Abstract Interpretation

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Week 1

`http://www.cs.au.dk/~jmi/AbsInt/`

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What is this course about?

Crudely simplified the history of program analysis (or static analysis) can be split in two:

- an American school of program analysis
- a French school of program analysis

I highly recommend the *Static Analysis course*, which gives a nice introduction mainly to the American approach.

This course is concerned with the alternative, French approach.

Which is the right approach?

None of them is right or wrong — it is simply an alternative view — an eye opener to a new world.

It can be used to explain existing approaches and extend or strengthen them

In 7 weeks, you will be in a position to make an informed opinion

It is not just an academic theory: it has been used to check/verify flight control software for both Airbus and Mars missions. By the end of this course, we will read papers about those.

It will get bloody — there will be mathematics — there will be semantics

What is abstract interpretation?

- It is a theory of *semantics-based program analysis*
- It was initially conceived in the late 1970's by Patrick and Radhia Cousot
- It has been refined over the last 40 years
  - to new applications
  - to new kinds of semantics
  - to new programming paradigms
  - by new abstract domains
  - …

Learning outcomes and competences

The participants must at the end of the course be able to:

- *describe* and *explain* basic analyses in terms of classical abstract interpretation.
- *apply* and *reason* about Galois connections.
- *implement* abstract interpreters on the basis of the derived program analyses.

Outline

- What and how of the course
- Transition systems
- Math: Posets, CPOs, complete lattices, Galois connections, fixed points
- Abstract interpretation basics
- OCaml intro

Transition systems

Transition systems - quick recap

You already know transition systems from dADS 1.

Definition. A transition system is a triple (quadruple) $\langle S, I, F, \rightarrow\rangle$ where

- S is a set of states
- $I \subseteq S$ is a set of initial states
- $F \subseteq S$ is an optional set of final states ($\forall s \in F, s' \in S : s \not\rightarrow s'$)
- $\rightarrow \subseteq S \times S$ is a transition relation relating a state to its (possible) successors

Example 1: Euclid's algorithm

Given two numbers $x, y \in \mathbb{N}$ we can describe Euclid's GCD algorithm as a transition system:

$$\begin{aligned}
S &= \mathbb{N} \times \mathbb{N} \\
I &= \{\langle x,\, y\rangle\} \\
F &= \{\langle n,\, n\rangle \mid n \in \mathbb{N}\} \\
\to : \langle n,\, m\rangle &\to \langle n - m,\, m\rangle \qquad\qquad \text{if } \; n > m \\
\langle n,\, m\rangle &\to \langle n,\, m - n\rangle \qquad\qquad \text{if } \; n < m
\end{aligned}$$

where we have written the transition relation using *infix notation*.
We can write it even more formally as:

$$\begin{aligned}
\to = \{(\langle n,\, m\rangle, \langle n - m,\, m\rangle) \mid n > m\} \\
\cup \{(\langle n,\, m\rangle, \langle n,\, m - n\rangle) \mid n < m\}
\end{aligned}$$

Example 2: Modeling a program

Modeling the program

```
x := 0;
while (x < 100) {
    x := x + 1;
}
```

as a transition system:

$$
\begin{aligned}
S &= \mathbb{Z} \\
I &= \{0\} \\
\rightarrow &= \{(x, x') \mid x < 100 \ \wedge \ x' = x + 1\}
\end{aligned}
$$

How to get from a program to a transition system is the topic of next week's lecture.

For now we assume that we can model the semantics (the meaning) of a program as a transition system.

Mathematical foundations

Partially ordered sets

Definition. A partially ordered set (poset) $\langle S; \sqsubseteq \rangle$ is a set S equipped with a binary relation $\sqsubseteq \subseteq S \times S$ with the following properties:

- Reflexive: $\forall a \in S : a \sqsubseteq a$
- Antisymmetric: $\forall a, b \in S : a \sqsubseteq b \ \wedge \ b \sqsubseteq a \implies a = b$
- Transitive: $\forall a, b, c \in S : a \sqsubseteq b \ \wedge \ b \sqsubseteq c \implies a \sqsubseteq c$

Example 1: $\langle \mathbb{N}; \leq \rangle$ is a poset

Example 2: $\langle \wp(S); \subseteq \rangle$ is a poset

Note: $\wp(S)$ is sometimes written 2^S

Upper and lower bounds

Let $\langle P; \sqsubseteq \rangle$ be a partially ordered set.

