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CONTRACTION-FREE SEQUENT CALCULI FOR INTUITIONISTIC LOGIC

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§0. Prologue. Gentzen's sequent calculus LJ, and its variants such as G3 [21], are (as is well known) convenient as a basis for automating proof search for IPC (intuitionistic propositional calculus). But a problem arises: that of detecting loops, arising from the use (in reverse) of the rule $\supset\Rightarrow$ for implication introduction on the left. We describe below an equivalent calculus, yet another variant on these systems, where the problem no longer arises: this gives a simple but effective decision procedure for IPC.

The underlying method can be traced back forty years to Vorob'ev [33], [34]. It has been rediscovered recently by several authors (the present author in August 1990, Hudelmaier [18], [19], Paulson [27], and Lincoln et al. [23]). Since the main idea is not plainly apparent in Vorob'ev's work, and there are mathematical applications [28], it is desirable to have a simple proof. We present such a proof, exploiting the Dershowitz-Manna theorem [4] on multiset orderings.

§1. Introduction. Consider the task of constructing proofs in Gentzen's sequent calculus LJ of intuitionistic sequents $\Gamma \Rightarrow G$, where Γ is a set of assumption formulae and G is a formula (in the language of zero-order logic, using the nullary constant f [absurdity], the unary constant $\neg$ [negation, with $\neg A =_{\text{def}} A \supset f$] and the binary constants &, $\vee$, and $\supset$ [conjunction, disjunction, and implication respectively]). By the Hauptsatz [15], there is an apparently simple algorithm which breaks up the sequent, growing the proof tree until one reaches axioms (of the form $\Gamma \Rightarrow A$ where A is in Γ), or can make no further progress and must backtrack or even abandon the search. (Gentzen's argument in fact was to use the subformula property derived from the Hauptsatz to limit the size of the search tree. Došen [5] improves on this argument.)

Some extra effort is required to ensure termination. Suppose we use a version of the calculus with a contraction rule: then, at every step, we may choose to *duplicate* a formula before breaking it up. With care, one can remove the contraction rule almost entirely, but a trace of it remains in the rule $\supset\Rightarrow$: in using this rule (in reverse) to extend the proof tree above a node labelled $A \supset B, \Gamma \Rightarrow G$ we must

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duplicate the formula $A \supset B$, generating children $A \supset B, \Gamma \Rightarrow A$ and $B, \Gamma \Rightarrow G$. (Consider trying to prove $\neg(A \vee \neg A) \Rightarrow f$ without doing such duplication.) This duplication forbids the argument that the size, by some measure, of the sequent has been diminished, and hence that the search must terminate. We could, following standard practice, use a stack to detect looping—but the looping tests are expensive, and complicate the task of extending the technique to the first-order case.

In the present work we give a sound complete calculus LJT for constructing such proofs, but with no primitive rule for contraction: the only essential change is to the rule $\supset\Rightarrow$. This provides a simple decision procedure for IPC.

Note that our description of LJT as "contraction-free" is based on our view of the sequent calculus as intended to be used from the root up: we have no qualms therefore about having primitive rules with the same formulae as antecedents in each of two premisses. Thus the apparent move from premisses $\Gamma \Rightarrow A$ and $\Gamma \Rightarrow B$ to the conclusion $\Gamma \Rightarrow A \mathbin{\&} B$ does *not* require a contraction to be done on the multiset Γ, Γ containing twice as many copies of everything in Γ as Γ itself has. The two copies of Γ are, on being observed to be equal as multisets, just merged. From the viewpoint of Girard's linear logic [16], this counts as a contraction: from ours, concerned with termination of an algorithm, it does not. Both linear logic and the present work can be seen as means of controlling the use of the contraction rule, but in different ways.

LJT does *not* have the subformula property. Nevertheless, it has it in an obvious weak sense: one can work out what formulae are able to appear in a proof of a particular end-sequent.

§2. Logical preliminaries. First, we present a complete set of sound rules for the sequent calculus LJ, essentially due to Gentzen [15]. Of various presentations, such as G3 of Kleene [21, p. 481], that by Dragalin [6] suits us best: but that covers the multi-succedent version GHPC.

Sequents are of the form

$$\text{FormulaMultiset} \Rightarrow \text{Formula},$$

where a *multiset* [4] is a set in which repetitions are allowed, but no account of order is taken. A, Γ denotes the multiset containing an occurrence of A, the remainder being the multiset Γ. We require no structural rules, other than those implicit in the use of multisets (so permutations are allowed). We easily see (using Dragalin's arguments, for example) that the rules of weakening, contraction and cut are admissible. These arguments apply when the use of the axiom rule (see below) is confined to atomic formulae A: it then follows that the more general usage is also admissible.

