \lambda_0)\text{abs}(w_j - z_j) = s.$$  

Moreover, $s > 0$ implies $\lambda_0 < 1$, and then $z_i = \lambda_0 w_i + (1 - \lambda_0) z_j \neq w_i$, since $w_i \neq z_i$. Summarizing, the assumptions that $w$ and $z$ are two solutions with $w_i \neq z_i$ and $\text{abs}(w_j - z_j) \leq s$ are satisfied. □

Lemma 3.14

Proof. Since $\phi (d|_1)$ is of the form $x = a$, the overall system $\phi (d|_1)$ is a linear system of equalities $x = a, A_c v = b_c$. By Gauss-Jordan elimination of variables in $v \setminus x$, the system can be transformed into the equivalent following form:

$$x = a, Iw = b' + A'z + B'x, 0 = b'' + B''x,$$

(4)
where $I$ is a diagonal matrix, $w$ and $z$ are a partition of variables in $v \setminus x$ with $z$
f ree to assume any value, and $0 = b''' + B'' x$ is the condition of satisfiability of the
system $(4)$. If we show the conclusion for $(4)$, then it holds for the system:

$$Iw = A'z + B'x, 0 = B''x,$$

since the proof obligations do not involve $a, b'$ nor $b'''$.

(Only-part). Assume $d|c ightarrow x : d$ valid. If $x$ is in $x$, we are done. $x$ cannot be
in $z$ otherwise it would be a variable free to assume any value, fixed any $u$ instance
of $a$ such that the system $(4)$ is satisfiable. Finally, let $x$ be $w_i$, for some $i$. Then
$x = b'_i + row(A', i)z + row(B', i)x$ is in $(4)$. If $r^2 = row(A', i) = 0$, we are done.
Otherwise, assuming $r_j \neq 0$ for some $j$, we can fix $z = 0$ except for $z_j$ which is free
to assume any value. This results in $x = b'_i + r_j z_j + row(B', i)x$, which leads to
infinitely many distinct solutions for $x$.

(If-part). For any $u$ instance of $a$ such that the system $(4)$ is satisfiable, every
variable in $x$ trivially assumes a single value. This property extends to variables
$w_i$ for which $row(A', i) = 0$, since $w_i$ is defined as:

$$w_i = b'_i + row(A', i)z + row(B', i)x$$

and $x = u$. □

A linear system $A^-x = b^-$, $A^+x \leq b^+$ as in Theorem 3.16 has a solution $x_0$ for
which $A^+x_0 < b^+$. $x_0$ is called an inner point.

**Lemma A.4.** Assume that $Sol(Ax \leq b) \neq \emptyset$, and let $A^-x = b^-$, $A^+x \leq b^+$
as in Theorem 3.16. There exists $x_0 \in Sol(Ax \leq b)$ such that $A^+x_0 < b^+$.
Moreover, for every $y_0$ such that $A^-y_0 = b^-$ there exists $0 < \lambda \leq 1$ such that for
every $0 < \lambda_0 \leq \lambda$, $z_0 = \lambda_0 y_0 + (1 - \lambda_0) x_0$ is in $Sol(Ax \leq b)$.

**Proof.** Assume that $A^+x \leq b^+$ consists of $c_i^T x \leq b_i$ for $i = 1, \ldots, n$. Let
$x_1, \ldots, x_n \in Sol(Ax \leq b)$ such that $c_i^T x_i < b_i$ for $i = 1, \ldots, n$. Let $x_0 = (x_1 + \ldots + x_n)/n$. We calculate $A^-x_0 = (A^-x_1 + \ldots + A^-x_n)/n = (nb^-)/n = b^-$. Moreover,
for $i = 1, \ldots, n$ we calculate $c_i^T x_0 = (c_i^T x_1 + \ldots + c_i^T x_n)/n \leq (c_i^T x_i + (n-1)b_i)/n < b_i$
since $c_i^T x_i < b_i$ and $c_i^T x_j \leq b_i$ for $j \neq i$.

Consider now the second part of the lemma. First, for any $0 \leq \lambda \leq 1$, we have
$A^-z_0 = \lambda A^-y_0 + (1 - \lambda) A^-x_0 = \lambda b^- + (1 - \lambda) b^- = b^-$. Let now fix:

$$\lambda = \min\{1 \cup \{(b_i - c_i^T x_0)/(c_i^T y_0 - c_i^T x_0) \mid i = 1, \ldots, n, c_i^T y_0 > c_i^T x_0\}\}.$$ 

We have $\lambda > 0$ since $c_i^T x_0 < b_i$ for $i = 1, \ldots, n$. Let us now calculate, for $i = 1, \ldots, n$:

- if $c_i^T y_0 \leq c_i^T x_0$, then $c_i^T z_0 = \lambda c_i^T y_0 + (1 - \lambda)c_i^T x_0 \leq \lambda c_i^T x_0 + (1 - \lambda)c_i^T x_0 = c_i^T x_0 < b_i$;
- if $c_i^T y_0 > c_i^T x_0$, then $c_i^T z_0 = \lambda c_i^T y_0 + (1 - \lambda)c_i^T x_0 = \lambda(c_i^T y_0 - c_i^T x_0) + c_i^T x_0 \leq (b_i - c_i^T x_0) + c_i^T x_0 = b_i$.

Summarizing, $z_0 \in Sol(A^-x = b^-, A^+x \leq b^+) = Sol(Ax \leq b)$. The same
conclusion readily follows for any $0 < \lambda_0 \leq \lambda$. □
LEMMA 3.18

PROOF. The if-part follows by the monotonicity Lemma 2.8, since \( R \models c \rightarrow \text{ie}(c) \). We show the only-if part by contraposition. Let \( A, v \leq b \) be the geometric representation of \( c \) and \( A^= v \leq b^- \) be the one of \( \text{ie}(c) \), where \( v = \text{vars}(c) \cup \text{vars}(d_i) \cup \{x\} \). Also, let \( i \) such that \( v_i = x \). Called \( x = \text{vars}(d_i) \), \( \phi(d_i) \) is of the form \( x = a \).

Assume that \( d_i \models \text{ie}(c) \rightarrow x \). ! is not valid. Since \( c \) is satisfiable, this implies that there exist \( u \) instance of \( a \), and \( w_0, w_1 \in \text{Sol}(x = u, A^= v \leq b^-) \) such that \( w_0 \neq w_1 \). Consider now the linear constraint \( c' = x = a \land c \) in the space of variables plus parameters. We have that \( \text{ie}(c') \) is \( x = a, A^= v \leq b^- \). Also, \( (w_0, u) \) and \( (w_1, u) \) belong to \( \text{Sol}(x = a, A^= v \leq b^-) \).

Let \( (v_0, u_0) \) be the inner point as from Lemma A.4 applied to \( c' \). Then there exists \( 0 < \lambda \leq 1 \) such that:

\[
\text{for every } 0 < \lambda_0 \leq \lambda, \left( \lambda_0 w_0 + (1 - \lambda_0) v_0 \right) \in \text{Sol}(c').
\]

Analogously, there exists \( 0 < \lambda' \leq 1 \) such that:

\[
\text{for every } 0 < \lambda_0' \leq \lambda', \left( \lambda_0' w_1 + (1 - \lambda_0') v_0 \right) \in \text{Sol}(c').
\]

For \( \lambda_1 = \text{min}(\lambda, \lambda') > 0 \), we then have:

\[
\left( \lambda_1 w_0 + (1 - \lambda_1) v_0 \right), \left( \lambda_1 w_1 + (1 - \lambda_1) v_0 \right) \in \text{Sol}(c').
\]

called \( u_1 = \lambda_1 u + (1 - \lambda_1) u_0 \), \( z_0 = \lambda_0 w_0 + (1 - \lambda_0) v_0 \) and \( z_1 = \lambda_0 w_1 + (1 - \lambda_0) v_0 \), we have that \( u_1 \) is an instance of parameters \( a \) such that \( z_0, z_1 \in \text{Sol}(x = u_1, A, v \leq b^-) \). Moreover, \( z_0 \neq z_1 \) since \( w_0 \neq w_1 \) and \( \lambda_1 > 0 \). Summarizing, \( d_i \models c \rightarrow x \) ! is not valid. \( \square \)

THEOREM 3.19 \([\text{IEInfer} - \text{soundness and completeness}]\)

PROOF. Working on \( d = d_i \) at Step 0 is sound and complete by Theorem 3.12. Also, by Lemma A.1, each variable \( x \) in \( v \) can be considered separately.

(Soundness) Step 2 is justified by Lemma 3.18.

Step 3 is justified by Soundness of \( \text{LPInfer} \).

Step 6 is justified by the if-part of Lemma 3.14.

(Completeness) Assume that \( d_i \models c \rightarrow x : \tau \) is valid.

If \( x \not\in v_c = \text{vars}(c) \) then the output of \( \text{LPInfer} \) is complete, since validity of \( d_i \models c \rightarrow x : \tau \) is equivalent to check \( n_f(d_i) \geq_t n_f(x : \tau) \), which is dealt with at Step 3 of \( \text{LPInfer} \), as shown in the proof of Theorem 3.7.

If \( \tau \not\square \) then the output of \( \text{LPInfer} \) is complete again: for \( \tau \not\square \) there cannot be any more general answer type; for \( \square >_t \tau \) there cannot be any more general answer, otherwise \( \text{LPInfer} \) would have answered \( \tau = \square \).

If \( x \in v_c \) and \( \tau = \square \) the test at Step 6 \((a, b)\) is complete by the only-if part of Lemma 3.14. \( \square \)

We conclude by showing how to compute the set of implicit equalities starting from the Minkowski’s form of a polyhedron.
Lemma 3.20

Proof. We have to show that: \( \{ c^T x \mid x = R\lambda + V\gamma, \lambda, \gamma \geq 0, \Sigma \gamma = 1^T \gamma = 1 \} = \{ b \} \) iff \( c^T R = 0 \) and \( c^T V = b1^T \). The if part easily follows since \( c^T R\lambda + c^T V\gamma = b1^T \gamma = b \). We show the only-if part by contraposition.