Definition. $u \in P$ is an *upper bound* of $S \subseteq P$ iff $\forall s \in S : s \sqsubseteq u$

Definition. $l \in P$ is an *lower bound* of $S \subseteq P$ iff $\forall s \in S : l \sqsubseteq s$

Definition. $u \in P$ is a *least upper bound* (lub) of $S \subseteq P$ iff it is an upper bound of S and it is less than all other upper bounds: $\forall u' \in P : (\forall s \in S : s \sqsubseteq u') \implies u \sqsubseteq u'$

Definition. $l \in P$ is a *greatest lower bound* (glb) of $S \subseteq P$ iff it is an lower bound of S and it is greater than all other lower bounds:
$\forall l' \in P : (\forall s \in S : l' \sqsubseteq s) \implies l' \sqsubseteq l$

Complete Partial Orders (CPOs)

Definition. A complete partial order is a poset such that all increasing chains $c_i, i \in \mathbb{N}$ ($\forall i \in \mathbb{N} : c_i \sqsubseteq c_{i+1}$) have a least upper bound:

$$\bigsqcup_{i \in \mathbb{N}} c_i$$

Non-example: $\langle \mathbb{N}; \leq \rangle$ is *not* a CPO. Why?

Example: $\langle \wp(S); \subseteq \rangle$ is a CPO.

Complete lattices

Definition. A complete lattice is a poset $\langle C; \sqsubseteq, \perp, \top, \sqcup, \sqcap \rangle$ such that

- the least upper bound $\sqcup S$ and
- the greatest lower bound $\sqcap S$ exists for every subset S of C.
- $\perp = \sqcap C$ denotes the infimum of C and
- $\top = \sqcup C$ denotes the supremum of C.

Example 1: $\langle \wp(S); \subseteq, \emptyset, S, \cup, \cap \rangle$ is a complete lattice.

Example 2: The integers (extended with $-\infty$ and $+\infty$) is a complete lattice $\langle \mathbb{Z} \cup \{-\infty, +\infty\}; \leq, -\infty, +\infty, \max, \min \rangle$.

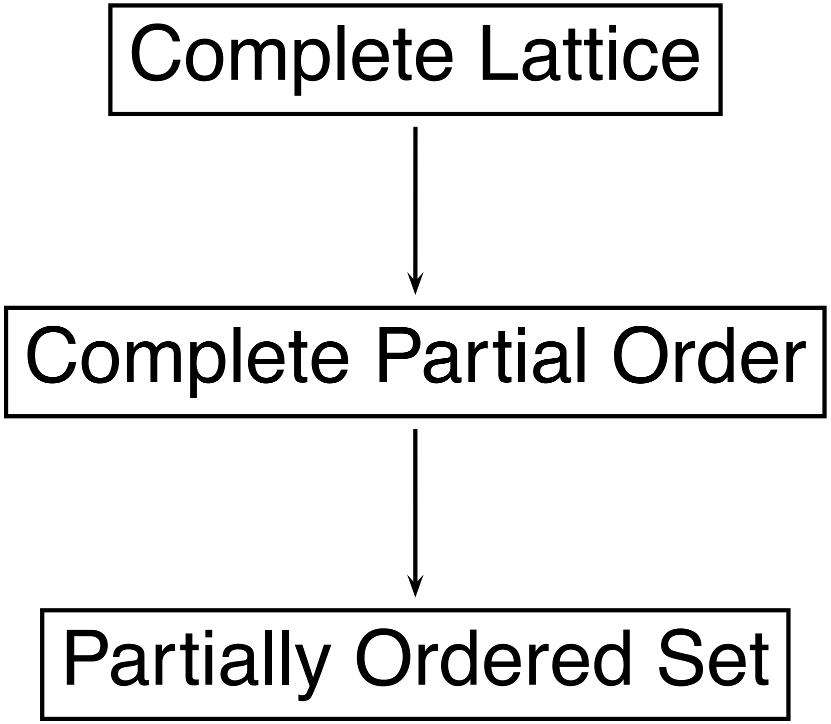
Example: A complete lattice of functions