Our presentation (Figure 1) differs a little from Gentzen's, but the differences are trivial enough to allow us the use of the name LJ of his system: modulo renaming of some variables (in the first-order case), proofs in our system translate directly into his system and vice versa. Note that we do *not* require contraction: its only essential use is built into our formulation of the rule $\supset\Rightarrow$. (Our reason for using multisets rather than sets is to avoid concealing a contraction rule in the

Axiom	$\overline{A, \Gamma \Rightarrow A}$	$f \Rightarrow$	$\overline{\boxed{f}, \Gamma \Rightarrow G}$
$\& \Rightarrow$	$\dfrac{A, B, \Gamma \Rightarrow G}{\boxed{A \& B}, \Gamma \Rightarrow G}$	$\Rightarrow \&$	$\dfrac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow \boxed{A \& B}}$
$\vee \Rightarrow$	$\dfrac{A, \Gamma \Rightarrow G \qquad B, \Gamma \Rightarrow G}{\boxed{A \vee B}, \Gamma \Rightarrow G}$	$\Rightarrow \vee$	$\dfrac{\Gamma \Rightarrow A}{\Gamma \Rightarrow \boxed{A \vee B}} \qquad \dfrac{\Gamma \Rightarrow B}{\Gamma \Rightarrow \boxed{A \vee B}}$
$\supset \Rightarrow$	$\dfrac{A \supset B, \Gamma \Rightarrow A \qquad B, \Gamma \Rightarrow G}{\boxed{A \supset B}, \Gamma \Rightarrow G}$	$\Rightarrow \supset$	$\dfrac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow \boxed{A \supset B}}$

FIGURE 1

notation.) Our objective is contraction elimination, and we shall show how, in the zero-order case, even this one (disguised) use can also be eliminated.

In the presentation of these rules, in each case the formula in the conclusion which is designated the *principal formula* is boxed. A similar convention applies in some of the proofs below.

Note the occurrence of $A \supset B$ in the major premiss of the $\supset\Rightarrow$ rule. But for this, all the rules have the property that each premiss is simpler than the conclusion. We shall replace this rule by some similar rules which allow the same set of theorems. Let LJT denote the formal system having the above rules, minus $\supset\Rightarrow$, augmented by the four rules of Figure 2, routinely seen to be admissible in LJ. (The T is for "terminating".)

$\supset\Rightarrow_1$	$\dfrac{B, A, \Gamma \Rightarrow G}{A \supset B, A, \Gamma \Rightarrow G}$ [A being atomic]
$\supset\Rightarrow_2$	$\dfrac{C \supset (D \supset B), \Gamma \Rightarrow G}{(C \& D) \supset B, \Gamma \Rightarrow G}$
$\supset\Rightarrow_3$	$\dfrac{C \supset B, D \supset B, \Gamma \Rightarrow G}{(C \vee D) \supset B, \Gamma \Rightarrow G}$
$\supset\Rightarrow_4$	$\dfrac{D \supset B, \Gamma \Rightarrow C \supset D \qquad B, \Gamma \Rightarrow G}{(C \supset D) \supset B, \Gamma \Rightarrow G}$

FIGURE 2

We regard the formula f as not being atomic. Note that the rule $\supset\Rightarrow_1$ is sound even if the formula A is not atomic: what matters is that we can show that its use just when A is atomic suffices. It is easy to show that weakening is (still) an admissible rule of this system: so therefore also is the rule that from $\Gamma \Rightarrow G$ one may infer $f \supset B, \Gamma \Rightarrow G$—a rule one might otherwise expect to be primitive in LJT.

Note that the first three of these four new rules are invertible—if the conclusion is provable in LJ, so is the premiss. The fourth rule has the property that if the conclusion is provable in LJ, so is the second premiss.

Our arguments will use various forms of induction, one of which is on the "size" of sequents. We measure the size of sequents as follows; but first we define an ordering on formulas. The *weight*, $\mathrm{wt}(A)$, of a formula A is a positive integer defined recursively as follows: $\mathrm{wt}(A) = \mathrm{wt}(f) = 1$ for atomic A, $\mathrm{wt}(A \vee B) = \mathrm{wt}(A \supset B) = \mathrm{wt}(A) + \mathrm{wt}(B) + 1$, and $\mathrm{wt}(A \,\&\, B) = \mathrm{wt}(A) + \mathrm{wt}(B) + 2$. This gives us a well-founded order relation $>$ on formulae, with $A > B$ iff $\mathrm{wt}(A) > \mathrm{wt}(B)$. Note for later use that, for example, $\mathrm{wt}((C \,\&\, D) \supset B) > \mathrm{wt}(C \supset (D \supset B))$.