If \( c^T R \neq 0 \) then for some column \( R_i \), \( c^T R_i \neq 0 \). By fixing any \( \gamma = 0 \) except for \( \gamma_i = 1 \), and \( \lambda = 0 \) except for the \( i^{th} \) element, we have \( c^T x = c^T R\lambda + c^T V\gamma = c^T R_i \lambda_i + c^T V \), which can assume infinitely many values, since \( \lambda_i \) can be any non-negative number. Hence, \( c^T x \) cannot constantly be equal to \( b \).

If \( c^T R = 0 \) and \( c^T V \neq b1^T \), then for some column \( V_i \), \( c^T V_i \neq b \). By fixing \( \gamma = 0 \) except for \( \gamma_i = 1 \), we have \( c^T x = c^T R\lambda + c^T V\gamma = c^T V_i \neq b \).

\( \square \)

A.5 Section 3.4

Theorem 3.22 [(Definiteness II)].

Proof. The if-part is immediate by Lemma 2.8 since \( d \geq \{ \frac{\text{max}}{} \} \).

Consider the only-if part. If \( c \) is unsatisfiable, the conclusion readily follows. Also, we can assume that \( d \) is in normal form. In fact, let \( d \vdash c \rightarrow x : \Box, \) be valid. By Lemma A.1, \( nf(d) \vdash c \rightarrow x : \Box, \) is valid as well. If we could show the conclusion for normal forms, we get that \( nf(d)|_{\Gamma} \vdash c \rightarrow x : \Box, \) is valid. Finally, due to the structure of \( \mathcal{B} \), we observe that \( d|_{\Gamma} \geq \{ \frac{\text{max}}{} \} \) holds, and then, by Lemma 2.8 again, we conclude that \( d|_{\Gamma} \vdash c \rightarrow x : \Box, \) is valid.

Let \( v = \text{vars}(c) \cup \text{vars}(d) \cup \{ x \} \), and let \( j \) such that \( v_j = x \). Since the order of \( v \)'s in \( d \) is irrelevant for validity, we can assume that \( d = d_1, d|_{\Gamma} \), with \( d_1 \) assigning no variable to types in \( \mathcal{B} \). We reason by induction on the length of \( d_1 \).

When \( d_1 \) is the empty sequence, the conclusion is immediate since \( d = d|_{\Gamma} \). Assume now that \( d_1 = y : \tau, d_2 \), with \( \tau \notin \mathcal{B} \), and let \( j \) such that \( v_j = y \). We will show that \( d_2, d|_{\Gamma} \vdash c \rightarrow x : \Box, \) is valid, and then, by inductive hypothesis, we have that \( d|_{\Gamma} \vdash c \rightarrow x : \Box, \) is valid. Let us consider the possible cases for \( \tau \).

\((\tau = *)\). This cannot occur, since \( d \) is in normal form.

\((\tau = \cap)\). Assume, by absurd, that \( d_2, d|_{\Gamma} \vdash c \rightarrow x : \Box, \) is not valid. Called \( d = d_2, d|_{\Gamma} \), there exists an instance \( u \) of parameters \( a = v(d) \) of the parameterized system \( \phi(d) \land c \) such that the instantiated system has at least two solution points \( w \) and \( z \) which differ on the value of \( x \) by at least \( r \), i.e., in formulas:

\[
A_w w \leq b_w, A_\cap w \leq B_\cap u \quad \text{and} \quad A_z z \leq b_z, A_\cap z \leq B_\cap u,
\]

and \( \text{abs}(w_j - z_j) > r \). Let \( m = \max\{w_j, z_j\} \). We have: \( A_w w \leq b_w, A_\cap w \leq B_\cap u, w_j \leq m \) and \( A_z z \leq b_z, A_\cap z \leq B_\cap u, z_j \leq m, \) where \( w_j \leq m \) and \( z_j \leq m \) correspond to the instantiation of \( \phi(y : \cap) \). Therefore, there exist a parameter instance \( (u m) \) in the parameter space augmented by one dimension (needed since \( y \) is not typed in \( d \) as \( d \) is in normal form) such that \( \phi(d) \land c \) admits two solutions \( w \) and \( z \) with \( \text{abs}(w_j - z_j) > r \). Summarizing, \( d \vdash c \rightarrow x : \Box, \) is not valid, contrarily to the hypothesis.

\((\tau = \cup)\). Analogous to \( \tau = \cap \), but considering \( m = \min\{w_j, z_j\} \).

\((\tau = \emptyset)\). Analogous to \( \tau = \cap \), but now consider both an upper bound \( m_1 = \max\{w_j, z_j\} \) and a lower bound \( m_2 = \min\{w_j, z_j\} \). \( \square \)
The next results characterizes the $\simeq_r$ relation in terms of the Minkowski’s form of a polyhedron.

**Lemma 3.27.**

**Proof.** Let us set $c = c_1 - c_2$ and $\alpha = \alpha_1 - \alpha_2$. It is immediate to observe that $c^T x + \alpha \succeq_r c_2^T x + \alpha_2$ iff $c^T x + \alpha \succeq 0$.

Consider the if-part. Since $\text{Sol}(Ax \leq b)$ is its generating and vertex matrices $R$ and $V$ exist. For every $x_0$ such that $Ax_0 \leq b$ holds, there exists $\lambda, \mu \geq 0$ with $\Sigma \mu = 1$ such that $x_0 = R\lambda + V\mu$. We calculate: $\text{abs}(c^T x_0 + \alpha) = \text{abs}(c^T R\lambda + c^T V\mu + \alpha) = \text{abs}(c^T V\mu + \alpha 1^T\mu) = \text{abs}((c^T V + \alpha 1^T)\mu)$, since $c^T R = 0$ and $1^T\mu = \Sigma \mu = 1$. Since $1 \geq \mu$, we have $\text{abs}(c^T x_0 + \alpha) = \text{abs}((c^T V + \alpha 1^T)\mu) \leq \|c^T V + \alpha 1^T\|_\infty = r$. Moreover, by fixing $\mu = 0$ except for $\mu_i = 1$, where $i$ is such that $\text{abs}(c^T V_i + \alpha 1^T_i) = r$, we have an $x_0 = V\mu$. such that $\text{abs}(c^T x_0 + \alpha) = \text{abs}((c^T V + \alpha 1^T)\mu) = \text{abs}(c^T V_i + \alpha 1^T_i) = r$.

Consider now the only-if part. Let $V_k$ be any column of $V$. Hence, $V_k \in \text{Sol}(Ax \leq b)$ and then $\text{abs}(c^T V_k + \alpha) \leq r$.

If $c^T R \neq 0$ then there exists a column $R_j$ of $R$ such that $c^T R_j \neq 0$. By choosing $\lambda_i = 0$ for $i \neq j$, $\mu_i = 0$ for $i \neq k$ and $\mu_k = 1$, for $x_0 = R\lambda + V\mu$ we get $c^T x_0 + \alpha = c^T R\lambda + c^T V\mu + \alpha = c^T R_j \lambda_j + c^T V_k + \alpha \neq r$, since $c^T R_j \neq 0$ and $\lambda_j$ can be any positive real number.

If $c^T R = 0$ and $\|c^T V + \alpha 1^T\|_\infty = r$, we distinguish two cases. If for some $i$, $\text{abs}(c^T V_i + \alpha 1^T_i) > r$, then $c^T x + \alpha \neq r$ over $Ax \leq b$ since every vertex $V_i$ is in $\text{Sol}(Ax \leq b)$. If for all $i$, $\text{abs}(c^T V_i + \alpha 1^T_i) < r$, then there cannot exist $x_0 \in \text{Sol}(Ax \leq b)$ such that $\text{abs}(c^T x_0 + \alpha) = r$. In fact, since $c^T R = 0$, such an $x_0$ can be rewritten as $x_0 = V\mu$ for some $\mu \geq 0$ with $\Sigma \mu = 1$. Then, $\text{abs}(c^T x_0 + \alpha) = \text{abs}(c^T V\mu + \alpha) \leq \max\{\text{abs}(c^T V_i + \alpha 1^T_i)\} < r$. □

The next result states that the range width of a linear function over the solutions of a polytope (a polyhedron whose generating matrix is empty), can be computed as the maximum range width between two vertices.

**Lemma A.5.** Let $\alpha \geq 0$, and $V$ be a matrix of $k$ columns such that for $1 \leq m < n \leq k$, $\text{abs}(c^T V_m - c^T V_n) \leq \alpha$. Then for every $\mu' \geq 0$, $\Sigma \mu' = 1$, $\mu'' \geq 0$, $\Sigma \mu'' = 1$, we have $\text{abs}(c^T V(\mu' - \mu'')) \leq \alpha$.

**Proof.** Without any loss of generality, we can assume $\text{abs}(c^T V(\mu' - \mu'')) = c^T V(\mu' - \mu'')$ (otherwise, simply swap $\mu'$ and $\mu''$).

When $k = 1$, we have $\mu' = \mu'' = 1$ and then $\text{abs}(c^T V(\mu' - \mu'')) = 0 \leq \alpha$.

When $k > 1$, we reason by contraposition. Assume $c^T V(\mu' - \mu'') > \alpha$ for some $\mu', \mu''$. Let $m$ be such that $c^T V_m$ is maximum, and $n$ be such that $c^T V_n$ is minimum. We have $\alpha < c^T V(\mu' - \mu'') = c^T V\mu' - c^T V\mu'' \leq c^T V_m - c^T V_n \leq \text{abs}(c^T V_m - c^T V_n) = \text{abs}(c^T V_n - c^T V_m)$. This show our conclusion for some $m \leq n$. Moreover, we claim that $m \neq n$. Otherwise we have $c^T V_i = c^T V_j$ for every $1 \leq i, j \leq k$, and then we can use any other index pair $i, j$ such that $i < j$. □

**Lemma 3.28.**

**Proof.** For a parameter instance $u$ let us denote by $V^u$ the set of (instances of the) parameterized vertices whose domain includes $u$, namely: $V^u = \{v^u(i) \mid i =
1..k, C_i u \leq c_i \}. If V^u \neq \emptyset, let us denote by V^u the matrix whose columns are the vectors in V^u.