Theorem. The set of total functions $D \to C$, whose codomain is a complete lattice $\langle C; \sqsubseteq, \bot, \top, \sqcup, \sqcap \rangle$, is itself a complete lattice $\langle D \to C; \dot{\sqsubseteq}, \dot{\bot}, \dot{\top}, \dot{\sqcup}, \dot{\sqcap} \rangle$ under the pointwise ordering $f \mathrel{\dot{\sqsubseteq}} f' \iff \forall x. f(x) \sqsubseteq f'(x)$, and with

- $\dot{\bot} = \lambda x.\, \bot$
- $\dot{\top} = \lambda x.\, \top$
- $f \mathbin{\dot{\sqcup}} g = \lambda x.\, f(x) \sqcup g(x)$
- $f \mathbin{\dot{\sqcap}} g = \lambda x.\, f(x) \sqcap g(x)$

Here $\lambda x. \ldots$ is a mathematical function with argument x.

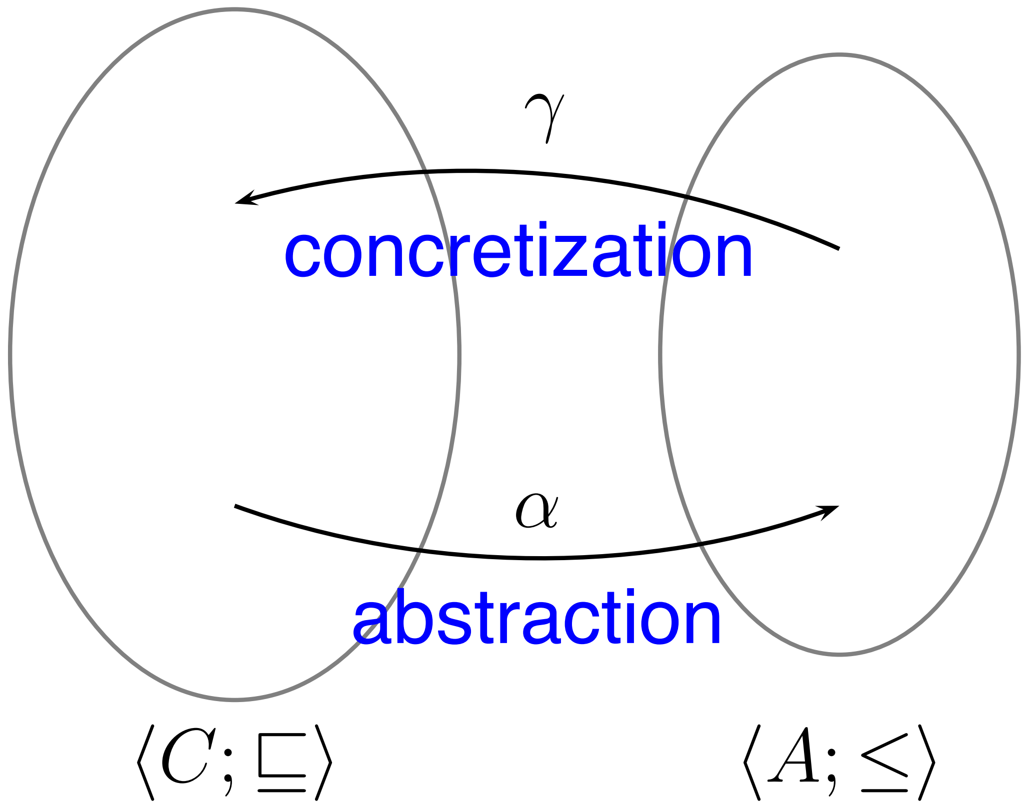
A quick comparison



Galois connections

Galois connections

Definition. A Galois connection is a pair of functions α, γ between two partially ordered sets:



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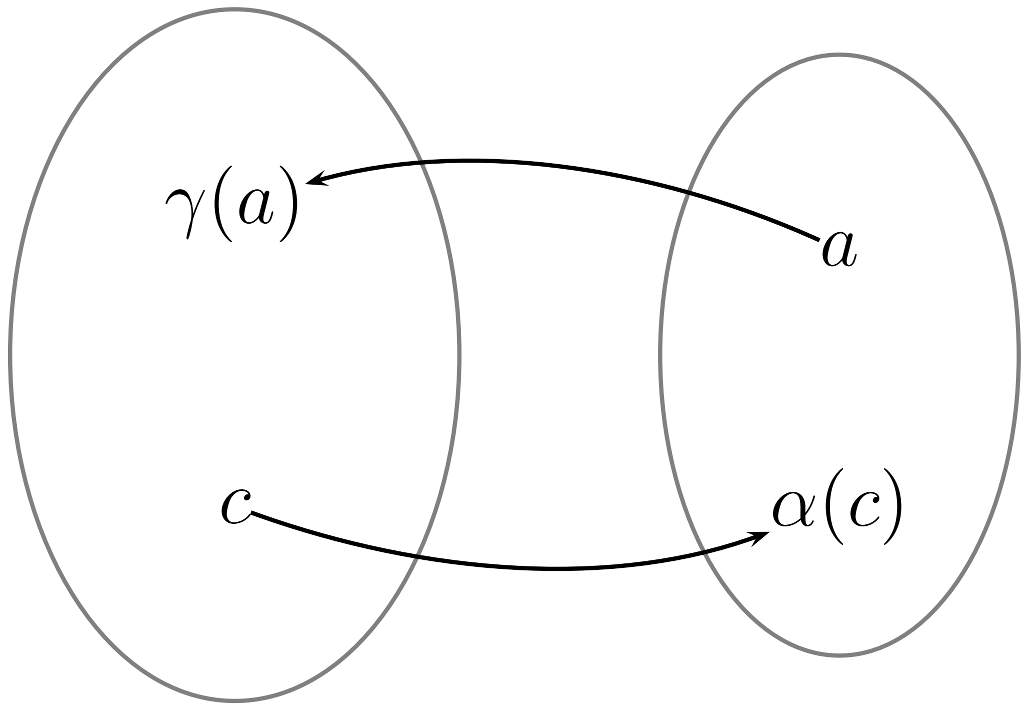


$\langle C; \sqsubseteq \rangle$ $\langle A; \leq \rangle$

such that: $\forall a \in A, c \in C : \alpha(c) \leq a \iff c \sqsubseteq \gamma(a)$

An equivalent definition

Definition. A Galois connection is a pair of functions α and γ satisfying

(a) α and γ are monotone
(for all $c, c' \in C : c \sqsubseteq c' \implies \alpha(c) \leq \alpha(c')$ and
for all $a, a' \in A : a \leq a' \implies \gamma(a) \sqsubseteq \gamma(a')$),

(b) $\alpha \circ \gamma$ is reductive (for all $a \in A : \alpha \circ \gamma(a) \leq a$),

(c) $\gamma \circ \alpha$ is extensive (for all $c \in C : c \sqsubseteq \gamma \circ \alpha(c)$).

Galois connections are typeset as $\langle C; \sqsubseteq \rangle \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \langle A; \leq \rangle$.

Galois connection properties (1/3)

Theorem. For a Galois connection between two complete lattices $\langle C; \sqsubseteq, \bot_c, \top_c, \sqcup, \sqcap \rangle$ and $\langle A; \leq, \bot_a, \top_a, \vee, \wedge \rangle$, α is a complete join-morphism (CJM):

for all $S_c \subseteq C : \alpha(\sqcup S_c) = \vee \alpha(S_c) = \vee \{\alpha(c) \mid c \in S_c\}$

and γ is a complete meet morphism (CMM):

for all $S_a \subseteq A : \gamma(\wedge S_a) = \sqcap \gamma(S_a) = \sqcap \{\gamma(a) \mid a \in S_a\}$