Now treat the antecedent formulae of a sequent together with its succedent as a multiset. Then, by definition, $\Gamma \gg \Delta$ iff the multiset Δ is obtained from the multiset Γ by replacing one or more formulas of Γ by zero or more formulas, each of which is of lower weight than one of the replaced formulas. This is the *multiset ordering* of Dershowitz and Manna, known [4] to be well-founded if the ordering on formulae is. Note that in each of the rules of LJT, the conclusion is in the relation $\gg$ to each of the associated premisses (if any).

Below, we shall talk informally about the "size" of a sequent: arguments using this undefined concept can be formalised as arguments by well-founded induction on the relation $\gg$. To see how to convert these (in the cases considered below) into arguments by induction on the natural numbers, see [4].

§3. **Main results.** The idea of the proof that LJT is equivalent to LJ is to consider an LJ-provable sequent, and use wherever possible one of the invertible rules of LJ to replace it by a simpler (with respect to the ordering $\gg$) LJ-provable sequent. In the case that the final step of the LJ proof of the given sequent is by means of $\supset\Rightarrow$, we use one of the rules $\supset\Rightarrow_1, \ldots, \supset\Rightarrow_4$. The only difficulty arises with the $\supset\Rightarrow_1$ rule, since its conclusion is not quite general enough: as well as the implication $A \supset B$ being introduced, the antecedent has to contain an occurrence of the atomic left-subformula A. We therefore prove a lemma which implies that in constructing the proof from the root up we may delay uses of this rule (with $A \supset B$ the principal formula, A being atomic) until the atom A occurs on the left of the sequent to be proved.

DEFINITION. A formula of the form $A \supset B$ is said to be *awkward* if A is atomic (recall that f is not regarded as atomic). A multiset Γ is said to be *irreducible* if it contains no pair $A, A \supset B$ where $A \supset B$ is awkward, and neither absurdity nor a conjunction nor a disjunction. A sequent is *irreducible* if its antecedent is irreducible. An LJ proof is *clumsy* if the principal formula of the final step is on the left and is awkward; otherwise it is *sensible*.

LEMMA 1. *Any* LJ-*provable irreducible sequent has a sensible proof.*

PROOF. Otherwise, some LJ-provable irreducible sequents have only clumsy proofs. Consider, among all proofs of all such sequents, one of the shortest (where "length" of proof is measured along the leftmost branch): let it be Π, with end-sequent $\Gamma \Rightarrow G$. So, the final step of Π has an awkward principal formula $A \supset B$ on the left. Then $\Gamma \equiv A \supset B, \Gamma'$ and Γ is irreducible (and so A is not in Γ'). So the proof looks like

$$(1) \qquad \dfrac{\dfrac{\Pi'}{A \supset B, \Gamma' \Rightarrow A} \qquad \dfrac{\Pi''}{B, \Gamma' \Rightarrow G}}{\boxed{A \supset B}, \Gamma' \Rightarrow G}$$

Since A does not occur in Γ', and cannot be identical to $A \supset B$, the proof Π' is nontrivial. Its end-sequent has irreducible Γ as its antecedent, so, by the inductive assumption, it has a sensible proof Π'''. Consider its final step. Since A is atomic, this step must be by $*\Rightarrow$, where $*$ is one of the logical constants f, &, $\vee$ or $\supset$. Since Γ is irreducible, $*$ must be $\supset$. Let $D \supset E$ be the principal formula of the final step: since Π''' is sensible, D is not atomic. The proof of $\Gamma \Rightarrow G$ now looks as follows:

$$(2) \qquad \dfrac{\dfrac{\dfrac{\Pi_0}{A \supset B, D \supset E, \Gamma'' \Rightarrow D} \quad \dfrac{\Pi_1}{E, A \supset B, \Gamma'' \Rightarrow A}}{A \supset B, \boxed{D \supset E}, \Gamma'' \Rightarrow A} \qquad \dfrac{\Pi''}{B, D \supset E, \Gamma'' \Rightarrow G}}{\boxed{A \supset B}, D \supset E, \Gamma'' \Rightarrow G}$$

and we permute this into

$$(3) \qquad \dfrac{\dfrac{\Pi_0}{A \supset B, D \supset E, \Gamma'' \Rightarrow D} \qquad \dfrac{\dfrac{\Pi_1}{A \supset B, E, \Gamma'' \Rightarrow A} \quad \dfrac{\Pi_2}{B, E, \Gamma'' \Rightarrow G}}{E, \boxed{A \supset B}, \Gamma'' \Rightarrow G}}{A \supset B, \boxed{D \supset E}, \Gamma'' \Rightarrow G} \supset\Rightarrow$$

where $\Pi_2/E, B, \Gamma'' \Rightarrow G$ is routinely derived by substituting E for $D \supset E$ at appropriate occurrences of the latter in $\Pi''/B, D \supset E, \Gamma'' \Rightarrow G$, dealing with the lowest principal occurrence by discarding the major premiss.