(If-part). We first show the for-all statement.

Let u be fixed. If S_u = \emptyset, we are done. Otherwise, by Theorem 3.24, V^u is not empty. Since c^T R = 0, we calculate:

\[
S_u = \{ c^T x \mid x = R \lambda + V^u \mu, \lambda \geq 0, \mu \geq 0, \Sigma \mu = 1 \}
= \{ z \mid z = c^T V^u \mu, \mu \geq 0, \Sigma \mu = 1 \}.
\]

Moreover, since u belongs to the domains of all the vertices in V^u, for every pair v^u(m), v^u(n) of them Sol(P_{m,n}) \neq \emptyset. By hypothesis, we then have:

\[
abs(c^T V^u(m) - c^T V^u(n)) \leq r. \tag{5}
\]

For x, y \in S_u, abs(x - y) can be rewritten as abs(c^T V^u(\mu_x - \mu_y)), for some \mu_x \geq 0, \Sigma \mu_x = 1, \mu_y \geq 0, \Sigma \mu_y = 1. By Lemma A.5, this and (5) imply abs(x - y) \leq r.

Next, let us show the existential statement. Notice that the hypotheses imply that r \geq 0.

If r = 0 the conclusion holds by considering any u such that S_u \neq \emptyset (at least one exists, since the parameterized polyhedron is assumed not to be empty). For any x \in S_u, we have abs(x - x) = 0.

If r > 0, by assumption there exists m, n such be that c^T V^u(m) \simeq c^T V^u(n) over P_{m,n}, with Sol(P_{m,n}) \neq \emptyset. By Definition 3.25, there exists u \in Sol(P_{m,n}) such that abs(c^T V^u(m) - c^T V^u(n)) = r. The conclusion then follows by noting that x = c^T V^u(m) and y = c^T V^u(n) belong to S_u since v^u(m) and v^u(n) are in V^u.

(Only-If part). We show the contrapositive of the three conclusions.

First, if c^T R \neq 0, consider any u in Sol(C_1 a \leq c_1). Notice that k \geq 1 since the parameterized polyhedron is assumed not to be empty, and Sol(C_1 a \leq c_1) \neq \emptyset by Theorem 3.24. We have:

\[
S_u = \{ c^T x \mid x = R \lambda + V^u \mu, \lambda \geq 0, \mu \geq 0, \Sigma \mu = 1 \}
= \{ z \mid z = c^T R \lambda + c^T V^u \mu, \mu \geq 0, \Sigma \mu = 1 \}.
\]

Let R_j be a column of R such that c^T R_j = \alpha \neq 0. By fixing \lambda_j = 0 for i \neq j, we get that c^T R \lambda = \alpha. By fixing \mu in any way such that \mu \geq 0, \Sigma \mu = 1, we also have c^T V^u \mu = \beta, for some \beta. Summarizing, S_u includes \alpha \lambda_j + \beta for every \lambda_j \geq 0. Hence, abs(x - y) cannot be bounded by any r for every x, y \in S_u.

Second, consider the case that there exist 1 \leq m < n \leq k such that Sol(P_{m,n}) \neq \emptyset and for every s \leq r, c^T V^u(m) \neq c^T V^u(n) over P_{m,n}. By Definition 3.25, this means that for every s \leq r: (a) for some u in Sol(P_{m,n}), abs(c^T V^u(m) - c^T V^u(n)) > s; or (b) for every u in Sol(P_{m,n}), abs(c^T V^u(m) - c^T V^u(n)) \neq s.

Let s' = max{abs(c^T V^u(m) - c^T V^u(n)) \mid u \in Sol(P_{m,n})}.

It turns out that s' \leq r does not hold, since neither (a) nor (b) are satisfiable for s = s' by definition of s'. Therefore s' = \infty or s' > r. This implies that there exists u in Sol(P_{m,n}) such that abs(c^T V^u(m) - c^T V^u(n)) > r. By fixing \mu_m = 0 for i \neq m and \mu_m = 1, we define x = V^u \mu = v^u(m). Analogously, by fixing \mu_0 = 0 for i \neq n and \mu_n = 1, we define y = V^u \mu = v^u(n). It turns out x = c^T x, y = c^T y belong to:

\[
\{ c^T x \mid x = V^u \mu, \mu \geq 0, \Sigma \mu = 1 \} = \{ z \mid z = c^T V^u \mu, \mu \geq 0, \Sigma \mu = 1 \} \subseteq S_u.
\]
and $\text{abs}(x - y) > r$.

Third, we have to show that if $r \neq \hat{r}$, where

$$\hat{r} = \max\{0\} \cup \{s \mid 1 \leq m < n \leq k, \text{Sol}(P_{m,n}) \neq \emptyset, c^T v^n(m) \succeq_s c^T v^n(n) \text{ over } P_{m,n}\},$$

then: (a) for some $u$, and some $x, y \in S_u$, $\text{abs}(x - y) > r$; or (b) for every $u$, and every $x, y \in S_u$, $\text{abs}(x - y) \neq r$. Without any loss of generality, we can assume that for $1 \leq m < n \leq k$, $\text{Sol}(P_{m,n}) = \emptyset$ or $c^T v^n(m) \succeq_s c^T v^n(n)$ over $P_{m,n}$ for some $s \leq r$ (otherwise, we can apply the previous part of the proof to obtain (a)). This trivially implies $\hat{r} \in R$ and $r \geq \hat{r}$. Therefore, we can assume $r > \hat{r}$. We will show (b) by absurd. Assume that (b) does not hold, namely there exist $u$ and $x, y \in S_u$, $\text{abs}(x - y) = r > \hat{r}$. Since we can assume $c^T R = 0$ (already shown), we calculate:

$$S_u = \{c^T x \mid x = R \lambda + V^n \mu, \lambda \geq 0, \mu \geq 0, \Sigma \mu = 1\} = \{z \mid z = c^T V^u \mu, \mu \geq 0, \Sigma \mu = 1\}.$$

Thus, $\text{abs}(x - y) > \hat{r}$ can be rewritten as $\text{abs}(c^T V^u(m_x - m_y)) > \hat{r}$, for some $\mu_x, \mu_y \geq 0, \Sigma \mu_x = 1, \mu_y \geq 0, \Sigma \mu_y = 1$. By Lemma A.5, there exist $m', n'$ such that $\text{abs}(c^T V^u_{m'} - c^T V^u_{n'}) > \hat{r}$. For some $m$, the column $V^u_m$ corresponds to an instance of a parameterized vertex $v^u(m)$, i.e., $V^u_m = v^u(m)$. Analogously, for some $n$, $V^u_n = v^u(n)$. Also, we can assume $m < n$ since we are considering absolute values.

Therefore, we can conclude that there exist $1 \leq m < n \leq k$ such that $\text{Sol}(P_{m,n}) \neq \emptyset$, since $u \in \text{Sol}(P_{m,n})$, and such that $\text{abs}(c^T v^u(m) - c^T v^u(n)) > \hat{r}$. This is impossible by definition of $\hat{r}$. \qed

**Theorem 3.30** [{\text{POLYInfer - soundness and completeness}}].

**Proof.** Working on $n = nf(d_1)$ at Step 0 is sound and complete by Lemma A.1; while working on $d = n_0$ is sound and complete by Theorem 3.22. Also, by Lemma A.1, each variable $x$ in $v$ can be considered separately. Finally, we recall that Lemma 3.28 provides necessary and sufficient conditions for computing the minimum $r$ such that $d \vdash c \rightarrow x : \square, r$ is valid.

*(Soundness)* Step 4 is justified by Soundness of LPInfer.

Step 5 is justified by the if-part of Lemma 3.28 instantiated by fixing $c_j = 0$ for $j \neq i$, and $c_i = 1$, so that $c^T v_c = x$. Notice that, since $r \geq 0$ by its definition, $\square, r$ is a type (in BT).

*(Completeness)* Assume that $d \vdash c \rightarrow x : \tau$ is valid.

If $x \notin v_c = \text{vars}(c)$ then the output of LPInfer is complete, since validity of $d \vdash c \rightarrow x : \tau$ is equivalent to validity of $d_1 \vdash c \rightarrow x : \tau$, which in turn is equivalent to checking $nf(d_1) \geq_t nf(x : \tau)$. This is dealt with at Step 3 of LPInfer, as shown in the proof of Theorem 3.7.

If $\tau \neq \square$ then the output of LPInfer is complete again: for $\tau = !$ there cannot be any more general answer type; for $\square >_t \tau$ there cannot be any more general answer, otherwise LPInfer would have answered $\tau = \square$.

If $x \in v_c$ and $\tau = \square$ the test at Step 5 (a,b) is complete by the only-if part of Lemma 3.28. \qed
A.6 Section 3.5

Next lemma states that if we are given a point \( x \) in an hyperplane and a point \( y \) outside it, the segment from \( x \) to \( y \) lies outside the hyperplane, except for \( x \).

**Lemma A.6.** Let \( c^T x = \alpha \) and \( c^T y \neq \alpha \). For every \( 0 \leq \lambda < 1 \), called \( z = \lambda x + (1 - \lambda)y \), we have \( c^T z \neq \alpha \).

**Proof.** We have \( c^T z > \alpha \) iff \( \lambda c^T x + (1 - \lambda)c^T y > \alpha \) iff \( \lambda\alpha + (1 - \lambda)c^T y > \alpha \) iff \( (1 - \lambda)c^T y > \alpha(1 - \lambda) \) and finally, since \( \lambda < 1 \), if \( c^T y > \alpha \). The case of \( c^T z < \alpha \) is analogous. \( \square \)

Let us provide a set oriented characterization of generalized constraints.

