« Basic Concepts of Abstract Interpretation »

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IFIP WCC — Topical day on Abstract Interpretation

Motivations

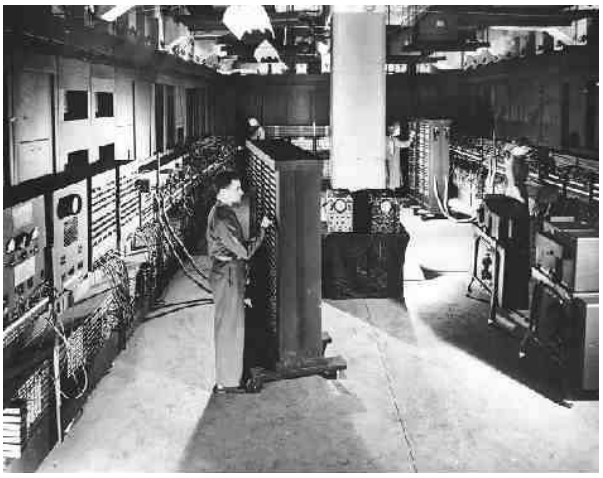
What is (or should be) the essential preoccupation of computer scientists?

What is (or should be) the essential preoccupation of computer scientists?

The production of reliable software, its maintenance and safe evolution year after year (up to 20 even 30 years).

Computer hardware change of scale

The 25 last years, computer hardware has seen its performances multiplied by 10^4 to $10^6/10^9$;

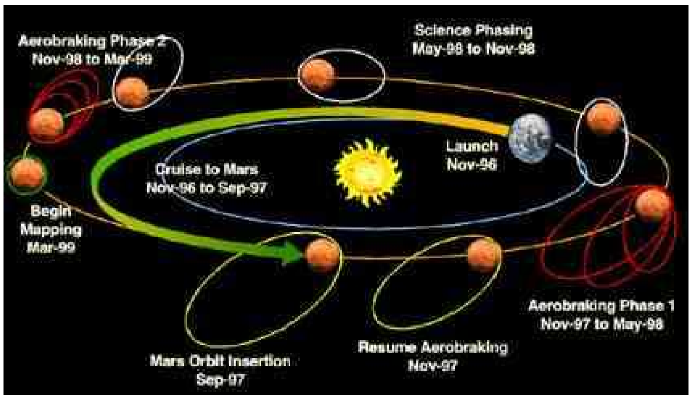

ENIAC (5000 flops)

Intel/Sandia Teraflops System (10^{12} flops)

The information processing revolution

A scale of 10^6 is typical of a significant **revolution**:

- Energy: nuclear power station / Roman slave;
- Transportation: distance Earth — Mars / Paris — Toulouse

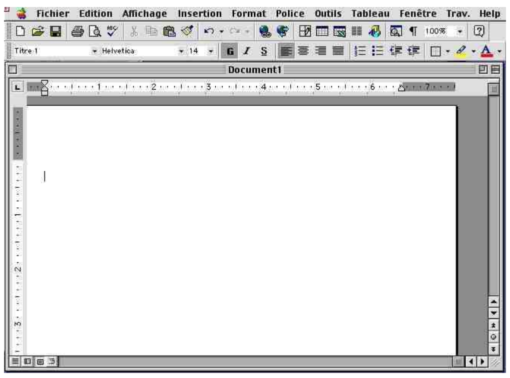


Computer software change of scale

– The size of the programs executed by these computers has grown up in similar proportions;

– **Example 1** (modern text editor for the general public):

- $>$ 1 700 000 lines of C[1];
- 20 000 procedures;
- 400 files;
- $>$ 15 years of development.

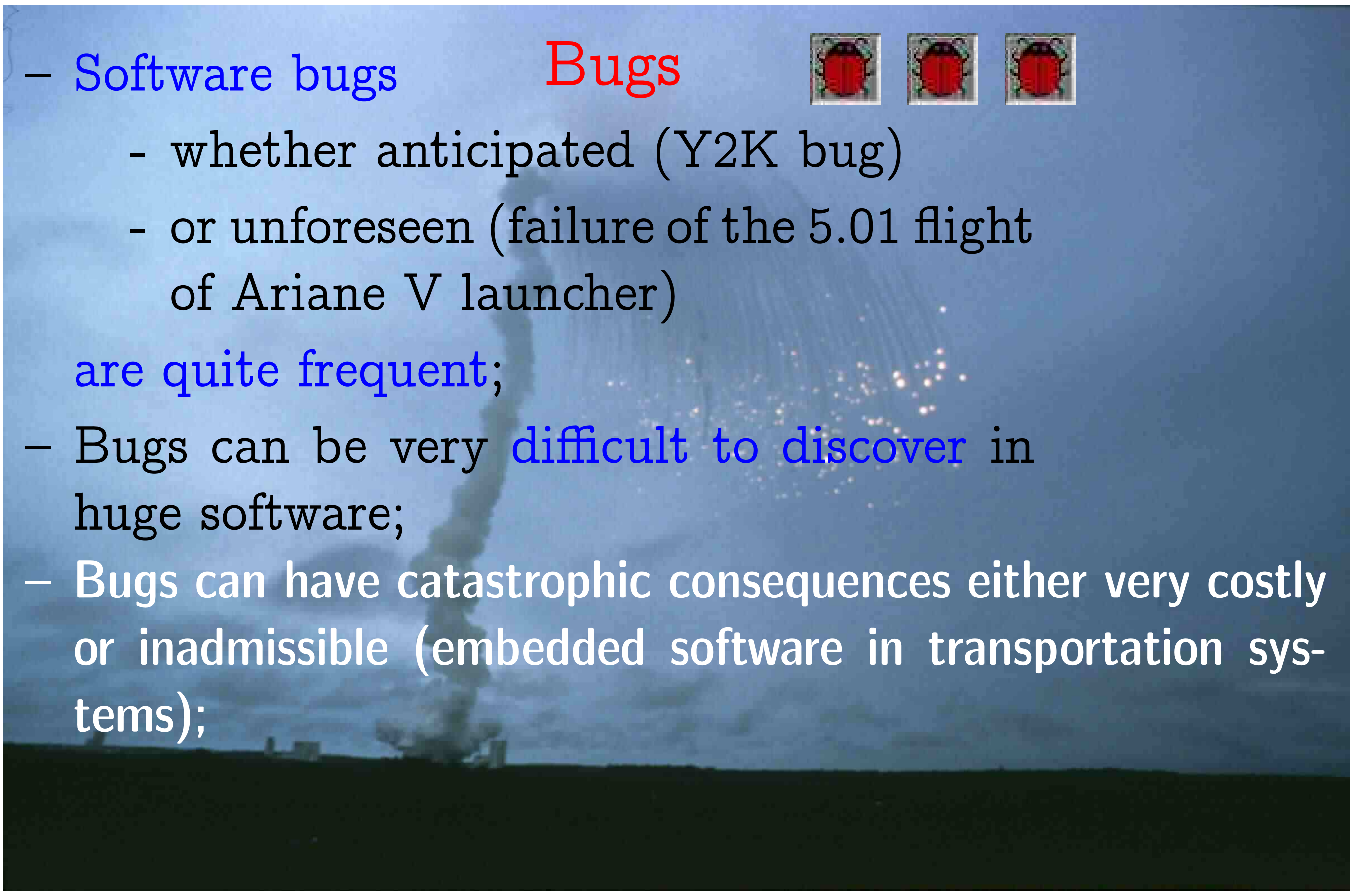


[1] full-time reading of the code (35 hours/week) would take at least 3 months!

Computer software change of scale (cont'd)

– **Example 2** (professional computer system):

- 30 000 000 lines of code;
- 30 000 (known) bugs!



Bugs

– Software bugs

- whether anticipated (Y2K bug)
- or unforeseen (failure of the 5.01 flight of Ariane V launcher)

are quite frequent;

– Bugs can be very difficult to discover in huge software;

– Bugs can have catastrophic consequences either very costly or inadmissible (embedded software in transportation systems);

The estimated cost of an overflow

- **500 000 000 $**;
- Including indirect costs (delays, lost markets, etc):

  2 000 000 000 $;

- The financial results of Arianespace were **negative** in 2000, for the first time since 20 years.

Who cares?

- No one is legally responsible for bugs:

  This software is distributed WITHOUT ANY WARRANTY; without even the implied warranty of MERCHANTABILITY or FITNESS FOR A PARTICULAR PURPOSE.

- So, no one cares about software verification
- And even more, one can even make money out of bugs (customers buy the next version to get around bugs in software)

Why no one cares?

– Software designers don't care because there is no risk in writing bugged software
– The law/judges can never enforce more than what is offered by the state of the art
– Automated software verification by formal methods is undecidable whence thought to be impossible
– Whence the state of the art is that no one will ever be able to eliminate all bugs at a reasonable price
– And so no one ever bear any responsability

Current research results

– Research is presently changing the state of the art (e.g. ASTRÉE)
– We can check for the absence of large categories of bugs (may be not all of them but a significant portion of them)
– The verification can be made automatically by mechanical tools
– Some bugs can be found completely automatically, without any human intervention

The next step (5 years)

– If these tools are successful, their use can be enforced by quality norms
– Professional have to conform to such norms (otherwise they are not credible)
– Because of complete tool automaticity, no one can be discharged from the duty of applying such state of the art tools
– Third parties of confidence can check software a posteriori to trace back bugs and prove responsabilities

A foreseeable future (10 years)

– The real take-off of software verification must be enforced
– Development costs arguments have shown to be ineffective
– Norms/laws might be much more convincing
– This requires effectiveness and complete automation (to avoid acquittal based on human capacity limitations arguments)

Why will "partial software verification" ultimately succeed?

– The state of the art will change toward complete automation, at least for common categories of bugs
– So responsabilities can be established (at least for automatically detectable bugs)
– Whence the law will change (by adjusting to the new state of the art)
– To ensure at least partial software verification
– For the benefit of all of us

Static analysis by abstract interpretation

Example of static analysis (input)

```
n := n0;

i := n;

while (i <> 0 ) do

      j := 0;

      while (j <> i) do

            j := j + 1

      od;

      i := i - 1

od
```

Example of static analysis (output)

```
{n0>=0}
   n := n0;
{n0=n,n0>=0}
   i := n;
{n0=i,n0=n,n0>=0}
   while (i <> 0 ) do
      {n0=n,i>=1,n0>=i}
         j := 0;
      {n0=n,j=0,i>=1,n0>=i}
         while (j <> i) do
            {n0=n,j>=0,i>=j+1,n0>=i}
               j := j + 1
            {n0=n,j>=1,i>=j,n0>=i}
         od;
      {n0=n,i=j,i>=1,n0>=i}
         i := i - 1
      {i+1=j,n0=n,i>=0,n0>=i+1}
   od
```

```
{n0=n,i=0,n0>=0}
```

Example of static analysis (safety)

```
{n0>=0}  <---------------------------- n0 must be initially nonnegative
   n := n0;                            (otherwise the program does not
{n0=n,n0>=0}                           terminate properly)
   i := n;
{n0=i,n0=n,n0>=0}
   while (i <> 0 ) do
      {n0=n,i>=1,n0>=i}
         j := 0;
      {n0=n,j=0,i>=1,n0>=i}
         while (j <> i) do
            {n0=n,j>=0,i>=j+1,n0>=i}
               j := j + 1      <---  j < n0 so no upper overflow
            {n0=n,j>=1,i>=j,n0>=i}
         od;
      {n0=n,i=j,i>=1,n0>=i}
         i := i - 1            <---  i > 0 so no lower overflow
      {i+1=j,n0=n,i>=0,n0>=i+1}
   od
{n0=n,i=0,n0>=0}
```

Static analysis by abstract interpretation

Verification: define and prove automatically a property of the possible behaviors of a complex computer program (example: program semantics);

Abstraction: the reasoning/calculus can be done on an abstraction of these behaviors dealing only with those elements of the behaviors related to the considered property;

Theory: abstract interpretation.