Thus, we have found a proof (3) of $\Gamma \Rightarrow G$ in which the principal formula $(D \supset E)$ of the final step is not awkward, as required. □

REMARK. Using the same methods, it is not hard to show that any LJ-provable sequent, not having a pair A, $A \supset B$ of antecedent formulae with A atomic, has a sensible proof. This result is stronger than we need for Theorem 1 below, but is required when extending the ideas to the first-order case. The lemma could, in part, be extracted from Lemma 7 in [22]. One could also go further and make all the subproofs sensible.

LEMMA 2. $\vdash_{\mathrm{LJ}} \Gamma, (C \supset D) \supset B \Rightarrow C \supset D$ *iff* $\vdash_{\mathrm{LJ}} \Gamma, D \supset B \Rightarrow C \supset D$.

PROOF. Trivial [34]. □

THEOREM 1. *The systems* LJ *and* LJT *are equivalent.*

PROOF. As noted earlier, it is routine to show that any sequent provable in LJT is provable in LJ, using the admissibility of cut, contraction, and weakening in the latter.

The important part is the converse. We argue by induction on the "size" of the sequent being proved (*not* on the structure of the proof).

Suppose $\vdash_{\mathrm{LJ}} \Gamma \Rightarrow G$. Several cases arise:

(a) Γ contains f: then there is a trivial proof (by $f\Rightarrow$) of $\Gamma \Rightarrow G$ in LJT.

(b) Γ contains a conjunction A & B. Let $\Gamma \equiv A \,\&\, B, \Gamma'$. Then $A, B, \Gamma' \Rightarrow G$ is also provable in LJ. By the inductive hypothesis, it has a proof in LJT; combine this proof in LJT with the use of the rule &$\Rightarrow$, proving $\Gamma \Rightarrow G$ in LJT.

(c) Γ contains a disjunction $A \vee B$: we deal with this in a manner similar to (b).

(d) Γ contains a pair A, $A \supset B$ where A is atomic. Let $\Gamma \equiv A, A \supset B, \Gamma'$. Then $A, B, \Gamma' \Rightarrow G$ is also provable in LJ, since $\vdash_{\text{LJ}} B \Rightarrow A \supset B$. By the inductive hypothesis, it has a proof in LJT; combine this proof in LJT with the use of the rule $\supset\Rightarrow_1$, proving $\Gamma \Rightarrow G$ in LJT.

(e) Otherwise, Γ is irreducible. By Lemma 1, $\Gamma \Rightarrow G$ has an LJ proof where the principal formula is either on the right, or is an implication $A \supset B$ on the left and A is not atomic. If the final step is by axiom, $\Rightarrow\&$, $\Rightarrow\vee$, or $\Rightarrow\supset$, we inductively have a proof in LJT, since in each case the premiss has (or the premisses have) lower size than the conclusion. The only difficulty arises with the $\supset\Rightarrow$ rule. Suppose the final step is by means of this rule, with $A \supset B$ being the principal formula, and $A \supset B, \Gamma' \Rightarrow G$ the conclusion. Consider the various possible forms of A.

(i) If A is a conjunction $(C \& D)$, then, since in LJ $(C \& D) \supset B$ iff $C \supset (D \supset B)$, we may also prove $C \supset (D \supset B), \Gamma' \Rightarrow G$ in LJ. By induction, we find a proof of this in LJT, to which we append an instance of the $\supset\Rightarrow_2$ rule.

(ii) Similarly, if A is a disjunction $(C \vee D)$, we inductively prove $C \supset B, D \supset B, \Gamma' \Rightarrow G$ in LJT and append an instance of the $\supset\Rightarrow_3$ rule.

(iii) Suppose A is an implication $(C \supset D)$. So, the conclusion

$$A \supset B, \Gamma' \Rightarrow G$$

follows in LJ from the LJ-provable premisses

$$A \supset B, \Gamma' \Rightarrow A \quad \text{and} \quad B, \Gamma' \Rightarrow G$$

But, in LJ, $A \supset B, \Gamma' \Rightarrow A$ iff $D \supset B, \Gamma' \Rightarrow A$, by Lemma 2. This second sequent is of lower size than $A \supset B, \Gamma' \Rightarrow G$, so has, by the inductive hypothesis, a proof in LJT. Similarly, so does $B, \Gamma' \Rightarrow G$. Combine these with the use of the rule $\supset\Rightarrow_4$.