**Lemma A.7.** Let \( g = c \land \bigwedge_{i=1}^{m} e_i \neq \alpha_i \) be a generalized linear constraint. We have:

\[
\text{Sol}(g) = \bigcap_{i=1}^{m} \text{Sol}(c) \setminus \text{Sol}(e_i = \alpha_i).
\]

**Proof.**

\[
\text{Sol}(g) = \text{Sol}(\bigwedge_{i=1}^{m} (c \land e_i \neq \alpha_i)) = \bigcap_{i=1}^{m} \text{Sol}(c \land e_i \neq \alpha_i)
\]

\[
= \bigcap_{i=1}^{m} \text{Sol}(c) \cap \text{Sol}(e_i \neq \alpha_i) = \bigcap_{i=1}^{m} \text{Sol}(c) \setminus \text{Sol}(e_i = \alpha_i).
\]

\( \square \)

The next result can be interpreted geometrically as follows. Given a point \( x \) in the solution set of a generalized linear constraints \( c \land \bigwedge_{i=1}^{m} e_i \neq \alpha_i \), and a point \( y \) in the polyhedron \( \text{Sol}(c) \), the hyperplanes \( e_i = \alpha_i \) intersect the segment from \( x \) to \( y \) in finitely many points.

**Lemma A.8.** Let \( g = c \land \bigwedge_{i=1}^{m} e_i \neq \alpha_i \) be a generalized linear constraint. For every \( x \in \text{Sol}(g) \) and \( y \in \text{Sol}(c) \) there exists \( 0 < \lambda < 1 \) such that, for every \( 0 < \lambda_0 \leq \lambda \), \( \lambda_0 x + (1 - \lambda_0)y \in \text{Sol}(g) \).

**Proof.** The proof is by induction on \( m \). For \( m = 0 \) the result is immediate for any \( 0 < \lambda < 1 \) since \( \text{Sol}(g) = \text{Sol}(c) \) is a convex set. Consider the inductive step.

If for every \( 0 < \lambda_0 < 1 \), \( \lambda_0 x + (1 - \lambda_0)y \in \text{Sol}(g) \) then we are done with any \( 0 < \lambda < 1 \). Assume then some \( 0 < \lambda_1 < 1 \) such that \( y_1 = \lambda_1 x + (1 - \lambda_1)y \notin \text{Sol}(g) \). Let \( J \) be the set of all \( j \in \{1, \ldots, m\} \) such that \( y_j \in \text{Sol}(e_j = \alpha_j) \). By Lemma A.7, \( J \neq \emptyset \). We observe that \( y \notin \text{Sol}(e_j = \alpha_j) \) for every \( j \in J \). Otherwise, since \( x \) is a linear combination of \( y \) and \( y_1 \) (namely, \( x = y_1/\lambda_1 - (1 - \lambda_1)/\lambda_1 y \)), it would belong to some \( \text{Sol}(e_j = \alpha_j) \) with \( j \in J \). But this is impossible since \( x \in \text{Sol}(g) \) implies, by Lemma A.7, \( x \notin \text{Sol}(e_j = \alpha_j) \) for \( j = 1..m \).

By Lemma A.6 applied to \( y_1 \) and \( y \), we have that for every \( 0 \leq \lambda_0 < 1 \) and \( j \in J \), \( z = \lambda_0 y_1 + (1 - \lambda_0)y \notin \text{Sol}(e_j = \alpha_j) \), i.e., the segment from \( y_1 \) to \( y \) does not belong to \( \text{Sol}(e_j = \alpha_j) \), except for \( y_1 \).

By inductive hypothesis on \( g_1 = c \land \bigwedge_{i \in \{1,\ldots,m\}\setminus J} e_i \neq \alpha_i \), \( y_1 \) (which is in \( \text{Sol}(g_1) \)) and \( y \) (which is in \( \text{Sol}(c) \)), there exists \( 0 < \lambda_2 < 1 \) such that, for every
0 < λ₀ ≤ λ₂, z = λ₀y₁ + (1 − λ₀)y ∈ Sol(g₁). In addition, we have shown that 
z ∉ Sol(e_j = α_j), for j ∈ J. Therefore, by Lemma A.7, z ∈ Sol(g). The conclusion 
now follows by observing that z can be rewritten as λ₀x + (1 − λ₀)y for 0 < λ₀ ≤ λ,
where λ = λ₁λ₂. □ 

Since Sol(g) can be an open set, we write sup{cᵀx | x ∈ Sol(g)} to denote the 
supremum, instead of max{cᵀx | x ∈ Sol(g)} to denote the maximum.

**Lemma A.9.** Let g = c ∧ "\[ \lambda_{i=1}^{m} e_i \] \neq \alpha_i" be a satisfiable 
generalized linear constraint. We have: sup{cᵀx | x ∈ Sol(g)} ∈ R iff max{cᵀx | x ∈ Sol(c)} ∈ R.

**Proof.** The if part follows since ∅ ⊂ Sol(g) ⊆ Sol(c). Consider now the only-if 
part. Let α = sup{cᵀx | x ∈ Sol(g)} ∈ R. Since g is satisfiable, there exists x₀ ∈ 
Sol(g), and α ≥ cᵀx₀. Since Sol(g) ⊆ Sol(c), we have that {cᵀx | x ∈ Sol(c)} ̸= ∅. 
We claim that max{cᵀx | x ∈ Sol(c)} ≤ α, which yields the conclusion. Assume, 
by absurd, that there exists x₁ ∈ Sol(c) such that cᵀx₁ = β > α ≥ cᵀx₀. By 
Lemma A.8, there exists 0 < λ < 1 such that z = λx₀ + (1 − λ)x₁ ∈ Sol(g) for 
every 0 < λ₀ ≤ λ. For λ₀ = min{λ, (β/2 − α/2)/(cᵀx₀)}, we have: cᵀz = 
λ₀cᵀx₀ + (1 − λ₀)cᵀx₁ = λ₀(cᵀx₀ − β) + β ≥ −(β/2 − α/2) + β = (β + α)/2 > α. 
But this is impossible by definition of α. □

**Theorem 3.31.**

**Proof.** The if-part is immediate: since R ⊃ g → c, by reasoning as in the proof 
of Lemma 2.8, which does not rely on linearity of the underlying constraints, we 
have the conclusion. Consider now the only-if part. The proof is by induction on 
the length of d₂. When it is empty, there is nothing to be shown. Assume now 
d₂ = x : τ, d₃. By inductive hypothesis, we have d₁ ⪰ c → d₄, Thus we are left 
with showing d₁ ⪰ c → x : τ. Let us consider the possible values for type τ.

(x : ⋁) There is nothing to be shown.

(x : ⊢) Since g is satisfiable, g ∧ φ(d₁) is satisfiable as well in the space of 
variables plus parameters. Let [x₀ | u₀] ∈ Sol(g ∧ φ(d₁)). Since d₁ ⪰ g → x : ⊢ is 
valid, sup{x | x ∈ Sol(g ∧ φ(d₁)) ∧ v(d₁) = u₀}) ∈ R. By Lemma A.9, we have: 
max{x | x ∈ Sol(c ∧ φ(d₁)) ∧ v(d₁) = u₀}) ∈ R.

Since c contains no disequality, it can be rewritten in the space of variables as 
max{x | x ∈ Sol(c ∧ φ(d₁), u₀}) ∈ R. We are now in the hypotheses of 
Theorem A.3 to conclude that max{x | x ∈ Sol(H)} = 0, where H is the homogeneous 
version of c ∧ φ(d₁). By Lemma 3.5, we conclude that d₁ ⪰ c → x : ⊢.

(x : ⊥) Same reasoning as in (x : ⋁) but maximizing −x instead of x.

(x : □) It reduces to the cases (x : ⋁) and (x : ⊥).

(x : ⊥r) Since g is satisfiable, g ∧ φ(d₁) is satisfiable as well in the space of 
variables plus parameters. Let [x₀ | u₀] ∈ Sol(g ∧ φ(d₁)).

Consider now an instance u of parameters in v(d₁). In order to conclude 
d₁ ⪰ c → x : ⊥r, we have to show that for every y, z ∈ \{x | x ∈ Sol(c ∧ φ(d₁), u)\}, 
abs(y − z) ≤ r. By absurd, assume that, without any loss of generality, y − z > r for 
some y, z. Then, there exist [y | u], [z | u] ∈ Sol(c ∧ φ(d₁)) such that the x dimension 
values of y and z are y and z respectively. By Lemma A.8, there exist 0 < λ₀ and
0 < \lambda_z such that for every 0 < \lambda_y \leq \lambda_z and 0 < \lambda_z \leq \lambda_z:

\[
\begin{bmatrix}
\lambda_y x_0 + (1 - \lambda_y) y \\
\lambda_y u_0 + (1 - \lambda_y) u \\
\lambda_z z_0 + (1 - \lambda_z) z \\
\lambda_z u_0 + (1 - \lambda_z) u
\end{bmatrix} \in \text{Sol}(g \land \phi(d_1)).
\]

By choosing \lambda < \min\{\lambda_y, \lambda_z, 1 - r/(y - z)\}, and called:

\[y_1 = \lambda x_0 + (1 - \lambda) y, \quad z_1 = \lambda x_0 + (1 - \lambda) z, \quad u_1 = \lambda u_0 + (1 - \lambda) u,\]
we have: \{y_1 | u_1], [z_1 | u_1] \in \text{Sol}(g \land \phi(d_1)). This implies that:

\[\{x | x \in \text{Sol}(g \land \phi(d_1), u_1)\} \supseteq \{y_1, z_1\},\]

where, called \(x_0\) the value of the \(x\) dimension of \(x_0\), \(y_1 = \lambda x_0 + (1 - \lambda) y\) and \(z_1 = \lambda x_0 + (1 - \lambda) z\). We have that \(y_1 - z_1 = (1 - \lambda)(y - z) > r\) since \(\lambda < 1 - r/(y - z)\). But this is impossible. \(\square\)

A.7 Section 3.6

First, we introduce notation for denoting type variables occurring in an atd \(x : \square_e\), and for denoting the primitive constraint \(e \geq 0\) which is part of \(\phi(x : \square_e)\).