Example of static analysis

Verification: absence of runtime errors;

Abstraction: polyhedral abstraction (affine inequalities);

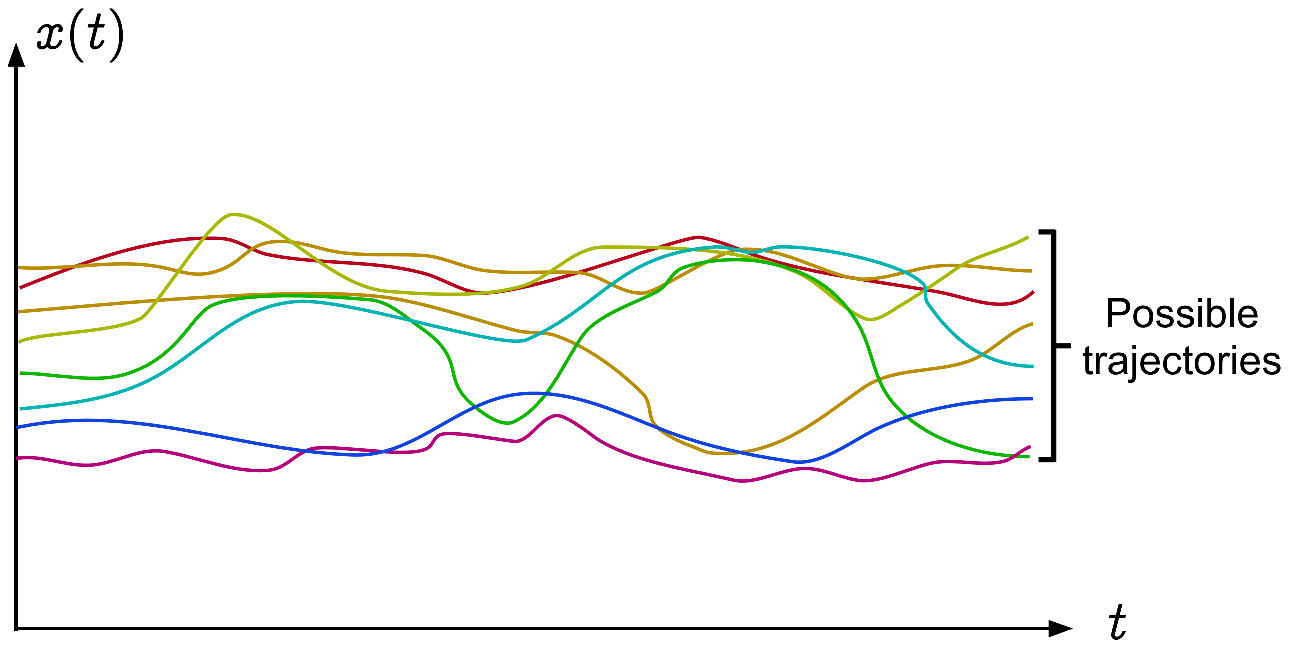
Theory: abstract interpretation.

A very informal introduction to the principles of abstract interpretation

Semantics

The *concrete semantics* of a program formalizes (is a mathematical model of) the set of all its possible executions in all possible execution environments.

Graphic example: Possible behaviors



Undecidability

- The concrete mathematical semantics of a program is an “tinfinite” mathematical object, *not computable*;
- All non trivial questions on the concrete program semantics are *undecidable*.

Example: termination

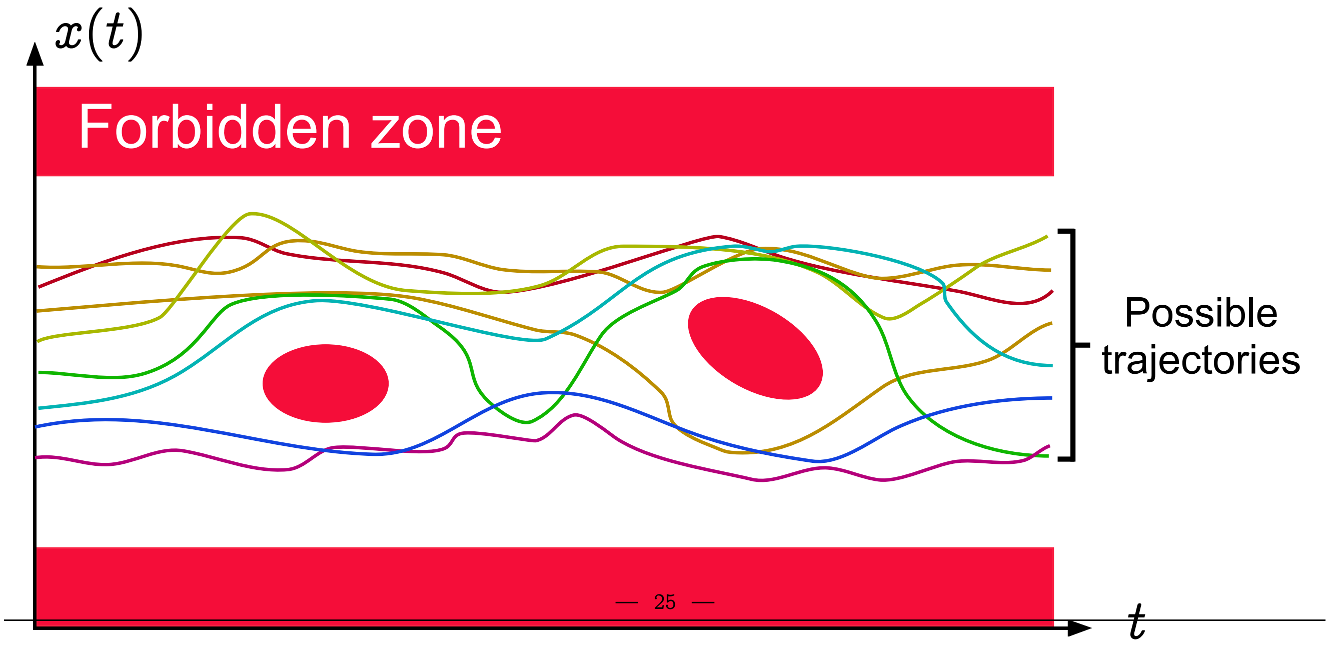
- Assume `termination(P)` would always terminates and returns `true` iff `P` always terminates on all input data;
- The following program yields a contradiction

$$\texttt{P} \equiv \texttt{while termination(P) do skip od.}$$

Graphic example: Safety properties

The *safety properties* of a program express that no possible execution in any possible execution environment can reach an erroneous state.

Graphic example: Safety property



Safety proofs

– A safety proof consists in proving that the intersection of the program concrete semantics and the forbidden zone is empty;

– Undecidable problem (the concrete semantics is not computable);

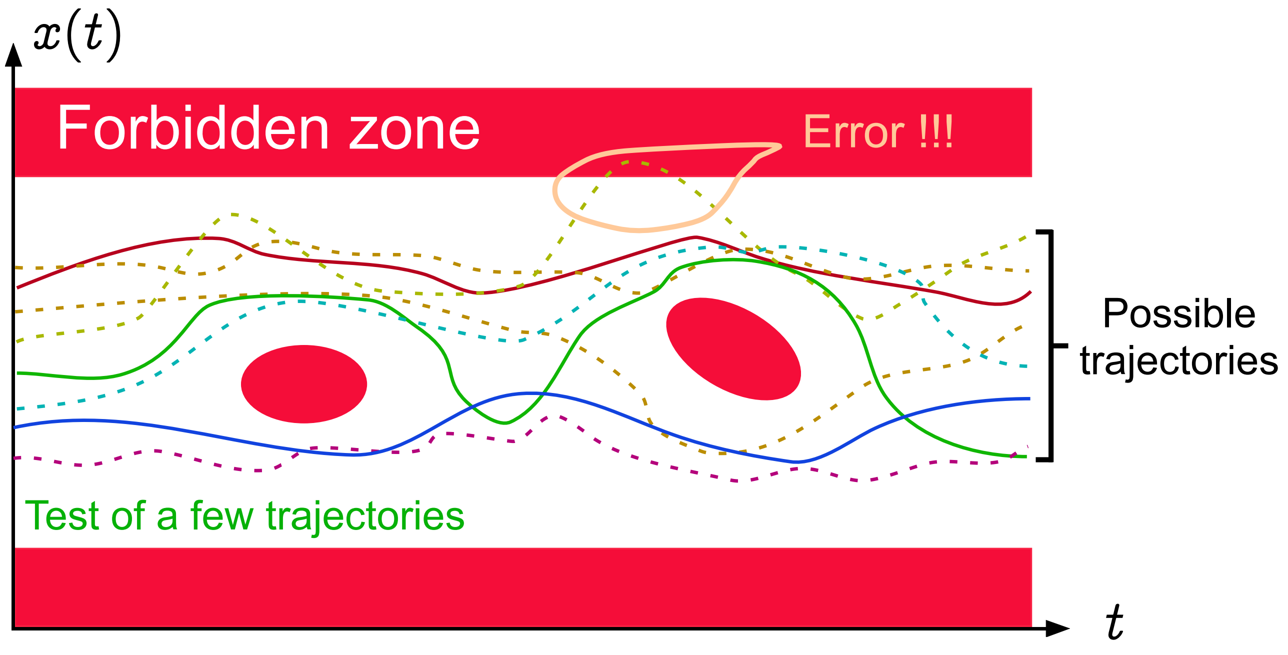
– Impossible to provide completely automatic answers with finite computer resources and neither human interaction nor uncertainty on the answer[2].

[2] e.g. probabilistic answer.

Test/debugging

– consists in considering a subset of the possible executions;
– not a correctness proof;
– absence of coverage is the main problem.

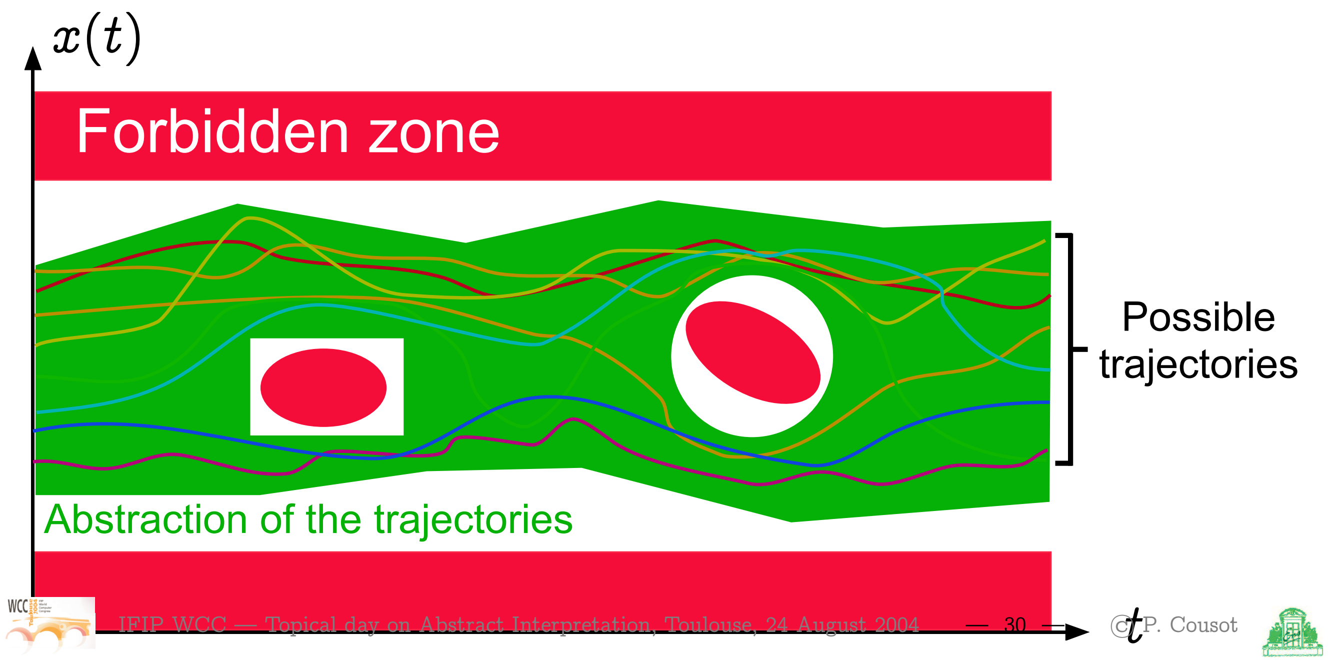
Graphic example: Property test/simulation



Abstract interpretation

- consists in considering an *abstract semantics*, that is to say a superset of the concrete semantics of the program;
- hence the abstract semantics covers all possible concrete cases;
- correct: if the abstract semantics is safe (does not intersect the forbidden zone) then so is the concrete semantics

Graphic example: Abstract interpretation



Formal methods

Formal methods are abstract interpretations, which differ in the way to obtain the abstract semantics:

– "*model checking*":
  - the abstract semantics is given manually by the user;
  - in the form of a finitary model of the program execution;
  - can be computed automatically, by techniques relevant to static analysis.