(iv) Suppose A is absurdity, f. Since $f \supset B$ is vacuously provable, $A \supset B, \Gamma' \Rightarrow G$ iff $\Gamma' \Rightarrow G$ in LJ. By the inductive hypothesis, the latter has a proof in LJT. Combine this with the use of the weakening rule in LJT. □

Since the rules of cut and contraction are admissible in LJ, these rules are admissible also in LJT. (A direct proof of cut-elimination for LJT seems difficult.)

The system LJT of rules, treated in reverse as problem reduction rules, is terminating: any search for a proof will terminate in either success or failure. There is no need to use a stack and check for looping: for each rule, the move from conclusion to premisses replaces a sequent by zero or more sequents of lower size, where "size" is as explained in §2. So we have a simple decision algorithm for zero-order intuitionistic logic. (In contrast to the case of classical logic, the order in which we choose to use these rules is important: some choices may have to be undone. Tennant [31] has described some heuristics for making such choices.)

Note that, with minor variations, this technique applies also to *minimal logic*.

§4. Contraction-free multi-succedent calculi. In practice, a multi-succedent calculus is convenient for implementation: one can then share code easily with an implementation for the classical case, and also postpone decisions about which disjunct (of a disjunctive formula on the right) should be chosen.

Dragalin [6] gives, as noted above, a convenient formalisation of such a calculus, GHPC. It looks as follows; the asterisks are to suggest "zero or more formulae" on the right:

Axiom*	$\overline{A, \Gamma \Rightarrow A, \Delta}$	$f\Rightarrow^*$	$\overline{\boxed{f}, \Gamma \Rightarrow \Delta}$
$\&\Rightarrow^*$	$\dfrac{A, B, \Gamma \Rightarrow \Delta}{\boxed{A \,\&\, B}, \Gamma \Rightarrow \Delta}$	$\Rightarrow\&^*$	$\dfrac{\Gamma \Rightarrow A, \Delta \qquad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow \boxed{A \,\&\, B}, \Delta}$
$\vee\Rightarrow^*$	$\dfrac{A, \Gamma \Rightarrow \Delta \qquad B, \Gamma \Rightarrow \Delta}{\boxed{A \vee B}, \Gamma \Rightarrow \Delta}$	$\Rightarrow\vee^*$	$\dfrac{\Gamma \Rightarrow A, B, \Delta}{\Gamma \Rightarrow \boxed{A \vee B}, \Delta}$
$\supset\Rightarrow^*$	$\dfrac{A \supset B, \Gamma \Rightarrow A \qquad B, \Gamma \Rightarrow \Delta}{\boxed{A \supset B}, \Gamma \Rightarrow \Delta}$	$\Rightarrow\supset^*$	$\dfrac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow \boxed{A \supset B}, \Delta}$

FIGURE 3

(In fact, [6] restricts the axiom* rule to atomic A, and then shows that the more general use is admissible.) The rules are valid with respect to LJ, if we consider the meaning of a sequent with several formulae in the succedent to be that of the sequent made by disjoining those formulae together. Note that all the rules are invertible, except for the last two: but the last *is* invertible for empty Δ. [6] shows that weakening, contraction and cut are admissible rules.

Now, $\supset\Rightarrow^*$ is the only rule we have to change: we do it just by changing all occurrences of the goal formula metavariable G in the new rules ($\supset\Rightarrow_1, \ldots, \supset\Rightarrow_4$) of LJT into the formula multiset metavariable Δ.

The new rules are now as follows:

$\supset\Rightarrow_1^*$	$\dfrac{B, A, \Gamma \Rightarrow \Delta}{A \supset B, A, \Gamma \Rightarrow \Delta}$ [A being atomic]
$\supset\Rightarrow_2^*$	$\dfrac{C \supset (D \supset B), \Gamma \Rightarrow \Delta}{(C \,\&\, D) \supset B, \Gamma \Rightarrow \Delta}$
$\supset\Rightarrow_3^*$	$\dfrac{C \supset B, D \supset B, \Gamma \Rightarrow \Delta}{(C \vee D) \supset B, \Gamma \Rightarrow \Delta}$
$\supset\Rightarrow_4^*$	$\dfrac{D \supset B, \Gamma \Rightarrow C \supset D \qquad B, \Gamma \Rightarrow \Delta}{(C \supset D) \supset B, \Gamma \Rightarrow \Delta}$

FIGURE 4

Our arguments of §§2 and 3 now go through with no essential changes. We thus obtain a contraction-free multi-succedent calculus for intuitionistic logic (with no contraction hidden in the $\supset\Rightarrow$ rule).