\textit{Definition A.10.} We define the set of type variables occurring in an atd as:

\(pvars(x : \square_e) = \text{vars}(e)\) and \(pvars(x : \tau) = \emptyset\) for \(\tau \in \mathcal{BT}_\Delta\).

Also, we define the constraint on parametric expressions in an atd as: \(\Phi(x : \square_e)\) is \(e \geq 0\), and \(\Phi(x : \tau)\) is \textit{true} for \(\tau \in \mathcal{BT}_\Delta\).

The notions readily extend to type declarations as follows:

\(pvars(d_1, \ldots, d_n) = \cup_{i=1..n} pvars(d_i)\)

\(\Phi(d_1, \ldots, d_n) = \land_{i=1..n} \Phi(d_i)\).

Let us show first two results dealing with local type variables.

\textbf{Lemma 3.36.}

\textbf{Proof.} Let us denote \(pv = pvars(d_1)\). Called \(s\) a local type variable in \(e\), and \(l\) the remaining local type variables (possibly none), \(e\) can be written as \(e = \alpha s + cl + cpv + r\), where \(\alpha \neq 0\) and \(r \in \mathcal{R}\). Validity of \(d_1 \vdash c \rightarrow x : \square_e\) consists of showing the formula (1) in Definition 2.4, which can be rewritten as follows:

\[\forall pv \exists a, l \forall v. (\phi(d_1) \land c) \rightarrow (a \leq x \leq a + e \land e \geq 0).\] (6)

If this is true, by defining \(b = a + e\) (all the variables in \(v(d_1)\) being fixed) the following holds:

\[\forall pv \exists v(d_1) \land pv \exists a, b \forall v. (\phi(d_1) \land c) \rightarrow (a \leq x \leq b),\]

namely \(d_1 \vdash c \rightarrow x : \square\) is valid. Conversely, if this is true, by defining \(l = 0, s = (b - a - c/l pv - r)/\alpha\), we have that \(e = b - a\) and then (6) holds. \(\square\)

\textbf{Lemma 3.38.}

\textbf{Proof.} Let us denote \(pv = pvars(d_1)\). Validity of \(d_1 \vdash c \rightarrow (d_2, d_3)\) consists of showing the formula (1) in Definition 2.4, which can be rewritten as follows:

\[\forall pv \exists v(d_1) \land pv \exists v(d_2) \land pv \exists v. (\phi(d_1) \land c) \rightarrow (\phi(d_2) \land \phi(d_3)).\]
No local type variable implies that an existentially quantified variable in $v(d_2) \setminus pqv$ appears either in $p(d_2)$ or in $p(d_3)$, but not in both. Therefore, the formula above can be rewritten as:

$$
\forall pqv \exists v(d_1) \setminus pqv \ (\exists v(d_1) \setminus pqv \cdot (\phi(d_1) \land c) \rightarrow \phi(d_2))
\land (\exists v(d_2) \setminus pqv \cdot (\phi(d_1) \land c) \rightarrow \phi(d_3)),
$$

which reduces to validity of both $d_1 \vdash c \rightarrow d_2$ and $d_1 \vdash c \rightarrow d_3$.

**Lemma 3.40.**

**Proof.** The only-if part is trivial. Concerning the if part, consider any solution $(v_0, u_0) \in Sol(A_dv \leq B_d) \neq \emptyset$. Let $u_0$ be the restriction of $a_0$ to variables in $pqv = pvars(d)$. Let $d'$ be obtained by replacing every $x : \Box x$ in $d$ by $x : \Box e[pqv/u_0]$.

Since $e[pqv/u_0] \geq 0$ is in $A_d v_0 \leq B_d u_0$, we have that $e[pqv/u_0]$ reduces to a non-negative real number. Thus, $d'$ is a type declaration in $BT$, and:

$$
\phi(e(d')) = \phi(e(d)|pv/u_0) \quad \forall(v') = v(d) \setminus pqv,
$$

modulo trivially true primitive constraints $e[pqv/u_0] \geq 0$. Consider now the parameterized linear system $P' = \phi(d') \land c$. By Lemma 3.4, $c$ satisfiable implies that there exists $u_1$ instance of parameters $v(d')$ such that $Sol(P', u_1) \neq \emptyset$. By observing that $P' = P[pqv/u_0]$, we conclude that $(u_0, u_1)$ is an instance of $v(d)$ such that $Sol(P', (u_0, u_1)) = Sol(P', u_1) \neq \emptyset$. □

Type assertions in the presence of parametric types can be instantiated by assigning real number values $u$ to type variables $pqv$. The instantiation is a type assertions in the type system $BT$ if all instantiated type expressions $e[pqv/u]$ satisfy the requirement $e[pqv/u] \geq 0$, namely if $\Box e[pqv/u] \in BT$. Next definition introduces such instances for type assertions without local variables.

**Definition A.11.** Let $d_1 \vdash c \rightarrow d_2$ be such that $pvars(d_2) \subseteq pvars(d_1)$.

A consistent instance of $pqv = pvars(d_1)$ is any $u \in R[pqv]$ such that $\Phi(d_1)[pv/u] \wedge \Phi(d_2)[pv/u]$ is true.

The requirement $pvars(d_2) \subseteq pvars(d_1)$ is equivalent to assume no local type variable in the type assertion. We are now in the position to restate validity of type assertions in the presence of parametric types in terms of validity of type assertions in the $BT$ type system and a further condition. Condition (ii) in the next lemma is needed to cover contrived type assertions, such as the one highlighted in Example 3.49, where for the type assertion $x : \Box x, y : \Box x \vdash z = 0 \rightarrow z : \Box x, r$, all consistent instances lead to valid type assertions, i.e., (i) holds; but the type assertion is not valid due to inconsistent instances such as $s = 0, r = 1$.

**Lemma A.12.** Let $d_1 \vdash c \rightarrow d_2$ be a type assertion such that $pvars(d_2) \subseteq pvars(d_1)$. $d_1 \vdash c \rightarrow d_2$ is valid iff called $pv = pvars(d_1)$:

(i). for every $u$ consistent instance of $pv$ we have that $d_1[pv/u] \vdash c \rightarrow d_2[pv/u]$ is valid in the $BT$ type system;

(ii). and, $Sol(e) = \emptyset$ or $\forall pv. \Phi(d_1) \rightarrow \Phi(d_2)$ is valid in the domain of the reals.
Proof. Validity of $d_1 \vdash c \rightarrow d_2$ consists of showing validity of the formula (1) in Definition 2.4, which can be rewritten as:

$$\forall p v \forall u (d_1) \backslash p v \exists v (d_2) \backslash p v \forall u . (\phi (d_1) \land c) \rightarrow \phi (d_2). \quad (7)$$

This holds iff for every $u \in \mathcal{R}[p v]$, the following holds:

$$\forall u (d_1) \backslash p v \exists v (d_2) \backslash p v \forall u . (\phi (d_1)[p v / u] \land c) \rightarrow \phi (d_2)[p v / u]. \quad (8)$$

(If-part). We show that (8) holds. When $u$ is a consistent instance, this follows from (i), since (8) is equivalent to the Definition 2.4 (1) formula of $d_1[p v / u] \vdash c \rightarrow d_2[p v / u]$, except for the fact that the $\Phi (d_1)$ and $\Phi (d_2)$ constraints appear in the formula. However, since $u$ is a consistent instance, they are trivially satisfied.

Assume now that $u$ is not a consistent instance.

If $Sol (c) = \emptyset$ then the conclusion readily follows. Also, it follows if $\Phi (d_1)[p v / u]$ is false, since it is a conjunct of $\phi (d_1)[p v / u]$. $\Phi (d_1)[p v / u]$ true and the hypothesis $\forall p v . \Phi (d_1) \rightarrow \Phi (d_2)$ imply that $\Phi (d_2)[p v / u]$ is true. However, this and $\Phi (d_1)[p v / u]$ true contradict the assumption that $u$ is not a consistent instance.

(Only-part). (i) follows from (8), which is equivalent to the Definition 2.4 (1) formula of $d_1[p v / u] \vdash c \rightarrow d_2[p v / u]$, except for the fact that the $\Phi (d_1)$ and $\Phi (d_2)$ constraints appear in the formula. However, since $u$ is a consistent instance, we have that they reduce to the $true$ constraint.

(ii) If $Sol (c) = \emptyset$ then (ii) readily follows. Assume then $Sol (c) \neq \emptyset$ and that for some $u_0$ instance of $p v$ we have that $\Phi (d_1)[p v / u_0] \rightarrow \Phi (d_2)[p v / u_0]$ is false, namely $\Phi (d_1)[p v / u_0] \land \neg \Phi (d_2)[p v / u_0]$ is true. Given $v_0 \in Sol (c)$, it can be extended to a solution of $\phi (d_1)[p v / u_0] \land c$ by setting parameters in $d_1 \vdash p v$ to the value of variables in $v_0$ they constraint. As an example, for $\phi (x : \square_a) = a \leq x \leq a + s \land s \geq 0$ we choose for $a$ the value of $x$ in $v_0$ (this choice satisfies $x \leq a + s$ since $s \geq 0$ is assumed by $\Phi (d_1)[p v / u_0]$). By (8), $\phi (d_1)[p v / u_0] \land c$ implies $\phi (d_2)[p v / u_0]$, and then $\Phi (d_2)[p v / u_0]$ which is part of it. This is absurd since $\neg \Phi (d_2)[p v / u_0]$ was assumed.

By using the characterization of validity of type assertions with parametric types in terms of validity of type assertions in $BT$, we can extend to parametric types results that hold in $BT$.

Theorem 3.43 [(Definiteness III)].