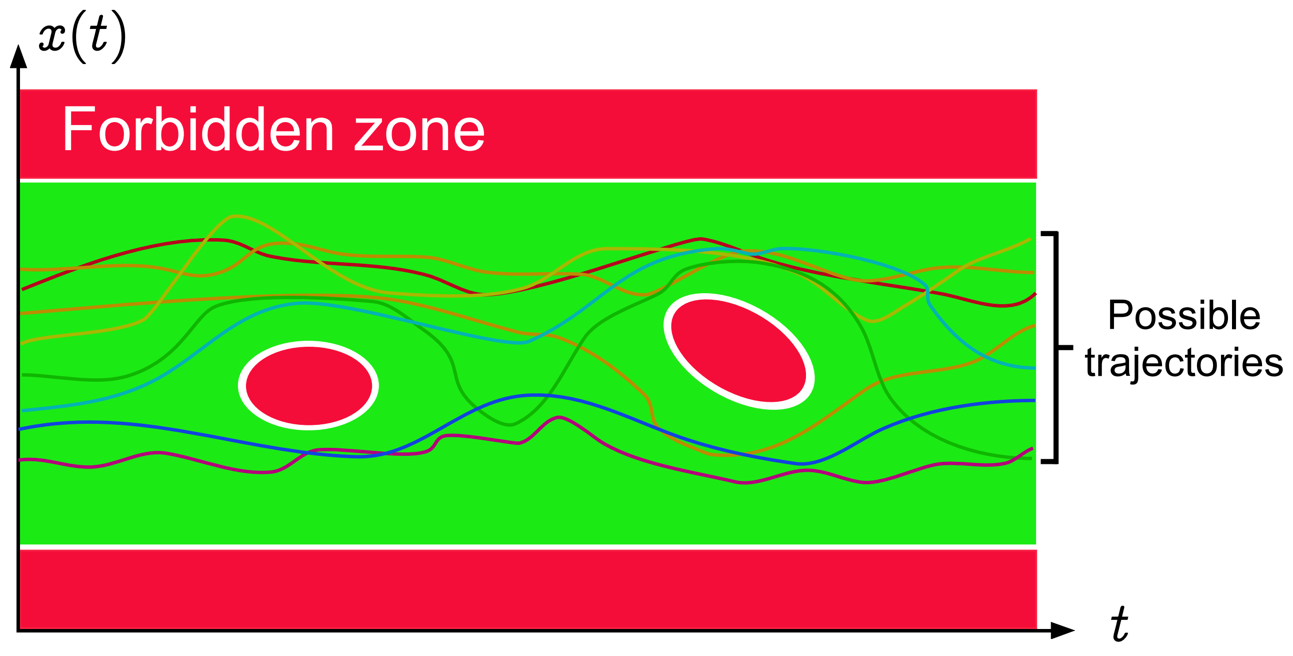
– "*deductive methods*":
  - the abstract semantics is specified by verification conditions;
  - the user must provide the abstract semantics in the form of inductive arguments (e.g. invariants);
  - can be computed automatically by methods relevant to static analysis.

– "*static analysis*": the abstract semantics is computed automatically from the program text according to predefined abstractions (that can sometimes be tailored automatically/manually by the user).

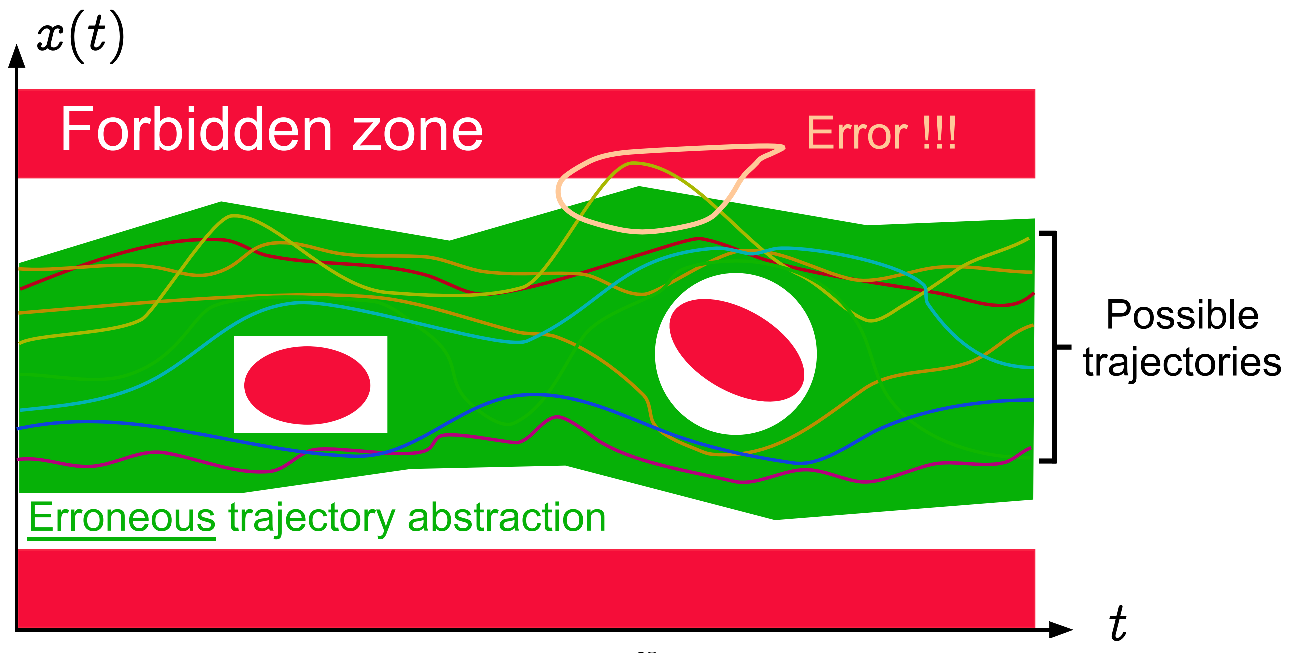
Required properties of the abstract semantics

- sound so that no possible error can be forgotten;
- precise enough (to avoid false alarms);
- as simple/abstract as possible (to avoid combinatorial explosion phenomena).

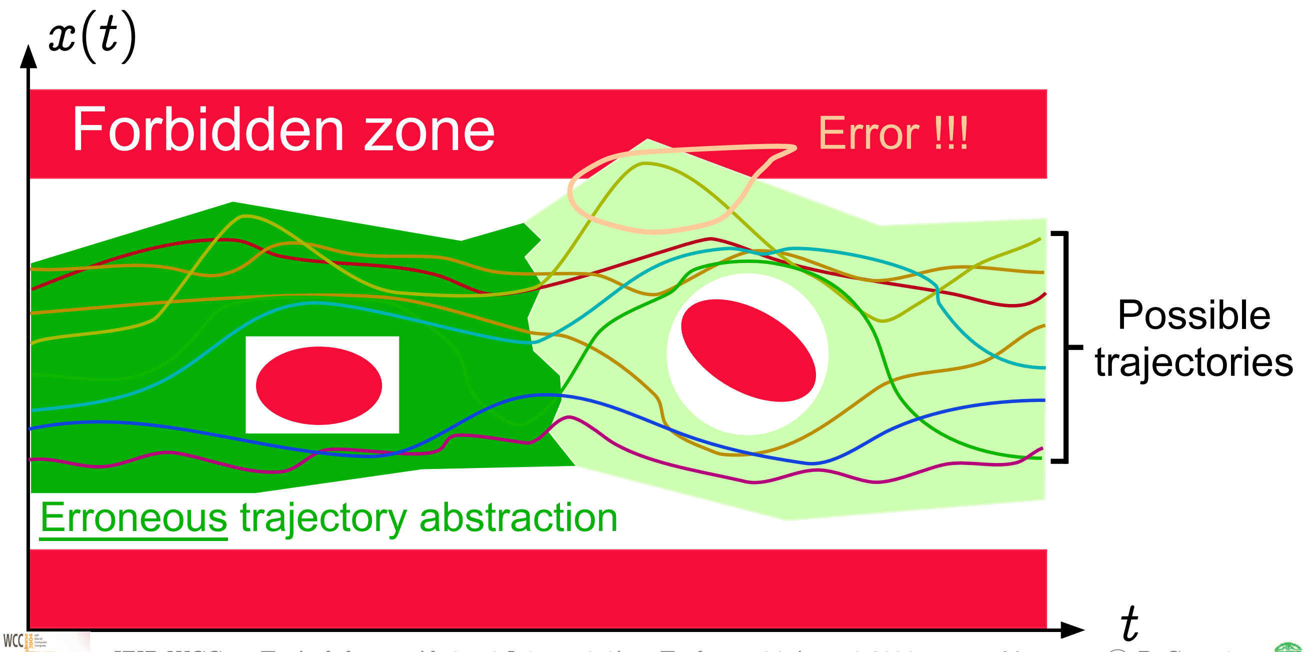
Graphic example: The most abstract correct and precise semantics



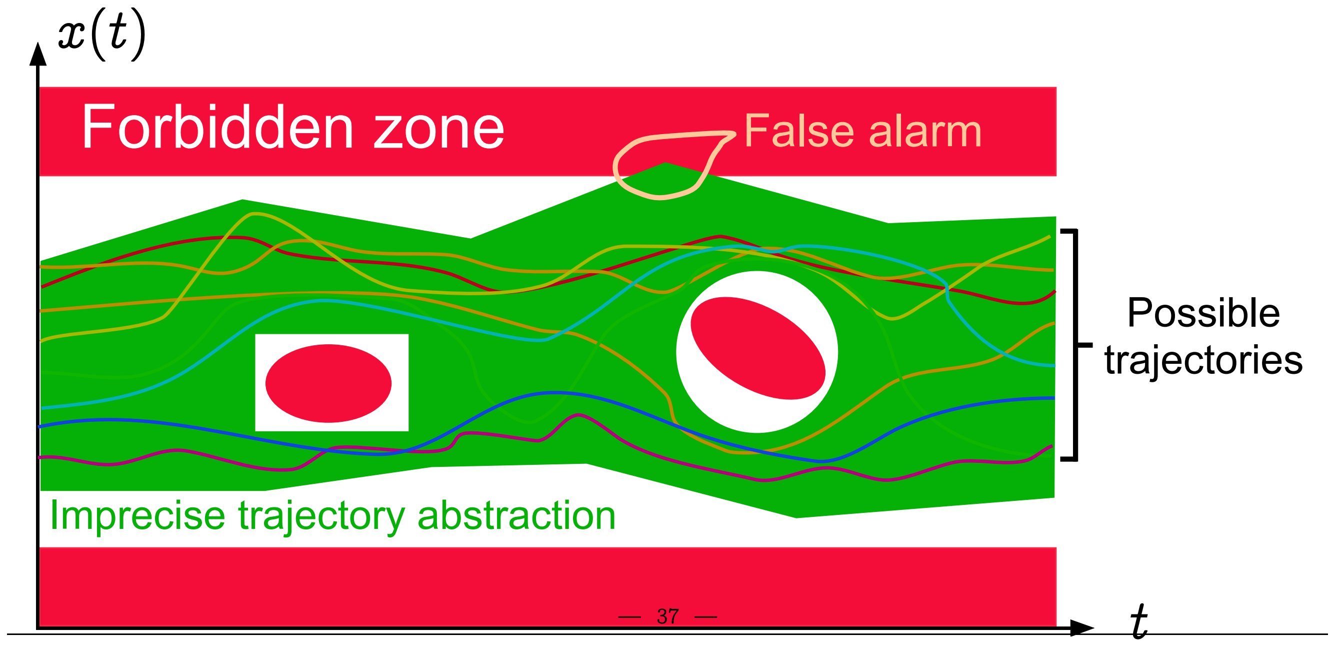
Graphic example: Erroneous abstraction — I