Note that, in contrast to the presentations in, for example, [7] and [35], the rule $\supset\Rightarrow^*$ of GHPC has no occurrence of Δ in the succedent of the major premiss. A variant of the rule having such an occurrence would be admissible, but is not necessary for the translation of LJ proofs into the multi-succedent calculus GHPC. However, there is an advantage in having such an occurrence of Δ: the rule is then invertible. On the other hand, the argument above would then no longer be correct: the difficulty would be in replacing the sequent $(C \supset D) \supset B, \Gamma' \Rightarrow C \supset D, \Delta$ by $D \supset B, \Gamma' \Rightarrow C \supset D, \Delta$ in (the multi-succedent variant of) the proof of Theorem 1. So, our argument is done in the context of Dragalin's system GHPC rather than in that of [7] or [35].

However, the following rule *is* admissible, and strengthens the rule $\supset\Rightarrow_4^*$:

$$\supset\Rightarrow_4^{**} \qquad \frac{D \supset B, \Gamma \Rightarrow C \supset D, \Delta \qquad B, \Gamma \Rightarrow \Delta}{(C \supset D) \supset B, \Gamma \Rightarrow \Delta}$$

Using it rather than the weaker rule $\supset\Rightarrow_4^*$ is therefore permitted, but not obligatory. Unfortunately, this rule is still not invertible, except when Δ is absurdity or empty. Which of the rules $\supset\Rightarrow_4^*$ and $\supset\Rightarrow_4^{**}$ should in practice be used is unclear.

§5. Natural deduction rules. Gentzen's formulation of LJ allows for the straightforward translation (cf. [1], [9], [16], and [29]) of LJ proofs into NJ deductions. For example, an instance of the rule $\&\Rightarrow$ is translated into an instance of the rule $\&E$ (in the form that from $A \& B$ we may infer both A and B). An instance of the rule $\supset\Rightarrow$ translates into an instance of the rule $\supset E$ (modus ponens) as follows:

$$\frac{\dfrac{\Pi_0}{A \supset B, \Gamma \Rightarrow A} \qquad \dfrac{\Pi_1}{B, \Gamma \Rightarrow G}}{\boxed{A \supset B}, \Gamma \Rightarrow G}$$

is translated to the deduction having $A \supset B$ and the formulae in Γ as assumptions, and G as conclusion: since from $A \supset B$ and Γ we have (in the translation of Π_0) deduced A, we may write B under the pair $A \supset B$ and A, appeal to modus ponens, and fill in the gap between B (with Γ) and G by entering the translation of Π_1.

The new rules of LJT correspond to the replacement, in NJ, of modus ponens by the following rules:

$$\frac{A \supset B \qquad A}{B} \,[A \text{ is atomic}]\ \mathrm{MP}_1$$

$$\frac{(C \,\&\, D) \supset B}{C \supset (D \supset B)}\,\mathrm{MP}_2 \qquad \frac{(C \vee D) \supset B}{C \supset B}\,\mathrm{MP}_{3a} \qquad \frac{(C \vee D) \supset B}{D \supset B}\,\mathrm{MP}_{3b}$$

$$\frac{(C \supset D) \supset B \qquad C \supset D[D \supset B]}{B}\,\mathrm{MP}_4$$

FIGURE 5

where the parenthesised $[D \supset B]$ in the rule MP_4 indicates its discharge as an assumption from the second premiss.

It is of interest to see how the deductions got by this means can be transformed into standard natural deductions. Abramsky [1] gives a convenient formulation of the standard translation of sequent calculus proofs into natural deduction proofs. Adopting roughly his notation (in his §2.2), and introducing appropriate new constants *apply_atom*, *curry*, *left*, *right*, and *apply_imp*, the new natural deduction rules (with annotations connoting proofs) look like

$$\frac{p : A \supset B \qquad q : A}{\text{apply\_atom}(p, q) : B} [A \text{ is atomic}]\ \text{MP}_1$$

$$\frac{p : (C \,\&\, D) \supset B}{\text{curry}(p) : C \supset (D \supset B)}\ \text{MP}_2$$

$$\frac{p : (C \vee D) \supset B}{\text{left}(p) : C \supset B}\ \text{MP}_{3a} \qquad \frac{p : (C \vee D) \supset B}{\text{right}(p) : D \supset B}\ \text{MP}_{3b}$$

$$\frac{p : (C \supset D) \supset B \qquad q(x) : C \supset D[x : D \supset B]}{\text{apply\_imp}(p, q) : B}\ \text{MP}_4$$