Galois connection properties (2/3)

Theorem. The composition of two Galois connections $\langle C; \sqsubseteq \rangle \xleftrightarrow[\alpha_1]{\gamma_1} \langle B; \subseteq \rangle$ and $\langle B; \subseteq \rangle \xleftrightarrow[\alpha_2]{\gamma_2} \langle A; \leq \rangle$ is itself a Galois connection:

$$\langle C; \sqsubseteq \rangle \xleftrightarrow[\alpha_2 \circ \alpha_1]{\gamma_1 \circ \gamma_2} \langle A; \leq \rangle$$

We can typeset this theorem as an inference rule:

$$\frac{\langle C; \sqsubseteq \rangle \xleftrightarrow[\alpha_1]{\gamma_1} \langle B; \subseteq \rangle \qquad \langle B; \subseteq \rangle \xleftrightarrow[\alpha_2]{\gamma_2} \langle A; \leq \rangle}{\langle C; \sqsubseteq \rangle \xleftrightarrow[\alpha_2 \circ \alpha_1]{\gamma_1 \circ \gamma_2} \langle A; \leq \rangle}$$

Hence Galois connections stack up like Lego bricks!

Galois connection properties (3/3)

Galois connections in which α is surjective / onto (or equivalently γ is injective) are typeset as:

$$\langle C; \sqsubseteq \rangle \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \langle A; \leq \rangle$$

and sometimes called Galois surjections (or insertions)

Galois connections in which α is injective / one-to-one (or equivalently γ is surjective) are typeset as:

$$\langle C; \sqsubseteq \rangle \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \langle A; \leq \rangle$$

and sometimes called Galois injections

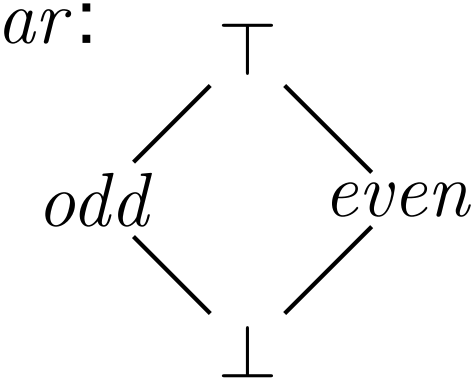
When both α and γ are surjective, the two domains are isomorphic.

Example: The Parity abstract domain

Consider the abstraction into the Parity domain:

$$\langle \wp(\mathbb{N}_0); \subseteq \rangle \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \langle Par; \sqsubseteq \rangle$$

Par:



The above *Hasse diagram* defines the Parity ordering.

The abstraction and concretization functions are:

$$\gamma(P) = \begin{cases} \emptyset & \text{if } P = \bot \\ \{n \in \mathbb{N}_0 \mid n \text{ is odd}\} & \text{if } P = odd \\ \{n \in \mathbb{N}_0 \mid n \text{ is even}\} & \text{if } P = even \\ \mathbb{N}_0 & \text{if } P = \top \end{cases} \qquad \alpha(N) = \begin{cases} \bot & \text{if } N = \emptyset \\ odd & \text{if } \forall n \in N : n \text{ is odd} \\ even & \text{if } \forall n \in N : n \text{ is even} \\ \top & \text{otherwise} \end{cases}$$

Example: an isomorphism

We can represent a set of pairs as a function from a first component to second components:

$$\langle \wp(A \times B); \subseteq \rangle \underset{\alpha}{\overset{\gamma}{\twoheadleftarrow\!\!\!\twoheadrightarrow}} \langle A \to \wp(B); \dot{\subseteq} \rangle$$

$$\text{where} \quad \begin{aligned} \alpha(R) &= \lambda a.\{b \mid (a, b) \in R\} \\ \gamma(F) &= \{(a, b) \mid b \in F(a)\} \end{aligned}$$