Graphic example: Erroneous abstraction — II



Graphic example: Imprecision ⇒ false alarms



Abstract domains

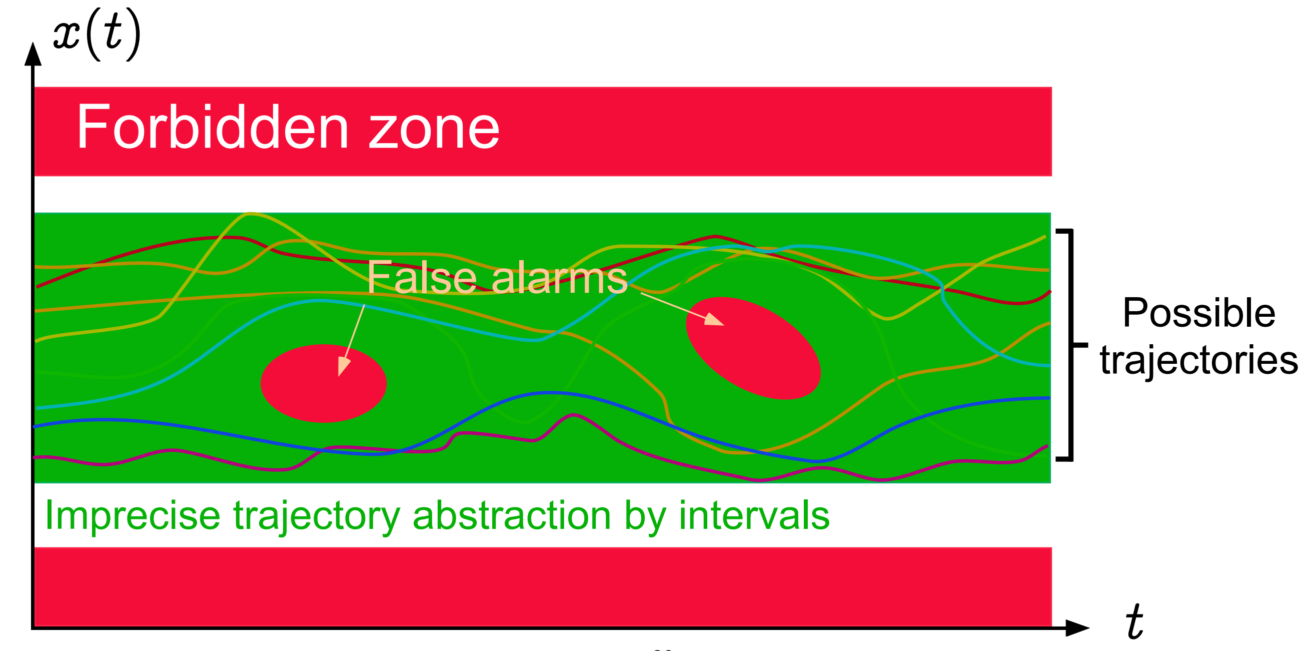
Standard abstractions

– that serve as a basis for the design of static analyzers:

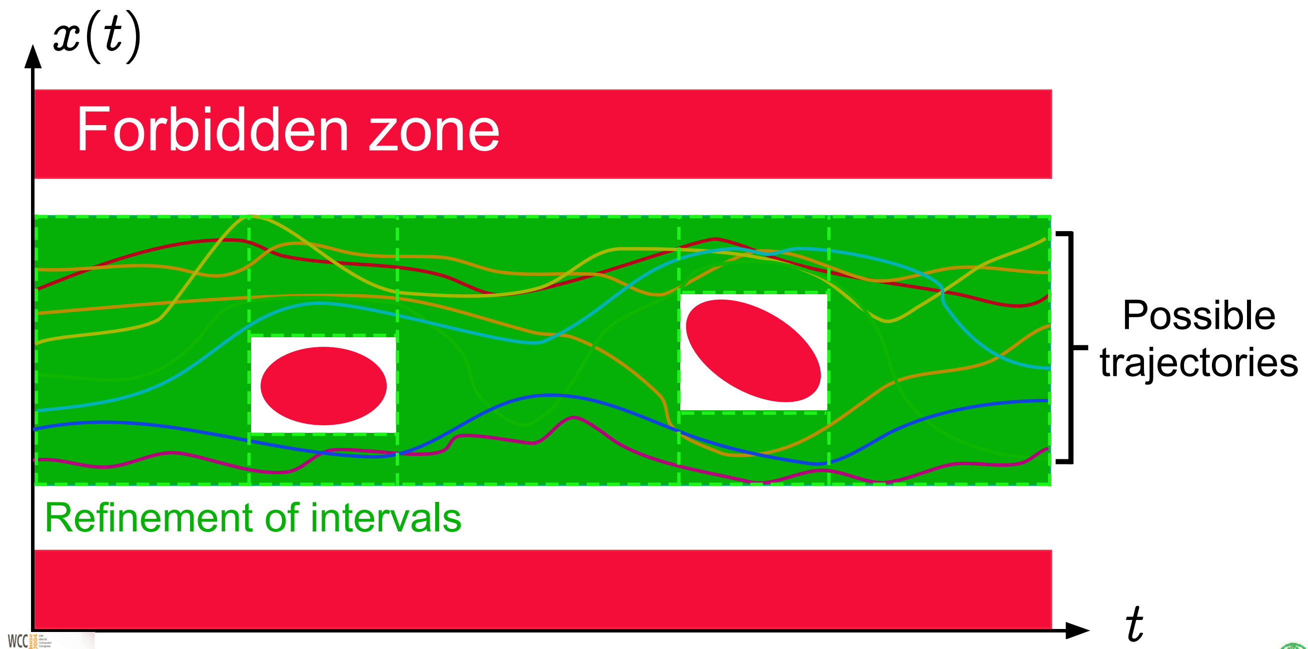
- abstract program data,
- abstract program basic operations;
- abstract program control (iteration, procedure, concurrency, . . .);

– can be parametrized to allow for manual adaptation to the application domains.

Graphic example: Standard abstraction by intervals



Graphic example: A more refined abstraction



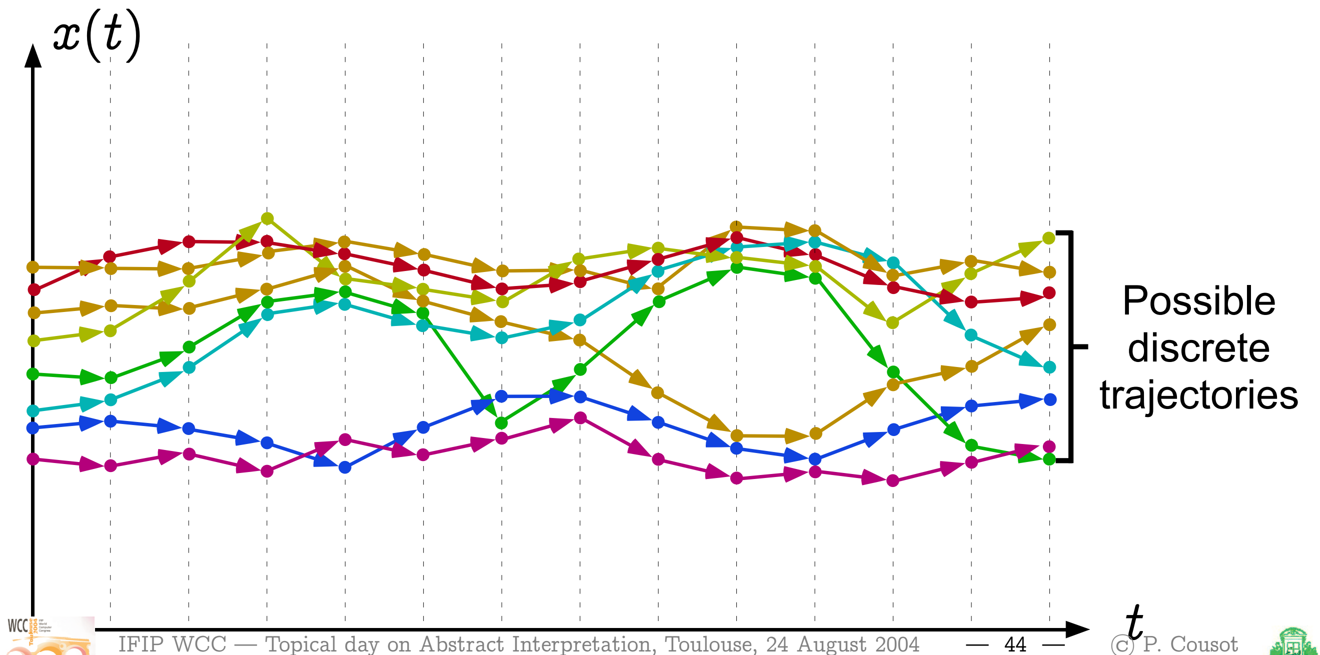
A very informal introduction to static analysis algorithms

Standard operational semantics

Standard semantics

– Start from a standard operational semantics that describes formally:
  - states that is data values of program variables,
  - transitions that is elementary computation steps;
– Consider traces that is successions of states corresponding to executions described by transitions (possibly infinite).

Graphic example: Small-steps transition semantics



Example: Small-steps transition semantics of an assignment

```
int x;
...
l:
    x := x + 1;
l':
```

$$\{\texttt{l} : \texttt{x} = v \rightarrow \texttt{l}' : \texttt{x} = v + 1 \mid v \in [\texttt{min\_int}, \texttt{max\_int} - 1]\}$$
$$\cup \{\texttt{l} : \texttt{x} = \texttt{max\_int} \rightarrow \texttt{l}' : \texttt{x} = \Omega\} \qquad \text{(runt me error)}$$

Example: Small-steps transition semantics of a loop

```
l1:
   x := 1;
l2:
   while x < 10 do
l3:
      x := x + 1
l4:
   od
l5:
```

$$\left.\begin{array}{c} \texttt{l1} : \ldots \\ \texttt{l1} : \texttt{x} = -1 \\ \texttt{l1} : \texttt{x} = 0 \\ \texttt{l1} : \texttt{x} = 1 \\ \texttt{l1} : \ldots \end{array}\right| \begin{array}{l} \searrow \\ \rightarrow \texttt{l2} : \texttt{x} = 1 \\ \nearrow \end{array}$$

$$\texttt{l2} : \texttt{x} = 1 \rightarrow \texttt{l3} : \texttt{x} = 1$$
$$\texttt{l3} : \texttt{x} = 1 \rightarrow \texttt{l4} : \texttt{x} = 2$$
$$\texttt{l4} : \texttt{x} = 2 \rightarrow \texttt{l3} : \texttt{x} = 2$$
$$\texttt{l3} : \texttt{x} = 2 \rightarrow \texttt{l4} : \texttt{x} = 3$$
$$\ldots$$
$$\texttt{l4} : \texttt{x} = 10 \rightarrow \texttt{l5} : \texttt{x} = 10$$

Example: Trace semantics of loop

$$\left.\begin{array}{c}\mathtt{l1}:\ldots\\ \mathtt{l1}:\mathtt{x}=-1\\ \mathtt{l1}:\mathtt{x}=0\\ \mathtt{l1}:\mathtt{x}=1\\ \mathtt{l1}:\ldots\end{array}\right| \begin{array}{c}\searrow\\ \rightarrow\\ \nearrow\end{array} \mathtt{l2}:\mathtt{x}=1 \rightarrow \mathtt{l3}:\mathtt{x}=1 \rightarrow \mathtt{l4}:\mathtt{x}=2 \rightarrow$$

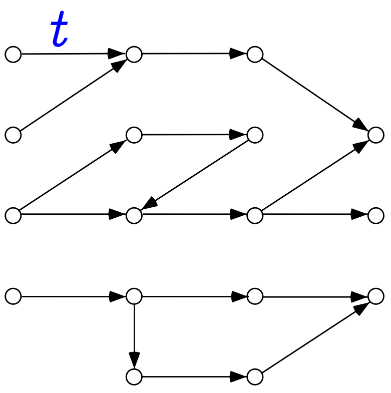
$$\mathtt{l3}:\mathtt{x}=2 \rightarrow \mathtt{l4}:\mathtt{x}=3\ldots \rightarrow \mathtt{l4}:\mathtt{x}=10 \rightarrow \mathtt{l5}:\mathtt{x}=10$$

```
l1:
    x := 1;
l2:
    while x < 10 do
l3:
        x := x + 1
l4:
    od
l5:
```

Transition systems

– $\langle S, \xrightarrow{t}\rangle$ where:

- S is a set of states/vertices/...
- $\xrightarrow{t} \in \wp(S \times S)$ is a transition relation/set of arcs/...