FIGURE 6

The appropriate labelling rules for the new sequent calculus rules are then

$$\supset\Rightarrow_1 \quad \frac{x : B, a : A, \Delta \Rightarrow g : G}{p : (A \supset B), a : A, \Delta \Rightarrow [\text{apply\_atom}(p, a)/x]g : G} [A \text{ being atomic}]$$

$$\supset\Rightarrow_2 \quad \frac{x : (C \supset (D \supset B)), \Gamma \Rightarrow g : G}{p : ((C \,\&\, D) \supset B), \Gamma \Rightarrow [\text{curry}(p)/x]g : G}$$

$$\supset\Rightarrow_3 \quad \frac{x : (C \supset B), y : (D \supset B), \Gamma \Rightarrow g : G}{p : ((C \vee D) \supset B), \Gamma \Rightarrow [\text{left}(p)/x, \text{right}(p)/y]g : G}$$

$$\supset\Rightarrow_4 \quad \frac{z : (D \supset B), \Gamma \Rightarrow q(z) : (C \supset D) \qquad x : B, \Gamma \Rightarrow g : G}{p : ((C \supset D) \supset B), \Gamma \Rightarrow [\text{apply\_imp}(p, q)/x]g : G}$$

FIGURE 7

By this means, we may label LJT proofs with typed lambda calculus terms as witnesses: these terms can be traversed to obtain natural deductions (in NJT). To obtain an NJ deduction it suffices to use the definitions

$$\begin{aligned}
\text{curry}(p) &=_{\text{def}} \lambda x.\ \lambda y.\ \text{apply}(p, \langle x, y \rangle) \\
\text{left}(p) &=_{\text{def}} \lambda x.\ \text{apply}(p, \text{inl}(x)) \\
\text{right}(p) &=_{\text{def}} \lambda x.\ \text{apply}(p, \text{inr}(x)) \\
\text{apply_atom}(p, q) &=_{\text{def}} \text{apply}(p, q) \\
\text{apply_imp}(p, q) &=_{\text{def}} \text{apply}(p, q(\lambda y.\ \text{apply}(p, \lambda x. y)))
\end{aligned}$$

interpreting the deductions of NJT into those of NJ. (These will not necessarily be normal deductions, but can be normalised in the usual way.)

Not every NJ deduction can be obtained in this way, even allowing for standard conversions. By König's lemma, since LJT is finitely branching and proofs of a given formula are of bounded depth, any formula can only have a finite number of proofs in LJT. But the formula $(p \supset p) \supset (p \supset p)$, where p is an atom, has infinitely many distinct deductions in NJ: the Church numerals (for p). (In fact, in LJT we can only construct proofs representing the numerals for zero and one.) This limitation is regrettable, since functional programming encourages us to look at ways of generating proofs mechanically.

§6. First-order intuitionistic calculi. These ideas extend to some extent to the first-order case. It is easy to handle formulae of the form $(\exists x)A(x) \supset B$, using the equivalence (provable even in minimal logic)

$$(\exists x)A(x) \supset B \leftrightarrow (\forall x)(A(x) \supset B),$$

where the bound variable x is not free in B. But formulae of the form $(\forall x)A(x) \supset B$ are problematic. An early version of this paper gave appropriate sound rules, which are complete in a weak sense: for any provable formula, a suitable choice of a parameter (similar to the *Q-depth* parameter in [12]) allowed a successful proof search. We refer to Hudelmaier's unpublished [18] for full details of a similar approach.

§7. Implementation. The system LJT of rules has been implemented in Prolog as part of the MacLogic system [8], used in teaching various first-order logics. This system includes some theorem provers (for classical, intuitionistic, and minimal logic), intended to be fast at solving problems but allowed to be slower when failing, in order to warn students of unsuccessful tactics for problem decomposition.

§8. Applications. Pitts [28] reports applications of LJT to show that, in intuitionistic logic, quantification over propositional variables can be modelled in what we call the zero-order calculus, with nice applications to the theory of Heyting algebras.

Hodas and Miller [17] illustrate the use of LJT in a generic theorem-prover based on linear logic, encoding LJT in about 15 lines of Prolog.

Tennant [31] describes a theorem-prover for minimal zero-order logic (with an extension to intuitionistic relevant logic in mind), where the looping tests are replaced by use of a natural deduction version of the rules of LJT. He reports that this led to a fourfold increase in performance on a wide range of difficult problems.