Proof. In the following we can ignore condition (ii) of Lemma A.12 since either it holds for both $d_1 \vdash c \rightarrow x : \square_e$ and $d_1[\emptyset] \vdash c \rightarrow x : \square_e$ or for none.

By Lemma A.12, $d_1 \vdash c \rightarrow x : \square_e$ is valid iff for every $u$ consistent instance of $p v = \text{params}(d_1)$, the type assertion $d[p v / u] \vdash c \rightarrow x : \square_e[p v / u]$ is valid. By Theorem 3.22, called $B' = BT \backslash BT_\square$, this holds iff $d[p v / u]_{\emptyset} \vdash c \rightarrow x : \square_e[p v / u]$ is valid. Since the restriction to $B'$ simply removes $\text{atd}$'s with types in $BT_\square$, we observe that $d[p v / u]_{\emptyset} = d[p v / u]$. Summarizing, $d_1[\emptyset] \vdash c \rightarrow x : \square_e$ is valid iff for every $u$ consistent instance of $p v$, the type assertion $d[p v / u] \vdash c \rightarrow x : \square_e[p v / u]$ is valid. By Lemma A.12, this is equivalent to validity of $d_1[\emptyset] \vdash c \rightarrow x : \square_e$. \qed
Lemma 3.46.

Proof. Let us set $c = c_1 - c_2$ and $\alpha = \alpha_1 - \alpha_2$. It is immediate to observe that $c^T x + \alpha \leq c_1^T x + \alpha_1$ iff $c^T x + \alpha \leq 0$.

(If-part). Since $\text{Sol}(Ax \leq b) \neq \emptyset$, its generating and vertex matrices $R$ and $V$ exist. For every $x_0$ such that $Ax_0 \leq b$ holds, there exists $\lambda, \mu \geq 0$ with $\Sigma \mu = 1^T \mu = 1$ such that $x_0 = \lambda x_0 + V \mu$. We distinguish two cases.

First, consider $\text{abs}(c^T x_0 + \alpha) = c^T x_0 + \alpha$. We have $c^T x_0 + \alpha = c^T R \lambda + c^T V \mu + 1^T \mu \alpha \leq c^T R \lambda + c^T V \mu + 1^T \mu r = c^T x_0 + r$ by exploiting the hypotheses $(c - c_2)^T R \leq 0$ and $(c - c_2)^T V + (\alpha - r) 1^T \leq 0$.

Second, consider $\text{abs}(c^T x_0 + \alpha) = -c^T x_0 - \alpha$. We have $-c^T x_0 - \alpha = -c^T R \lambda - c^T V \mu - 1^T \mu \alpha \leq -c^T R \lambda + c^T V \mu + 1^T \mu r = c^T x_0 + r$ by exploiting the hypotheses $(-c - c_2)^T R \leq 0$ and $(-c - c_2)^T V + (\alpha - r) 1^T \leq 0$.

(Only-if part) Let $V_k$ be any column of $V$.

If $(c - c_2)^T R \leq 0$ then there exists a column $R_j$ of $R$ such that $(c - c_2)^T R_j > 0$. By choosing $\lambda_j = 0$ for $i \neq j$, $\mu_i = 0$ for $i \neq k$ and $\mu_k = 1$, for $x_0 = \lambda R \lambda + V \mu$ we get $c^T x_0 + \alpha = c^T R \lambda + c^T V \mu + \alpha = c^T R_j \lambda_j + c^T V_k \alpha + \alpha \leq c^T R \lambda + c^T V \mu + \alpha = c^T x_0 + r$, since $(c - c_2)^T R_j > 0$ and $\lambda_j$ can be any positive real number. For some $\lambda_j$, we can then conclude $\text{abs}(c^T x_0 + \alpha) \geq c^T x_0 + \alpha > c^T x_0 + r$.

If $(c - c_2)^T R \geq 0$ then there exists a column $R_j$ of $R$ such that $(-c - c_2)^T R_j > 0$. By choosing $\lambda_j = 0$ for $i \neq j$, $\mu_i = 0$ for $i \neq k$ and $\mu_k = 1$, for $x_0 = \lambda R \lambda + V \mu$ we get $-c^T x_0 - \alpha = -c^T R \lambda - c^T V \mu - \alpha = -c^T R_j \lambda_j - c^T V_k \alpha - \alpha \leq c^T R \lambda + c^T V \mu + \alpha = c^T x_0 + r$, since $(-c - c_2)^T R_j > 0$ and $\lambda_j$ can be any positive real number. For some $\lambda_j$, we can then conclude $\text{abs}(c^T x_0 + \alpha) \geq -c^T x_0 - \alpha > c^T x_0 + r$.

Lemma 3.47.

Proof. For a parameter instance $u$ let us denote by $V^u$ the set of (instances of the) parameterized vertices whose domain includes $u$, namely: $V^u = \{V^n(i) \mid i = 1..k, C \mu \leq c_i \}$. If $V^u \neq \emptyset$, let us denote by $V^u$ the matrix whose columns are the vectors in $V^u$.

Consider the if-part. Let $u$ be fixed. If $S_u = \emptyset$, we are done. Otherwise, $V^u$ cannot be empty. Since $c^T R = 0$ (assumption (i)), we calculate:

$$S_u = \{ c^T x \mid x = R \lambda + V^u \mu, \lambda \geq 0, \mu \geq 0, \Sigma \mu = 1 \}
= \{ z \mid z = c^T V^u \mu, \mu \geq 0, \Sigma \mu = 1 \}.$$

Notice that for $x, y \in S_u$, $abs(x - y)$ can then be rewritten as $abs(c^T V^u(\mu_x - \mu_y))$, for some $\mu_x \geq 0, \Sigma \mu_x = 1, \mu_y \geq 0, \Sigma \mu_y = 1$. Moreover, by (ii), $c^T u + r \geq 0$. These last two statements and assumption (ii) allow for applying Lemma A.5 to conclude $abs(x - y) \leq c^T u + r$ for every $x, y \in S_u$. 


Consider now the only-if part. We show the contrapositive.

(i). If \( c^T R \neq 0 \), consider any \( u \) in \( \text{Sol}(C_1 a \leq c_1) \). Notice that \( k \geq 1 \) since the parameterized polyhedron is assumed not to be empty, and \( \text{Sol}(C_1 a \leq c_1) \neq \emptyset \) by Theorem 3.24. We have:

\[
S_u = \{ c^T x \mid x = R\lambda + V u, \lambda \geq 0, \mu \geq 0, \Sigma \mu = 1 \} = \{ z \mid z = c^T R\lambda + c^T V u, \lambda \geq 0, \mu \geq 0, \Sigma \mu = 1 \}.
\]

Let \( R_j \) be a column of \( R \) such that \( c^T R_j = \alpha \neq 0 \). By fixing \( \lambda_i = 0 \) for \( i \neq j \), we get that \( c^T R\lambda = \alpha \lambda_j \). By fixing \( \mu \) in any way such that \( \mu \geq 0, \Sigma \mu = 1 \), we also have \( c^T V u = \beta \), for some \( \beta \). Summarizing, \( S_u \) includes \( \alpha \lambda_j + \beta \) for every \( \lambda_j \geq 0 \).

Hence, \( abs(x - y) \) cannot be bounded by \( c^T u + r \) for every \( x, y \in S_u \).

(ii). Consider a pair \( 1 \leq m < n \leq k \) such that \( c^T V u^m(n) \neq 0 \). By fixing \( \mu_i = 0 \) for \( i \neq m \) and \( \mu_n = 1 \), we define \( x = V u^m(n) \). Analogously, by fixing \( \mu_i = 0 \) for \( i \neq n \) and \( \mu_n = 1 \), we define \( y = V u^m(n) \). It turns out \( x \neq c^T x, y = c^T y \) belong to:

\[
\{ c^T x \mid x = V u^m(n), \mu \geq 0, \Sigma \mu = 1 \} = \{ z \mid z = c^T V u^m(n), \mu \geq 0, \Sigma \mu = 1 \} \subseteq S_u,
\]

and \( abs(x - y) > c^T u + r \).

(iii) By Theorem 3.24, \( \text{Sol}(C_1 a \leq c_i) \neq \emptyset \) for \( i = 1..k \). Therefore, the negation of condition (iii) is equivalent to assume that for some \( i \) in \( 1..k \), there exists \( u \in \text{Sol}(C_1 a \leq c_i) \) such that \( c^T u + r < 0 \). By Theorem 3.24 again, \( S_u \neq \emptyset \) at least \( V u^m(i) \) belongs to it. Therefore, for any \( x \in S_u \), we have \( abs(x-x) > 0 > c^T u + r \). \( \square \)

**Lemma 3.48.**

**Proof.** \( x_0 \in \text{Sol}(A x \leq b) \) iff \( x_0 = R\lambda + V\mu \) for some \( \lambda, \mu \geq 0 \) and \( \Sigma \mu = 1 \). Then: \( c^T x_0 + r = c^T R\lambda + c^T V\mu + r1^T \mu = c^T R\lambda + (c^T V + r1^T)\mu \). The if part is immediate, namely \( c^T x_0 + r \geq 0 \) since \( c^T R\lambda \), \( (c^T V + r1^T)\mu \) and \( \mu \) contains only non-negative elements. The only-if part follows by contraposition. If for some \( x_0 \), we have \( c^T x_0 + r < 0 \) then \( c^T R\lambda \) or \( (c^T V + r1^T)\mu \) is a negative number. Since \( \lambda, \mu \geq 0 \), this implies that \( c^T R \leq 0 \) or \( (c^T V + r1^T) \geq 0 \). \( \square \)

**Theorem 3.50 ([ParCheck - soundness and completeness]).**

**Proof.** Working on \( d = d_1 \| g \) is sound and complete by Theorem 3.43.

(Soundness) Step 2 is justified by the if-part of Lemma 3.40.