Fixed points

Fixed points, briefly

Definition. a *fixed point* of a function f, is a point x such that $f(x) = x$

Assume $f : P \to P$ operates over a poset $\langle P; \sqsubseteq \rangle$

Definition. a *pre-fixed point* is a point x such that $x \sqsubseteq f(x)$

Definition. a *post-fixed point* is a point x such that $f(x) \sqsubseteq x$

Definition. a *least fixed point* (lfp) is a fixed point l such that for all other fixed points $l' : (f(l') = l') \implies l \sqsubseteq l'$

Definition. a *greatest fixed point* (gfp) is a fixed point l such that for all other fixed points $l' : (f(l') = l') \implies l' \sqsubseteq l$

Tarski's fixed point theorem

Theorem. If L is a complete lattice and $f : L \to L$ is a monotone function, f's fixed points themselves form a complete lattice.

Hence Tarski tells us that *there exists a least fixed point*.

Abstract interpretation basics

Abstract interpretation basics

Canonical abstract interpretation approximates the *collecting semantics* of a transition system.

A standard example of a collecting semantics is the *reachable states* from a given set of initial states I.
Given a transition function T defined as:

$$T(\Sigma) = I \cup \{\sigma \mid \exists \sigma' \in \Sigma : \sigma' \to \sigma\}$$

we can express the reachable states of T as the least fixed point $\mathrm{lfp}\, T$ of T.
For a fixed point $T(\Sigma) = \Sigma$ of T:

$$I \subseteq \Sigma \quad \wedge \quad \forall \sigma' \in \Sigma : \sigma' \to \sigma \implies \sigma \in \Sigma$$

which expresses the transitive closure of the states reachable from I.

Abstract interpretation basics

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A standard example of a collecting semantics is the *reachable states* from a given set of initial states I. Given a transition function T defined as:

$$T(\Sigma) = I \cup \{\sigma \mid \exists\sigma' \in \Sigma : \sigma' \to \sigma\}$$

we can express the reachable states of T as the least fixed point $\operatorname{lfp} T$ of T.
We can compute $\operatorname{lfp} T$ by Kleene iteration[1]:

$$\bot, T(\bot), T^2(\bot), T^3(\bot), \ldots$$

[1] In general we can only compute $\operatorname{lfp} f$ if f is contiguous $f(\sqcup S) = \sqcup f(S)$

The strength of the collecting semantics

- The collecting semantics is ideal, i.e., it is the *most precise analysis*.
- Unfortunately it is in general uncomputable: it is as hard as interpreting (i.e., running) a program
- We therefore approximate the collecting semantics, by computing a fixed point over an alternative and perhaps simpler domain: an *abstract* interpretation