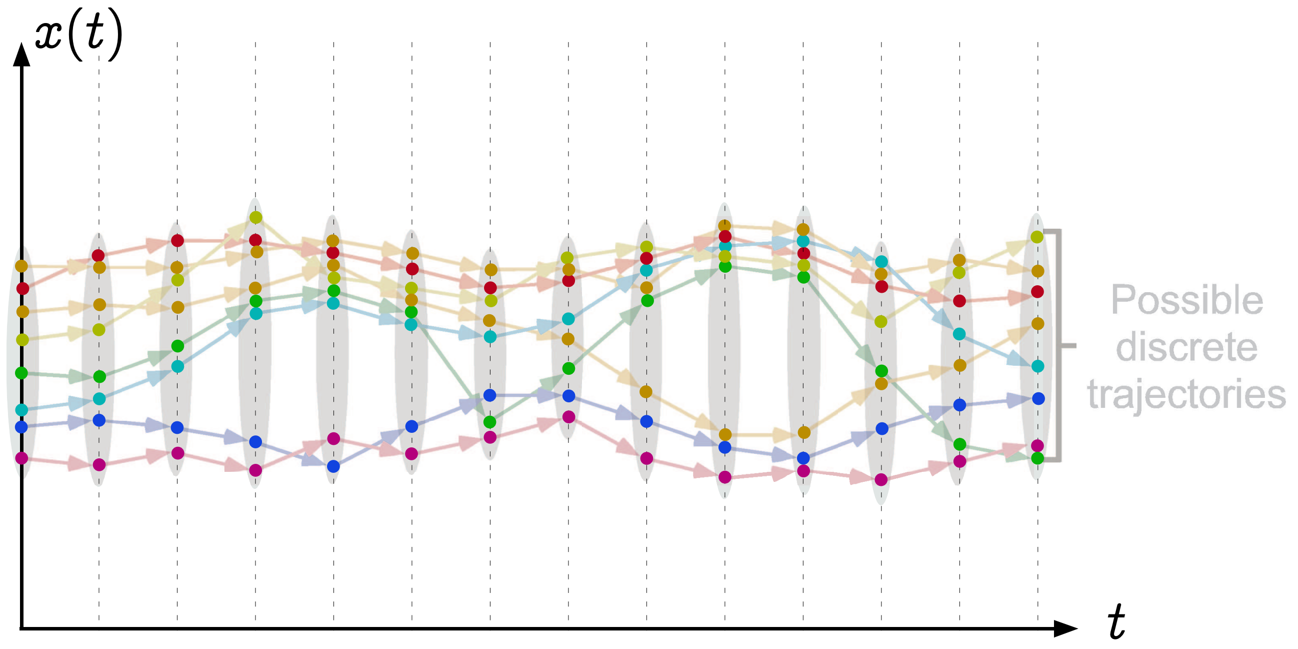


Collecting semantics in fixpoint form

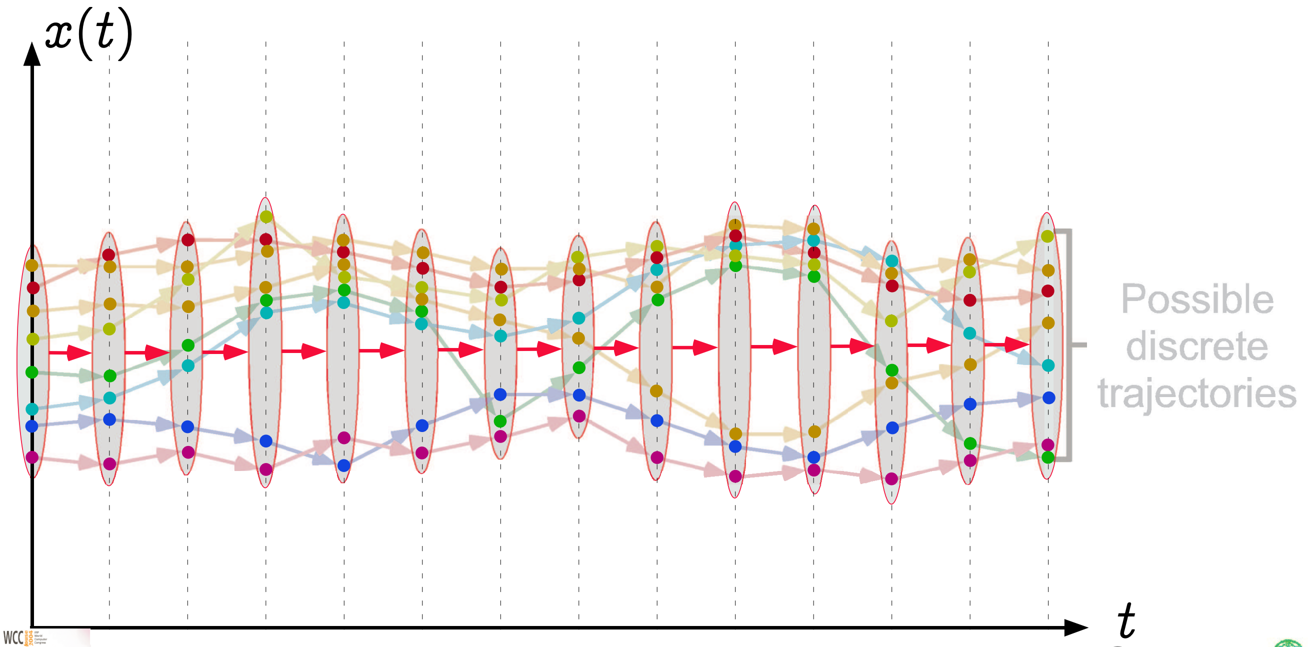
Collecting semantics

– consider all traces simultaneously;
– collecting semantics:
  - sets of states that describe data values of program variables on all possible trajectories;
  - set of states transitions that is simultaneous elementary computation steps on all possible trajectories;

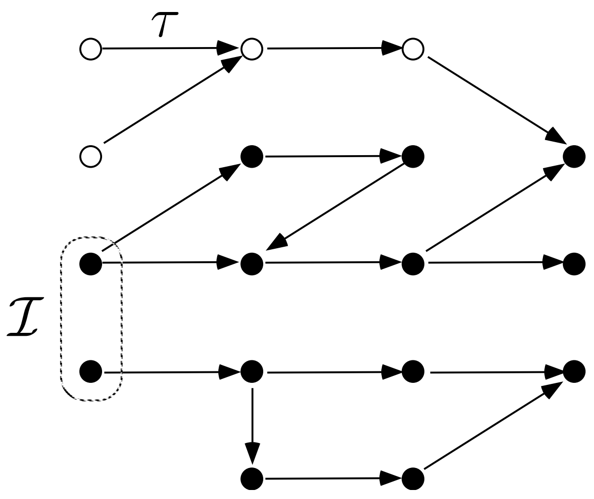
Graphic example: sets of states



Graphic example: set of states transitions



Example: Reachable states of a transition system



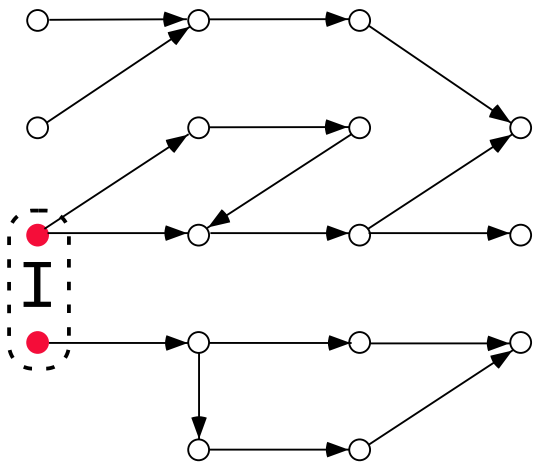
Reachable states in fixpoint form

$$F(X) = \mathcal{I} \cup \{s' \mid \exists s \in X : s \xrightarrow{t} s'\}$$

$$\mathcal{R} = \mathbf{lfp}^{\subseteq}_{\emptyset}\ F$$

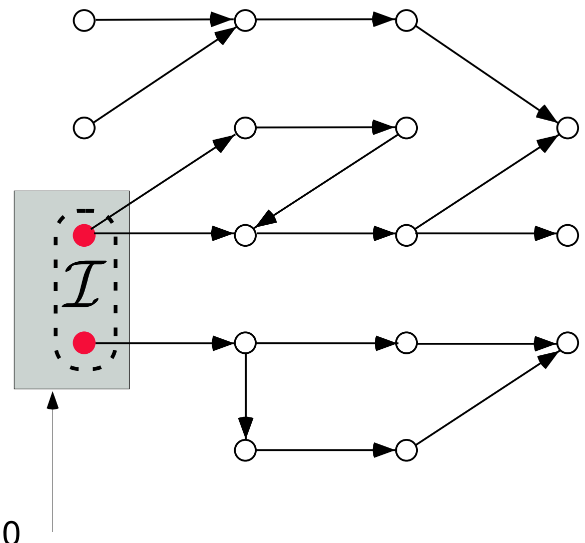
$$= \bigcup_{n=0}^{+\infty} F^n(\emptyset) \qquad \text{where} \qquad \begin{aligned} f^0(x) &= x \\ f^{n+1}(x) &= f(f^n(x)) \end{aligned}$$

Example of fixpoint iteration for reachable states $\mathsf{lfp}_{\emptyset}^{\subseteq} \lambda X \cdot \mathcal{I} \cup \{s' \mid \exists s \in X : s \xrightarrow{t} s'\}$



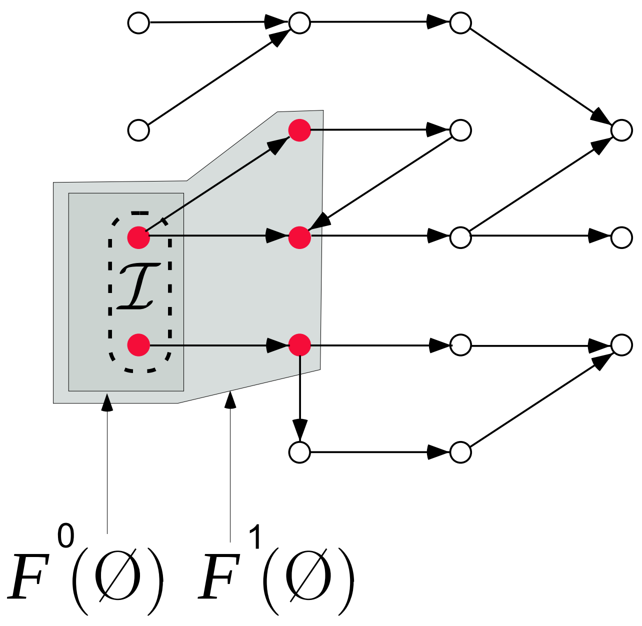
$\emptyset$

Example of fixpoint iteration for reachable states $\mathsf{lfp}_{\emptyset}^{\subseteq} \lambda X \cdot \mathcal{I} \cup \{s' \mid \exists s \in X : s \xrightarrow{t} s'\}$

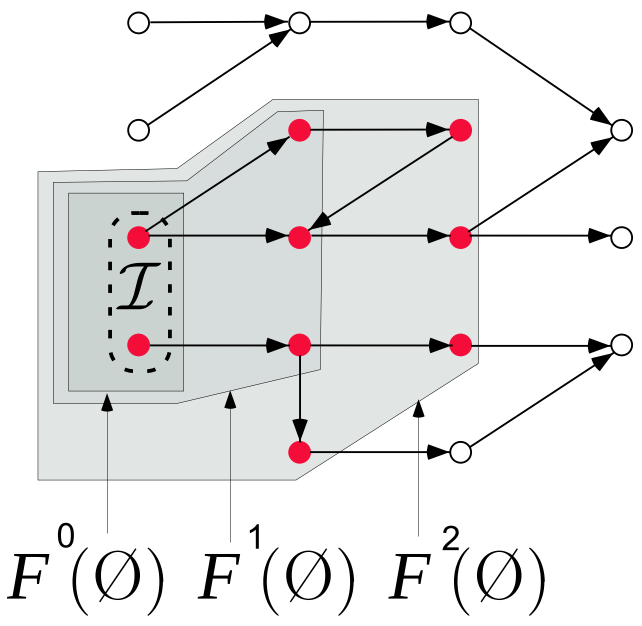


$F^0(\emptyset)$

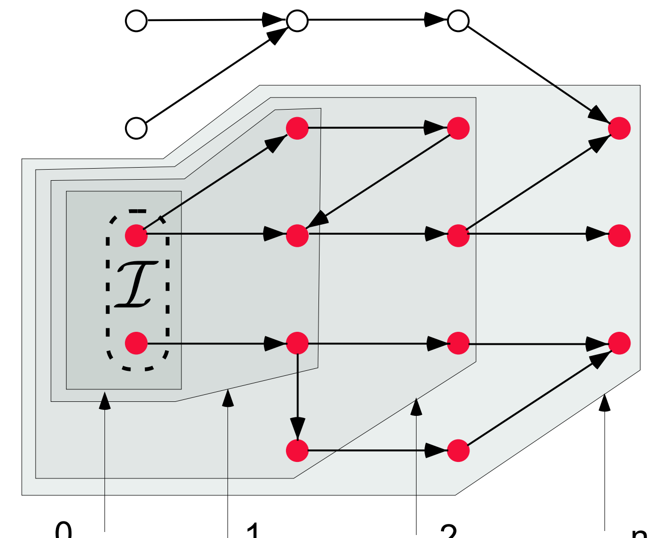
Example of fixpoint iteration for reachable states $\mathsf{lfp}_{\emptyset}^{\subseteq} \lambda X \cdot \mathcal{I} \cup \{s' \mid \exists s \in X : s \xrightarrow{t} s'\}$