Step 4 is justified by the if-part of Lemma 3.47 instantiated by fixing \( c_j = 0 \) for \( j \neq i \), and \( c_i = 1 \), so that \( c^T V = x \). Also, since there is no local type variable in \( x : \Box e \), \( e \) can be written as a linear combination of type variables in \( d_1 \). Since \( pears(d_1) \subseteq v(d_1) = a \), the assumption of Lemma 3.47 that \( c \) is of the form \( c^T a + r \) is satisfied.

(Completeness) Assume that \( d \vdash c : \Box e \) is valid, and consider the parameterized system \( P = \phi(d) \land c \).

Unsatisfiability of \( P \) is checked by Step 2, using the conditions of Lemma 3.40.

Step 4 deals with the case that \( P \) is satisfiable by checking the conditions of Lemma 3.47. \( \square \)
Lemma 3.51.

**Proof.** Let \( \mathcal{P} \) be the parameterized linear system \( \phi(d_1) \land c \). Since \( \phi(d_1) \) is satisfiable, by Lemma 3.40, \( \mathcal{P} \) is satisfiable iff \( c \) is satisfiable. Therefore, for \( \mathcal{P} \) unsatisfiable the conclusion holds since both \( d_1 \vdash c \rightarrow \tau \) and \( d'_1 \vdash c \rightarrow \tau \) are valid. Assume now \( \mathcal{P} \) satisfiable. Let \( \mathcal{P}' \) be the parameterized linear system \( \phi(d'_1) \land c \). By construction of \( d'_1 \), the homogeneous versions of \( \mathcal{P} \) and \( \mathcal{P}' \) coincide. For instance, the homogeneous version of \( \phi(x:2e) = a \leq x \leq a + e \land 0 \leq e \) is \( x = 0 \land 0 \leq 0 \), which (up to the true \( 0 \leq 0 \)) coincides with the homogeneous version of \( \phi(x:) = a \leq x \leq b \).

By Lemma 3.5: \( d'_1 \vdash c \rightarrow x: \top \) is valid iff \( \max\{v_i | v_i \in \text{Sol}(H')\} = 0 \), where \( v_i = x \) and \( H' \) is the homogeneous version of \( \mathcal{P}' \). Since \( \mathcal{P} \) is satisfiable, an immediate extension of Lemma 3.5 allows for concluding: \( d_1 \vdash c \rightarrow x: \top \) is valid iff \( \max\{v_i | v_i \in \text{Sol}(H)\} = 0 \), where \( H \) is the homogeneous version of \( \mathcal{P} \). Our conclusion then follows since \( H = H' \).

A.8 Section 3.7

**Theorem 3.52.**

**Proof.** By Theorem 3.19 and Lemma 2.17, CHECK(IEInfer) is a decision procedure, complete for \( BT \). Its complexity is polynomial by observing that:

—LPInfer has polynomial time complexity when adopting a polynomial time algorithm for linear programming [Khachiyan 1979];

—computing the set of implicit equalities for a system of \( n \) inequalities can be done by solving at most \( n \) linear programming problems [Greenberg 1996, Corollary 7.4], which, again, requires polynomial time;

—Gaussian elimination has polynomial time complexity [Schrijver 1986].

A.9 Section 4.2

The operational semantics of CLP(\( R \)) programs is specified in terms of derivations from states to states [Jaffar and Maher 1994; Jaffar et al. 1998]. Here, we consider derivations via the leftmost selection rule. A state \( \langle \leftarrow c_1, A_1, \ldots, A_n \parallel c' \rangle \) is reduced to another state, called a resolvent, as follows:

\( \mathcal{R}1. \) If a linear constraint \( c \) appears at the left of the query, and \( c \land c' \) is satisfiable, the resolvent is \( \langle \leftarrow A_1, \ldots, A_n \parallel c' \land c \rangle \).

\( \mathcal{R}2. \) If an atom \( A_1 = p(x_1, \ldots, x_h) \) appears at the left of the query, and \( p(y_1, \ldots, y_h) \leftarrow c, B_1, \ldots, B_k \) is a renamed apart clause from \( P \), then the resolvent is:

\[ \langle \leftarrow c, B_1, \ldots, B_k, A_2, \ldots, A_n \parallel c' \land \bigwedge_{i=1..h} x_i = y_i \rangle. \]

A derivation from a state \( S \) is a (finite or infinite) maximal sequence of states \( S_0 = S, S_1, \ldots, S_n, \ldots \) such that there is a reduction from \( S_i \) to \( S_{i+1} \), for \( i \geq 0 \). A derivation for a query \( Q \) is a derivation from \( \langle Q \parallel \text{true} \rangle \). The last state of a finite derivation is of the form \( \langle Q \parallel c \rangle \). If \( Q \) is not empty, the derivation is failed. Otherwise, it is successful, or a refutation.
Theorem 4.5.

Proof. (Persistency). It is a special case of the more general Theorem 4.17, which is shown in Appendix A.11.

(Call patterns). The state \( \langle \leftarrow q(x : \tau \times \mu), R \| c \rangle \) is well-moded by persistency. From the definition of well-moded states \((i = 1 \text{ and } c = \text{true})\), we have that \( \vdash c' \rightarrow x : \tau \) is valid.

(Answers). Consider the program \( P' \) defined as \( P \) plus the unit clause:

\[
p(x_1 : \mu_1 \times \star, \ldots, x_n : \mu_n \times \star) \leftarrow \text{true}.
\]

where \( p \) is a fresh predicate symbol, and the query \( Q' = Q p(x_1, \ldots, x_n) \). \( P' \) and \( Q' \) are well-moded since \( P \) and \( Q \) are well-moded. Under our hypotheses, there exists a derivation from the query \( Q' \) to the state \( \langle \leftarrow p(x_1, \ldots, x_n) \| c' \rangle \). By the call pattern conclusion, we obtain \( \vdash c' \rightarrow x_1 : \mu_1, \ldots, x_n : \mu_n \).

\( \square \)

A.10 Section 4.3

We recall that a term substitution \( \theta \) is represented by a finite set \( \{x_1/t_1, \ldots, x_n/t_n\} \) with \( x_1, \ldots, x_n \) distinct variables, and \( t_1, \ldots, t_n \) terms, and \( n \geq 0 \). Also, we can assume \( t_i \neq x_i \). \( \text{dom}(\theta) = \{x_1, \ldots, x_n\} \) is called the domain of variables in \( \theta \).

Lemma 4.11.

Proof. By Lemma A.1, we have to show that \( d_i \theta \vdash c \rightarrow (x : \tau) \theta \) is valid for every \( x : \tau \) in \( d_2 \). Also, we can assume \( \tau \neq \star \), otherwise the conclusion is trivial. Moreover, notice that the assumption \( c = c' \) implies that \( \text{vars}(c) \cap \text{dom}(\theta) = \emptyset \).

First, consider the case \( x/t \in \theta \) for some \( t \). Thus, \( x \not\in \text{vars}(c) \). By Lemma A.2 and \( \tau \neq \star \), \( d_1 \vdash c \rightarrow x : \tau \) valid implies that there exists \( x : \mu \) in \( \text{nf}(d_1) \) such that \( \mu \geq \tau \).

By Definition 4.9, for every \( y \in \text{vars}(t) \), \( y : \mu \) is in \( \text{nf}(d_1) \). By Lemma 2.11, this \( \mu \geq \tau \) imply that \( d_1 \theta \vdash \text{true} \rightarrow y : \tau \) is valid. By monotonicity, \( d_1 \theta \vdash c \rightarrow y : \tau \) is valid for every \( y \in \text{vars}(t) \). By Definition 4.9 and Lemma A.1, this is equivalent to validity of \( d_1 \theta \vdash c \rightarrow (x : \tau) \theta \).

Consider now the case \( x \not\in \text{dom}(\theta) \). Thus, \( (x : \tau) \theta = x : \tau \).

If \( x \not\in \text{vars}(c) \), by Lemma A.2 and \( \tau \neq \star \), \( d_1 \vdash c \rightarrow x : \tau \) valid implies that there exists \( x : \mu \) in \( \text{nf}(d_1) \) such that \( \mu \geq \tau \).

Since \( x \not\in \text{dom}(\theta) \), \( x : \mu \) is in \( \text{nf}(d_1) \) as well. Again by Lemma A.2, we conclude that \( d_1 \theta \vdash c \rightarrow x : \tau \) valid.

If \( x \in \text{vars}(c) \), by Lemma A.2, \( d_1 \vdash c \rightarrow x : \tau \) valid implies \( d \vdash c \rightarrow x : \tau \) valid, where \( d = d_1|_{\text{vars}(c)} \). Since \( \text{vars}(c) \cap \text{dom}(\theta) = \emptyset \), we have \( d|_{\theta} = d \), and then \( d \theta \vdash c \rightarrow x : \tau \) is valid. By monotonicity, from \( d_1 \theta \geq \tau \), we conclude that \( d_1 \theta \vdash c \rightarrow x : \tau \) is valid. \( \square \)

A.11 Section 4.4

Theorem 4.17 [(Persistency)].

Proof. Let \( S = \langle \leftarrow [c], A_1, \ldots, A_n \| c' \rangle \) be well-moded using type substitutions \( \vartheta_1, \ldots, \vartheta_n \). For \( A_i = p(x : \tau \times \mu), i = 1..n \), we denote \( \text{pre}(A_i) = x : \vartheta_i^x(\tau) \) and \( \text{post}(A_i) = x : \vartheta_i^y(\mu) \). We will show that every resolvent of \( S \) and a well-moded clause is well-moded. The final result follows since the initial state \( Q || \text{true} \) is well-moded by hypothesis. If rule \( R_1 \) is applied, the result is trivial since, by definition, \( S \) is well-moded iff \( \langle \leftarrow A_1, \ldots, A_n \| c' \land c \rangle \) is well-moded.