§9. Related work. Vorob'ev [33], [34] described a decision algorithm for IPC based on similar considerations. The present article may be regarded in part as a restatement of this relatively ancient Soviet work: it is offered however as a clarification and simplification, in the knowledge that the technique is now being reinvented and exploited. The sequent calculus lying behind Vorob'ev's algorithm in [34] is concealed by the pre-processing of sequents into a normal form (using the distributive laws); his algorithm also takes advantage of the equivalence (for negated goals) of the intuitionistic decision problem with the classical one. See [25] for a summary of some of the related Soviet work.

Hudelmaier [18], [19] invented the same calculus as LJT. His argument is similar to ours, but (like Vorob'ev) avoids the use of multiset orderings in favour of constructive techniques, from which one can extract explicit (but useless) theoretical bounds on the relationship between the depths of the LJ proof and of the

LJT proof. He has refined it into an $O(n \log n)$-SPACE solution [20] to the decision problem for IPC. (From Statman's work, this problem is known to be PSPACE-complete. In the worst case, the obvious algorithm based on LJT uses space exponential in the size of the sequent being proved. Our own view is that this is not a serious problem in practice.)

Lincoln et al. [23] report on the use of a calculus IIL* nearly identical to LJT, for a translation of intuitionistic sequents to linear logic sequents (in the calculus IMALL): this differs from Girard's translation [16] in not using the modal operators (!, ?), but still preserves provability. IIL* is restricted to the implicational fragment of intuitionistic logic, and differs from LJT in the rule for introduction of the implication $A \supset B$ on the left when A is atomic: rather than allowing this only when A is already in the remaining context Γ, it just requires the derivability of A from Γ. IIL* is obviously sound, and its completeness follows trivially from that of LJT (but not conversely). For automated proof search, we prefer the use of LJT, since it avoids work done trying to derive A in favour of waiting until it is obvious: but IIL* is the right calculus to use for the translation to IMALL.

As a solution to the looping problem, Van Gelder [32] proposes the *tortoise and hare* technique: this requires maintenance of a stack of subproblems being solved, and two pointers thereinto, which move at different speeds: if there is a loop, then the pointers will eventually point at different occurrences of the same problem. We have not examined this approach in any detail, believing it to have similar disadvantages to the maintenance of a stack with a more expensive and more frequent check, but with earlier detection of looping.

Gabbay [14] proposes the *bounded restart* rule: this requires that a history be kept of the search, but allows that when an atomic goal is to be proved, one may restart the search at points after earlier occasions when the same goal was investigated.

Slaney [30] reports that his theorem prover (for minimal logic) first tries to apply the $\supset\Rightarrow$ rule without keeping the principal formula in the antecedent, and only on failure tries the proper rule. He reports that "it uses a simple loop detector which in practice does not seem to slow it down much, though on really large problems it probably would be expensive".

One possible decision method for IPC is to use the Gödel-McKinsey-Tarski translation of intuitionistic problems into S4, and then to use a decision procedure for S4. Fitting [10], [11] argues that although one can in principle use a periodicity test as part of such a decision procedure, in practice "such a test would be quite expensive", and he therefore recommends use of a modal depth counter. Wallen [35] describes (for zero-order S5) how to calculate the *multiplicity* (essentially an upper bound on the number of times any formula may need to be duplicated), and comments that similar techniques could be developed for the other modal logics.

It is not clear how effective such a multiplicity is at constraining the search. Franzén [13] shows how to calculate such multiplicities for IPC; but he abandoned this approach in favour of a technique (*implication-locking*) of retaining implicative formulae in the major premiss of the $\supset\Rightarrow$ rule, but inhibiting their use until more information is available. Tennant [31] describes a similar technique, *fettering*.

Beeson [2] describes a (first-order) theorem-prover GENTZEN, implemented in Prolog, with loop-checking to avoid nontermination, and discusses related work.

It would be interesting to see whether a combination of the rules for LJT with the matrix methods of Bibel, Andrews and Wallen [35] would be effective.

Other authors use techniques which avoid the problem by other means: for example, recent Soviet work [24] has been based on pre-processing to normal form, and then the use of a form of "resolution". Such techniques are outwith the scope of this paper's consideration of techniques for controlling or avoiding duplication.

Note also that there is no connection with the work of Dardzhania [3], or that of Ono and Komori [26], on variants of intuitionistic logic with no contraction rule: these weaken the logic, rather than reformalise the same logic.

§10. Conclusion. We have shown the correctness and completeness of a variant of Gentzen's calculus LJ for zero-order intuitionistic logic, having no contraction rule and so having good termination properties when viewed as a system of problem reduction rules. This can be used as the basis for a simple decision procedure for this logic, which, in contrast to the direct use of LJ, needs no tests for looping. Proofs obtained by this means translate easily into natural deductions. The same variation can be applied to two multi-succedent calculi.

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