Example of fixpoint iteration for reachable states $\mathsf{lfp}^{\subseteq}_{\emptyset}\, \lambda X \cdot \mathcal{I} \cup \{s' \mid \exists s \in X : s \xrightarrow{t} s'\}$



Example of fixpoint iteration for reachable states $\mathsf{lfp}^{\subseteq}_{\emptyset}\, \lambda X \cdot \mathcal{I} \cup \{s' \mid \exists s \in X : s \xrightarrow{t} s'\}$



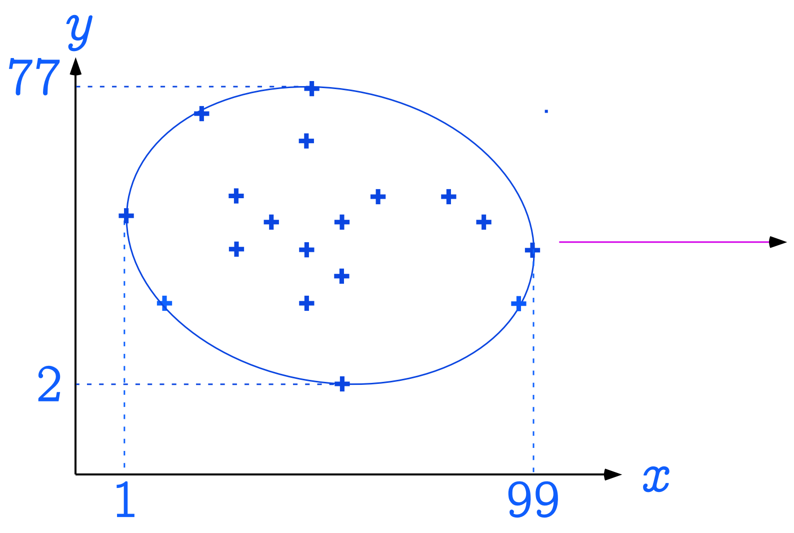
$F^0(\emptyset)$ $F^1(\emptyset)$ $F^2(\emptyset)$ $F^n(\emptyset)$, $n \geq 0$

Abstraction by Galois connections

Abstracting sets (i.e. properties)

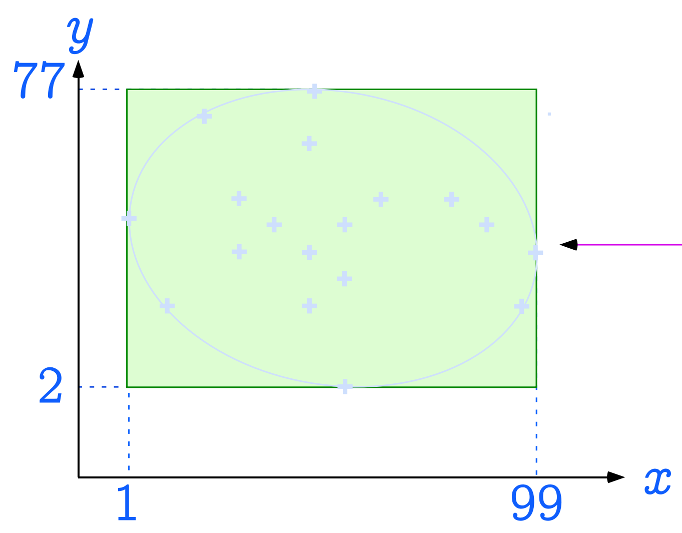
- Choose an abstract domain, replacing sets of objects (states, traces, ...) S by their abstraction $\alpha(S)$
- The abstraction function α maps a set of concrete objects to its abstract interpretation;
- The inverse concretization function γ maps an abstract set of objects to concrete ones;
- Forget no concrete objects: (abstraction from above) $S \subseteq \gamma(\alpha(S))$.

Interval abstraction α



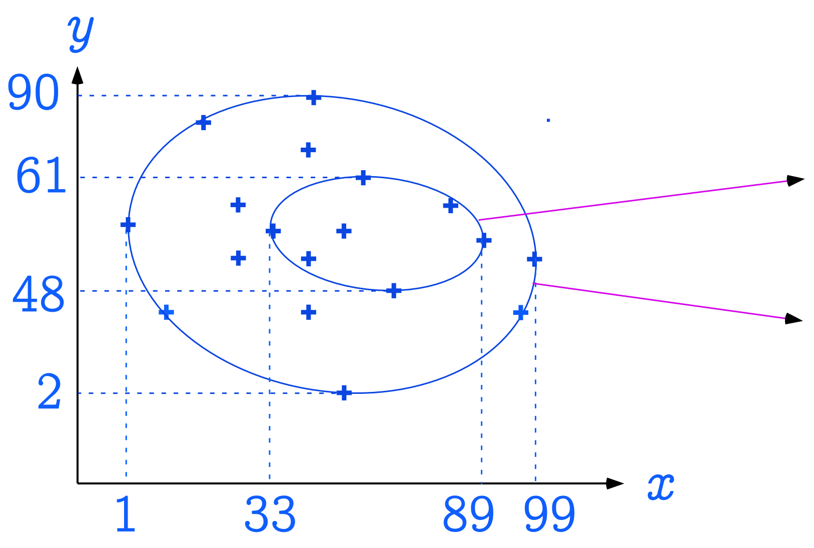
$\{x : [1, 99], y : [2, 77]\}$

Interval concretization γ



$\{x : [1, 99], y : [2, 77]\}$

The abstraction α is monotone



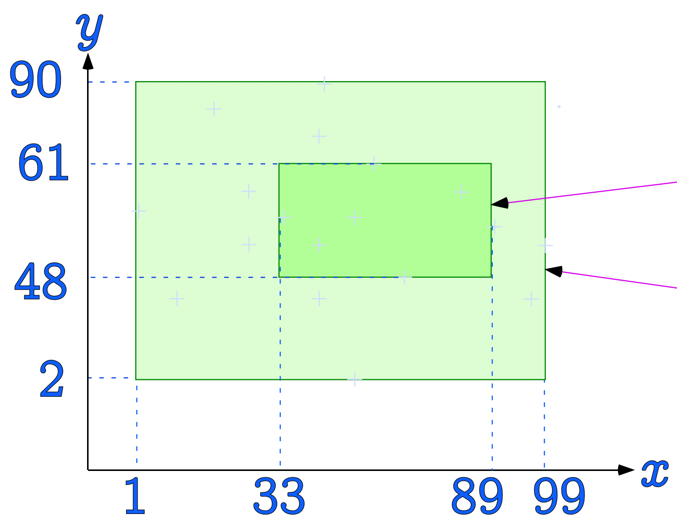
$\{x : [33, 89], y : [48, 61]\}$

$\sqsubseteq$

$\{x : [1, 99], y : [2, 90]\}$

$$X \subseteq Y \Rightarrow \alpha(X) \sqsubseteq \alpha(Y)$$

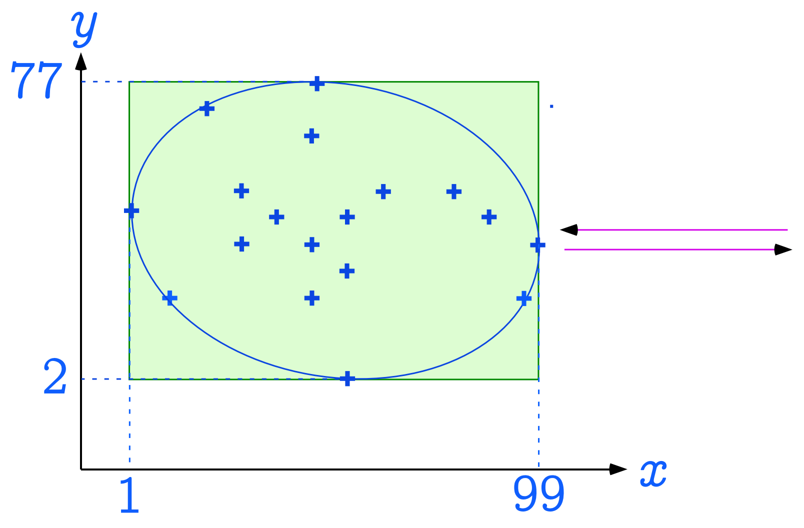
The concretization γ is monotone



$$\{x : [33, 89], y : [48, 61]\}$$
$$\sqsubseteq$$
$$\{x : [1, 99], y : [2, 90]\}$$

$$X \sqsubseteq Y \Rightarrow \gamma(X) \subseteq \gamma(Y)$$

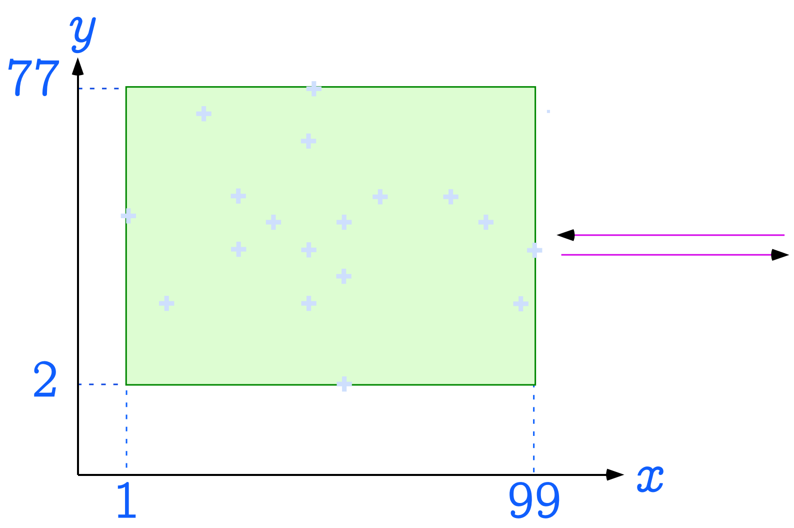
The $\gamma \circ \alpha$ composition is extensive



$$\{x : [1, 99], y : [2, 77]\}$$

$$X \subseteq \gamma \circ \alpha(X)$$

The $\alpha \circ \gamma$ composition is reductive



$\{x : [1, 99], y : [2, 77]\}$
$=/\sqsubseteq$
$\{x : [1, 99], y : [2, 77]\}$

$$\alpha \circ \gamma(Y) =/\sqsubseteq Y$$

Correspondance between concrete and abstract properties

– The pair $\langle \alpha, \gamma \rangle$ is a Galois connection:

$$\langle \wp(S), \subseteq \rangle \xleftrightarrow[\alpha]{\gamma} \langle \mathcal{D}, \sqsubseteq \rangle$$

– $\langle \wp(S), \subseteq \rangle \underset{\alpha}{\overset{\gamma}{\leftarrows\!\!\twoheadrightarrow}} \langle \mathcal{D}, \sqsubseteq \rangle$ when α is onto (equivalently $\alpha \circ \gamma = 1$ or γ is one-to-one).