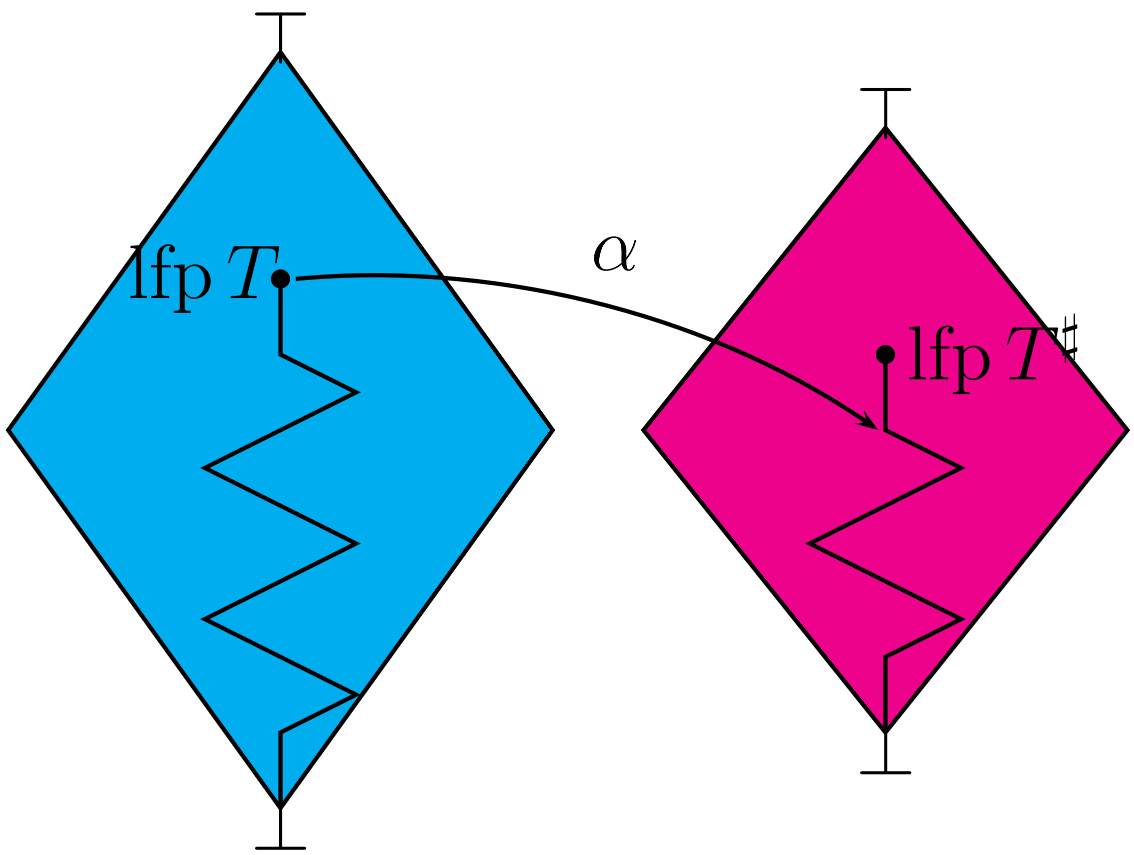
Abstractions are represented as Galois connections which connect complete lattices through α and γ.

We can derive an analysis systematically by composing the transition function with these functions: $\alpha \circ T \circ \gamma$ and gradually refine the collecting semantics into a computable analysis function by mere calculation.

Hence instead of *inventing* a static analysis, we arrive at one by a *structured abstraction* of the set of states $\wp(S)$.

Galois connection-based analysis

By the *fixed point transfer theorem* we can compute a sound approximation of the collecting semantics:



Theorem. Let $\langle C; \sqsubseteq \rangle \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \langle A; \leq \rangle$ be a Galois connection between complete lattices. If T and $T^\sharp$ are monotone and $\alpha \circ T \circ \gamma \mathrel{\dot{\leq}} T^\sharp$ then $\alpha(\operatorname{lfp} T) \leq \operatorname{lfp} T^\sharp$

Variations

An alternative approach

Rather than simplifying the abstract domains into finite ones, *widening* and *narrowing* permits infinite ones.

A first widening iteration overshoots the least fixed point but still ensures termination.

A second narrowing iteration improves the results of the widening iteration.

Widening

We compute instead the limit of the sequence:

$$X_0 = \bot$$
$$X_{i+1} = X_i \triangledown T(X_i)$$

where $\triangledown$ denotes the *widening operator*: an operator with the following properties:

- For all $x, y : x \sqsubseteq (x \triangledown y) \ \wedge \ y \sqsubseteq (x \triangledown y)$
- For any increasing chain $Y_0 \sqsubseteq Y_1 \sqsubseteq Y_2 \sqsubseteq \ldots$ the alternative chain defined as $Y_0' = Y_0$ and $Y_{i+1}' = Y_i' \triangledown Y_{i+1}$ stabilizes after a finite amount of steps.

Narrowing

We can compute the limit of the sequence:

$$X_0 = \lim_i Y_i$$
$$X_{i+1} = X_i \vartriangle T(X_i)$$

where $\vartriangle$ denotes the *narrowing operator*: an operator with the following properties:

- For all $x, y : (x \vartriangle y) \sqsubseteq x$
- For all $x, y, z : (x \sqsubseteq y \ \wedge \ x \sqsubseteq z) \implies x \sqsubseteq (y \vartriangle z)$
- For any chain Y_i the alternative chain defined as $Y_0' = Y_0$ and $Y_{i+1}' = Y_i' \vartriangle Y_{i+1}$ stabilizes after a finite amount of steps.