\( \square \)
Consider now rule $\mathcal{R}_2$ and let $A_1 = p(x_1, \ldots, x_h)$ and let:

$$\langle \leftarrow c, B_1, \ldots, B_k, A_2, \ldots, A_n \| c' \land \bigwedge_{i=1..h} x_i = y_i \rangle$$

be the resolvent obtained by a renamed apart clause $H \leftarrow c, B_1, \ldots, B_k$, where $H = p(y_1, \ldots, y_h)$. It is readily checked that renaming apart a well-modeled clause yields a well-modeled clause as well. Let $\vartheta_1, \ldots, \vartheta_k$ be the type substitutions as in Definition 4.16. For $B_i = p(x: \tau \times \mu), i = 1..k$, we denote $\text{pre}(B_i) = x: \vartheta_i^1(\tau)$ and $\text{post}(B_i) = x: \vartheta_i^\circ(\mu)$. Also, for $H = p(y: \tau \times \mu)$, we define $\text{pre}(H) = y: \tau$ and $\text{post}(H) = y: \mu$.

We will show that the resolvent is well-modeled by using the type substitutions $\vartheta_1, \ldots, \vartheta_{k+n-1}$ defined as follows: $\vartheta_i = \vartheta_i^1 \circ \vartheta_i^\circ$ for $i = 1..k$; $\vartheta_{k+i-1} = \vartheta_i^\circ$ for $i = 2..n$.

By well-moding of $S, \vdash c' \rightarrow \text{pre}(A_1)$ is valid and has no local type variable. Since $c'$ is satisfiable (it is a simple induction on the length of derivations), this implies that $\text{pre}(A_1)$ is a type declaration in $\mathcal{B}T$, namely $\vartheta_i^1(e) \geq 0$ for every $\Box_i$ in $\text{pre}(A_1)$. Called $c_1 = c' \land \bigwedge_{i=1..k} x_i = y_i$, this implies that $\vdash c_1 \rightarrow \overline{\text{pre}(H)}$ is valid, where, for $H = p(y: \tau \times \mu)$, $\overline{\text{pre}(H)} = y: \vartheta_i^1(\tau)$ is a type declaration in $\mathcal{B}T$.

Let us split the rest of the proof in two steps (a) and (b).

(a) For $i = 1..k$, from well-moding of the (renamed-apart) clause, we have that:

$$\text{pre}(H), \text{post}(B_1), \ldots, \text{post}(B_{i-1}) \vdash c \rightarrow \text{pre}(B_i)$$

is valid and has no local type variable. Since $\mathcal{R} \models c_1 \land c \rightarrow c$, this implies that:

$$\text{pre}(H), \text{post}(B_1), \ldots, \text{post}(B_{i-1}) \vdash c_1 \land c \rightarrow \text{pre}(B_i)$$  \hspace{1cm} (9)

is valid and has no local type variable. By recalling that no local to output type variable is admitted in parametric modes (see Definition 4.12), a simple induction on $i$ reveals that all type variables in (9) are included in $\text{pre}(H)$. We instantiate all of them by $\vartheta_i^\circ$ as follows: for $i = 1..k$ and $B_i = p(x: \tau \times \mu)$, we denote $\overline{\text{pre}(B_i)} = x: \vartheta_i^1(\vartheta_i^\circ(\tau))$ and $\overline{\text{post}(B_i)} = x: \vartheta_i^1(\vartheta_i^\circ(\mu))$. The following is an instance of (9):

$$\overline{\text{pre}(H)}, \overline{\text{post}(B_1)}, \ldots, \overline{\text{post}(B_{i-1})} \vdash c_1 \land c \rightarrow \overline{\text{pre}(B_i)}$$  \hspace{1cm} (10)

and it is valid. This comes directly from Definition 2.4 of validity of type assertions. Moreover, since $\overline{\text{pre}(H)}$ is in $\mathcal{B}T$, and (10) is an instance of (9), there is no type variable at all in (10). Assume for the moment that it is a type assertion in $\mathcal{B}T$. By monotonicity in $\mathcal{B}T$, $\vdash c_1 \rightarrow \overline{\text{pre}(H)}$ valid implies that:

$$\overline{\text{post}(B_1)}, \ldots, \overline{\text{post}(B_{i-1})} \vdash c_1 \land c \rightarrow \overline{\text{pre}(H)}$$  \hspace{1cm} (11)

is valid, and in $\mathcal{B}T$. By the transitivity Lemma 2.12, (10) and (11) imply that:

$$\overline{\text{post}(B_1)}, \ldots, \overline{\text{post}(B_{i-1})} \vdash c_1 \land c \rightarrow \overline{\text{pre}(B_i)}$$  \hspace{1cm} (12)

is valid. This concludes the proof, since this is the proof obligation of well-moding for type substitutions $\vartheta_1, \ldots, \vartheta_{k+n-1}$ and $i = 1..k$.

Assume now that (10) is not a type assertion in $\mathcal{B}T$. If $c_1 \land c$ is not satisfiable, then validity of (12) is immediate. Let $c_1 \land c$ be satisfiable. Since (10) contains no type variable, it is not in $\mathcal{B}T$ only if for some type $\Box_i$ in $\overline{\text{post}(B_1)}, \ldots, \overline{\text{post}(B_{i-1})}$, $\overline{\text{pre}(B_i)}$ we have $e < 0$, namely (10) is not in $\mathcal{B}T$ due to malformed syntax.
If some type \( \Box_e \) in \( \overline{\text{post}}(B_1), \ldots, \overline{\text{post}}(B_{i-1}) \) is such that \( e < 0 \), then (12) is valid, since \( \phi(\overline{\text{post}}(B_1), \ldots, \overline{\text{post}}(B_{i-1})) \) has no solution (it includes the always false constraint \( e \geq 0 \)).

Finally, consider the case that for all types \( \Box_e \) in \( \overline{\text{post}}(B_1), \ldots, \overline{\text{post}}(B_{i-1}) \), we have \( e \geq 0 \). This means that there is some type \( \Box_e \) in \( \overline{\text{pre}}(B_i) \) such that \( e < 0 \). Since \( c_1 \land c \) is satisfiable, we can extend one of its solutions to a solution of \( \phi(d) \land c_1 \land c \), where \( d \) is \( \overline{\text{pre}}(H) \), \( \overline{\text{post}}(B_1), \ldots, \overline{\text{post}}(B_{i-1}) \). This solution would show that \( d \vdash c_1 \land c \rightarrow \overline{\text{pre}}(B_i) \) is valid (as a type assertion in \( PT \)), which is absurd since such a type assertion is (10), which was shown to be valid.

(b) Since \( S \) is well-moded, for \( i = 1 \ldots n \):

\[
\overline{\text{post}}(A_1), \overline{\text{post}}(A_2), \ldots, \overline{\text{post}}(A_{i-1}) \vdash e' \rightarrow \overline{\text{pre}}(A_i)
\]  

is valid and has no local type variable. By recalling that no local to output type variable is admitted in parametric modes (see Definition 4.12), a simple induction on \( i \) reveals that there is no type variable at all. Also, by reasoning as in (a):

\[
\overline{\text{post}}(B_1), \ldots, \overline{\text{post}}(B_k) \vdash c_1 \land c \rightarrow \overline{\text{post}}(H)
\]

is valid and contains no type variable, where \( \overline{\text{post}}(H) = y : \tau \times \mu \) for \( H = p(y : \tau \times \mu) \). Since \( A_1 = p(x : \tau \times \mu) \), \( \overline{\text{post}}(A_1) = x : \tau \times \mu \), and \( x = y \) in \( c_1 \), validity of the last type assertion implies that:

\[
\overline{\text{post}}(B_1), \ldots, \overline{\text{post}}(B_k) \vdash c_1 \land c \rightarrow \overline{\text{post}}(A_1)
\]  

is valid and contains no type variable. Assume for the moment that the type assertions (13) and (14) are in \( BT \). First, by monotonicity on (13), for \( i = 2 \ldots n \):

\[
\overline{\text{post}}(B_1), \ldots, \overline{\text{post}}(B_k), \overline{\text{post}}(A_1), \overline{\text{post}}(A_2), \ldots, \overline{\text{post}}(A_{i-1}) \vdash c_1 \land c \rightarrow \overline{\text{pre}}(A_i)
\]  

is valid. Second, by monotonicity on (14), for \( i = 2 \ldots n \):

\[
\overline{\text{post}}(B_1), \ldots, \overline{\text{post}}(B_k), \overline{\text{post}}(A_2), \ldots, \overline{\text{post}}(A_{i-1}) \vdash c_1 \land c \rightarrow \overline{\text{post}}(A_i)
\]  

is valid. This concludes the proof, since this is the proof obligation of well-moding for type substitutions \( \vartheta_1, \ldots, \vartheta_{k+n-1} \) and \( i = 2 \ldots n \).

Assume now that (13) or (14) is not in \( BT \). If \( c_1 \land c \) is unsatisfiable then (17) is trivially valid. Let \( c_1 \land c \) be satisfiable. Since (13) and (14) contain no type variable, this means that for some type \( \Box_e \) in (13) or in (14), we have \( e < 0 \).

If for some type \( \Box_e \) in \( \overline{\text{post}}(B_1), \ldots, \overline{\text{post}}(B_k), \overline{\text{post}}(A_2), \ldots, \overline{\text{post}}(A_{i-1}) \) we have \( e < 0 \), then (17) is trivially valid.

Also, \( \overline{\text{post}}(A_1) \) does not contain any \( \Box_e \) with \( e < 0 \). Otherwise, since \( c_1 \land c \) is satisfiable, we can extend one of its solutions to a solution of \( \phi(d) \land c_1 \land c \), where \( d \) is \( \overline{\text{post}}(B_1), \ldots, \overline{\text{post}}(B_k) \). This solution would show that \( d \vdash c_1 \land c \rightarrow \overline{\text{post}}(A_1) \) is invalid (as a type assertion in \( PT \)), which is absurd since such a type assertion is (14), which was shown to be valid.

Finally, with the same reasoning we can show that \( \overline{\text{pre}}(A_i) \) in (13) does not contain any \( \Box_e \) with \( e < 0 \). □