Galois connection

$$\langle \mathcal{D}, \subseteq \rangle \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \langle \overline{\mathcal{D}}, \sqsubseteq \rangle$$

iff $\forall x, y \in \mathcal{D} : x \subseteq y \Longrightarrow \alpha(x) \sqsubseteq \alpha(y)$

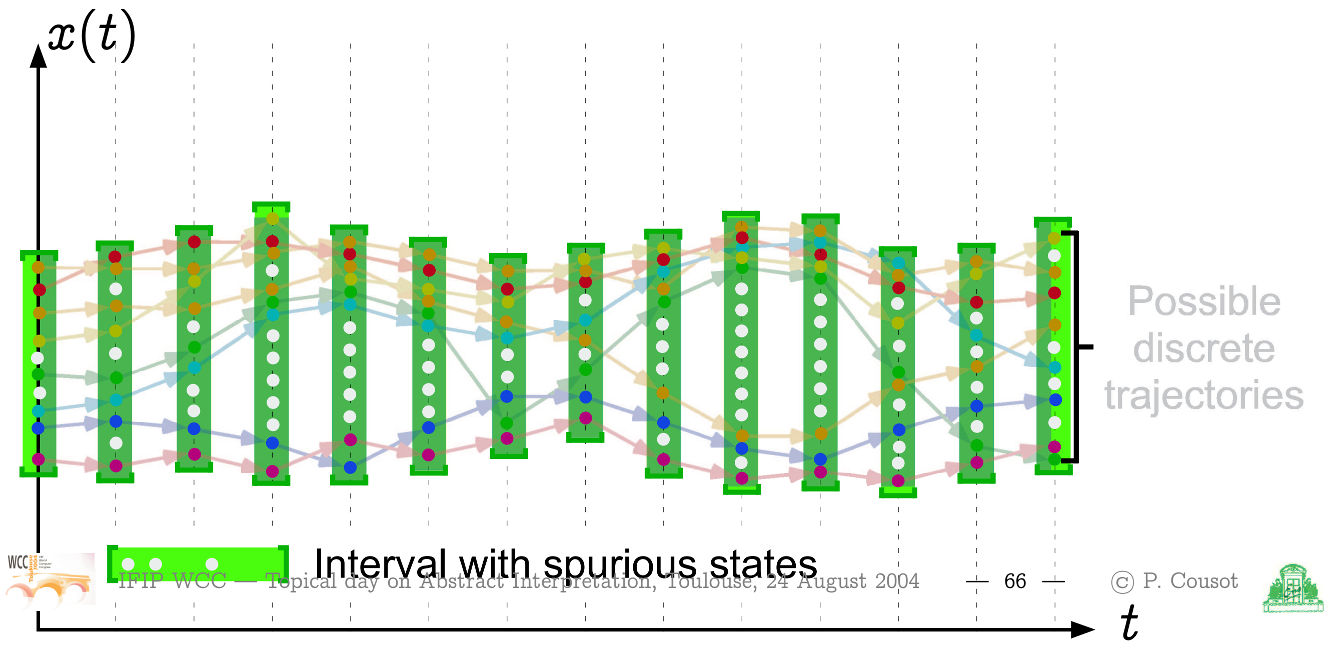
$\wedge\ \forall \overline{x}, \overline{y} \in \overline{\mathcal{D}} : \overline{x} \sqsubseteq \overline{y} \Longrightarrow \gamma(\overline{x}) \subseteq \gamma(\overline{y})$

$\wedge\ \forall x \in \mathcal{D} : x \subseteq \gamma(\alpha(x))$

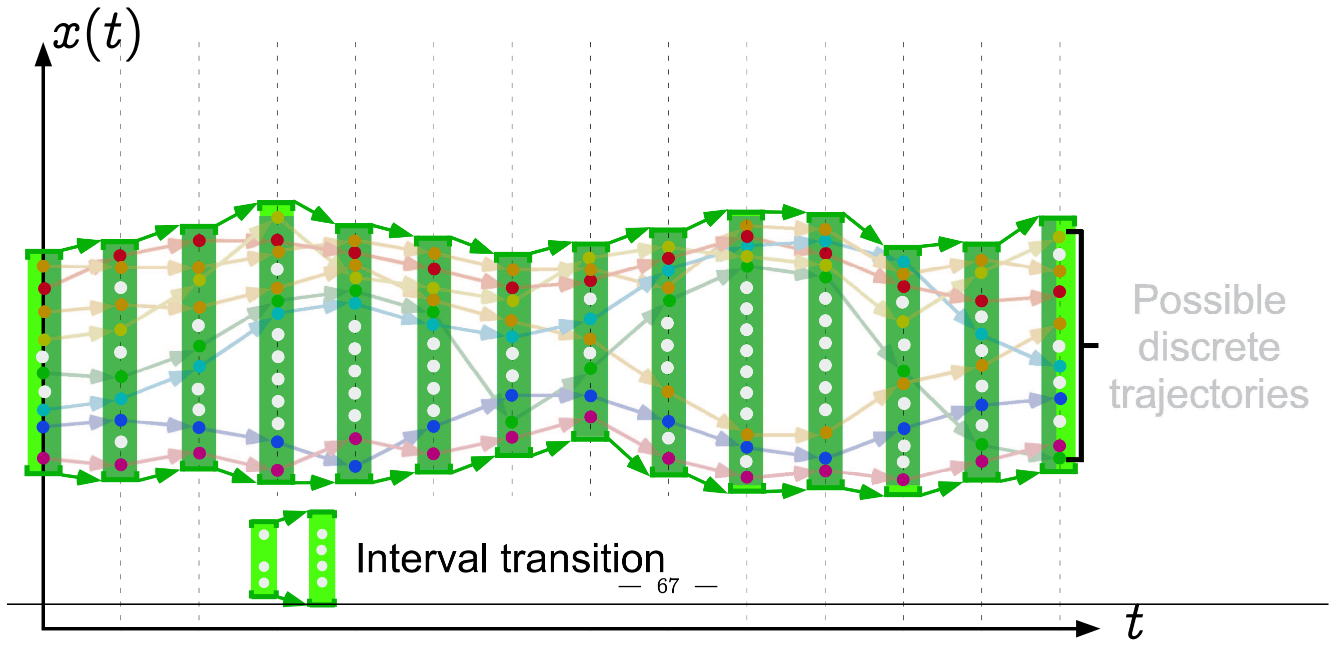
$\wedge\ \forall \overline{y} \in \overline{\mathcal{D}} : \alpha(\gamma(\overline{y})) \sqsubseteq \overline{x}$

iff $\forall x \in \mathcal{D}, \overline{y} \in \overline{\mathcal{D}} : \alpha(x) \sqsubseteq y \Longleftrightarrow x \subseteq \gamma(y)$

Graphic example: Interval abstraction



Graphic example: Abstract transitions

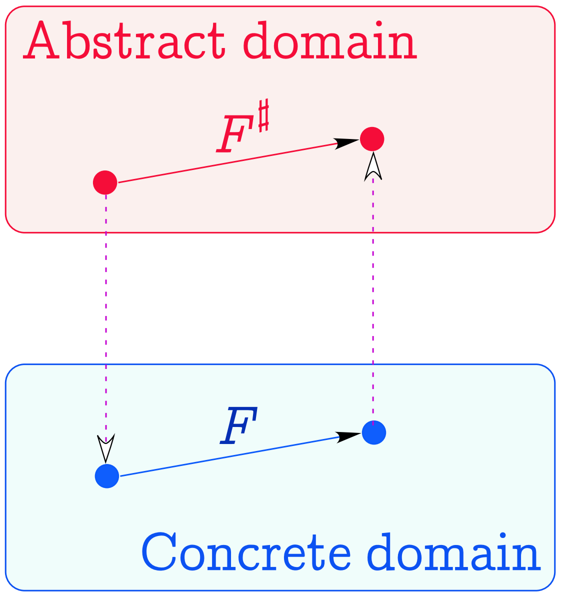


Example: Interval transition semantics of assignments

```
int x;
...
l:
    x := x + 1;
l':
```

$$\{\mathtt{l} : \mathtt{x} \in [\ell, h] \to \mathtt{l}' : \mathtt{x} \in [l + 1, \mathrm{m\ n}(h + 1, \mathtt{max\_int})] \cup \{\Omega \mid h = \mathtt{max\_int}\} \mid \ell \leq h\}$$

where $[\ell, h] = \emptyset$ when $h < \ell$.



Function abstraction

$$F^\sharp = \alpha \circ F \circ \gamma$$

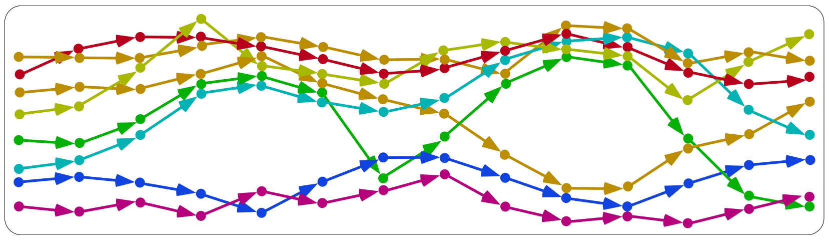
.e. $F^\sharp = \rho \circ F$

$$\langle P,\ \subseteq\rangle \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \langle Q,\ \sqsubseteq\rangle \Rightarrow$$

$$\langle P \overset{\text{mon}}{\longmapsto} P,\ \dot{\subseteq}\rangle \underset{\lambda F\,.\,\alpha\circ F\circ\gamma}{\overset{\lambda F^\sharp\,.\,\gamma\circ F^\sharp\circ\alpha}{\leftrightarrows}} \langle Q \overset{\text{mon}}{\longmapsto} Q,\ \dot{\sqsubseteq}\rangle$$

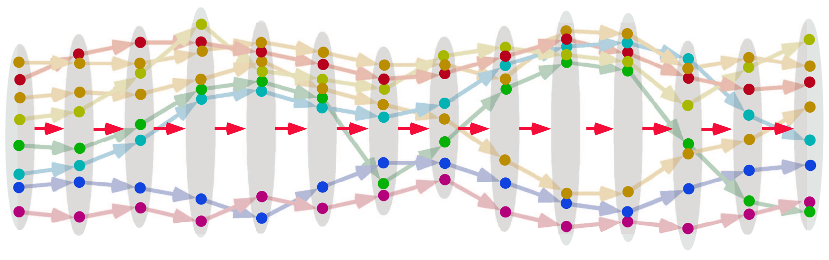
Example: Set of traces to trace of intervals abstraction

Set of traces:



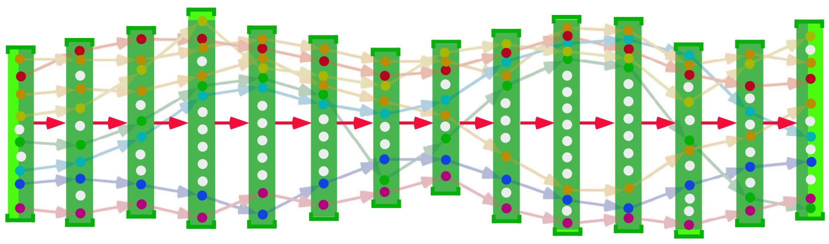
$\alpha_1 \downarrow$

Trace of sets:



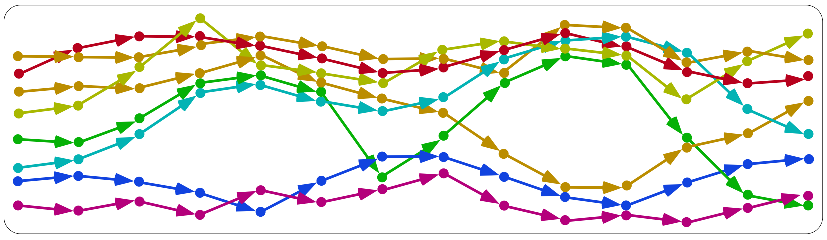
$\alpha_2 \downarrow$

Trace of intervals



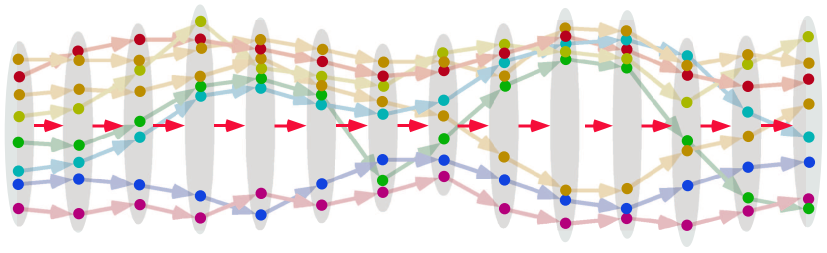
Example: Set of traces to reachable states abstraction

Set of traces:



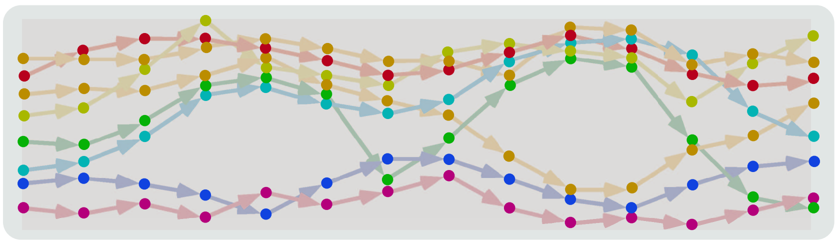
$\alpha_1 \downarrow$

Trace of sets:



$\alpha_3 \downarrow$

Reachable states



Composition of Galois Connections

The composition of Galois connections:

$$\langle L,\ \leq\rangle \xleftarrow[\alpha_1]{\gamma_1} \!\!\!\!\!\!\xrightarrow{} \langle M,\ \sqsubseteq\rangle$$

and:

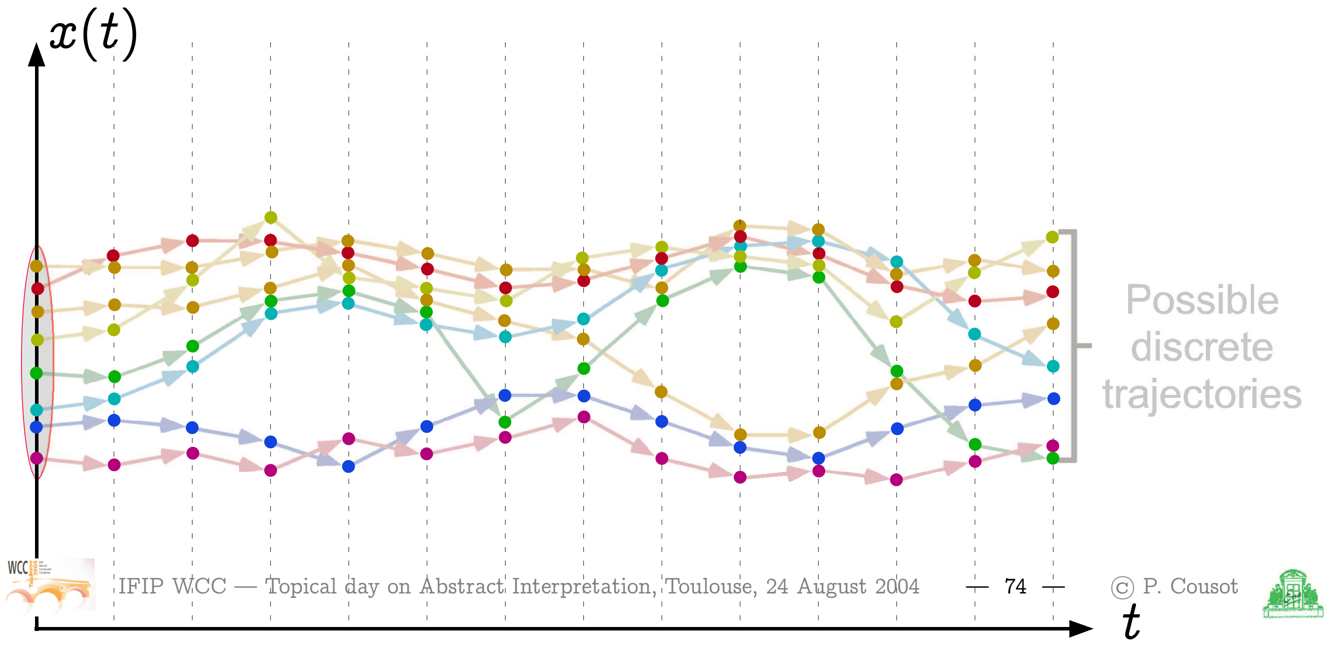
$$\langle M,\ \sqsubseteq\rangle \xleftarrow[\alpha_2]{\gamma_2} \!\!\!\!\!\!\xrightarrow{} \langle N,\ \preceq\rangle$$

is a Galois connection:

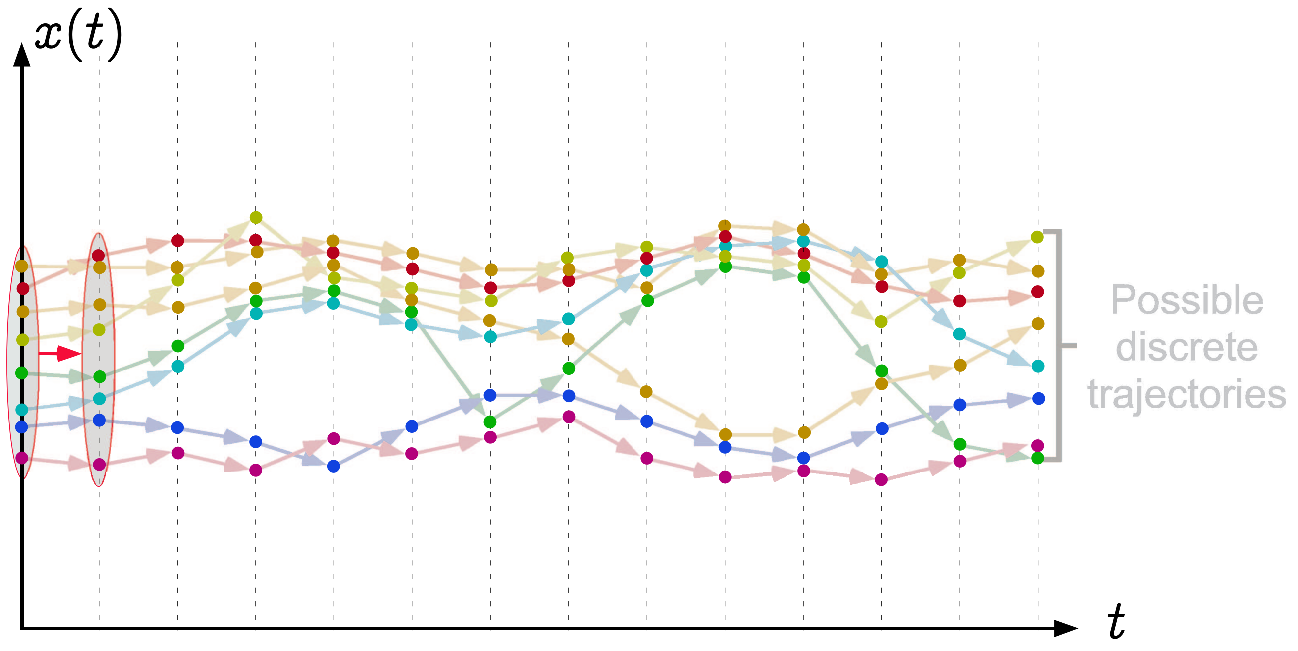
$$\langle L,\ \leq\rangle \xleftarrow[\alpha_2 \circ \alpha_1]{\gamma_1 \circ \gamma_2} \!\!\!\!\!\!\xrightarrow{} \langle N,\ \preceq\rangle$$

Abstract semantics in fixpoint form

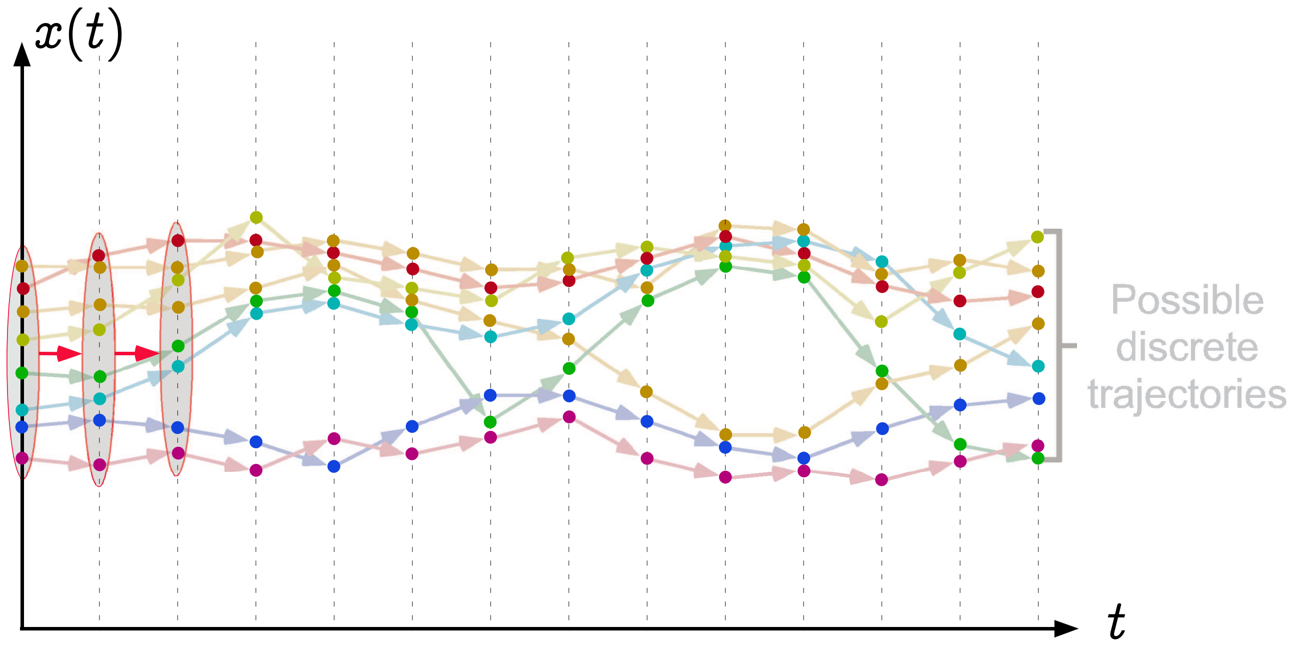
Graphic example: traces of sets of states in fixpoint form



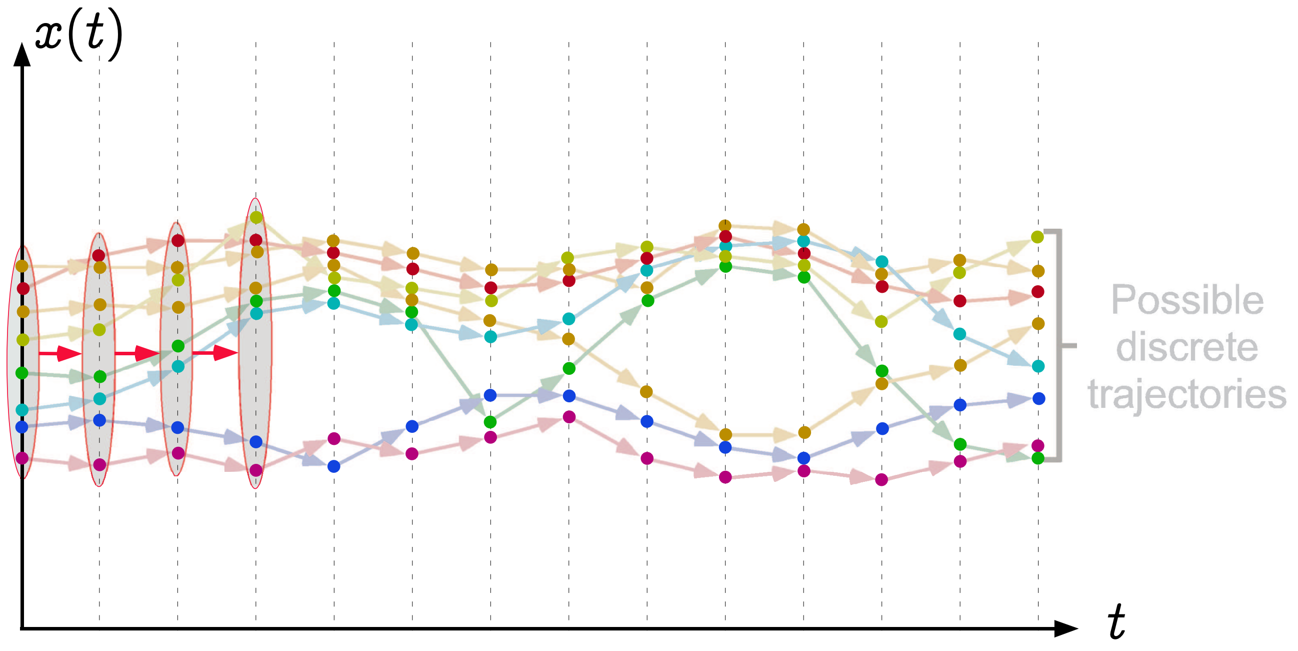
Graphic example: traces of sets of states in fixpoint form



Graphic example: traces of sets of states in fixpoint form



Graphic example: traces of sets of states in fixpoint form



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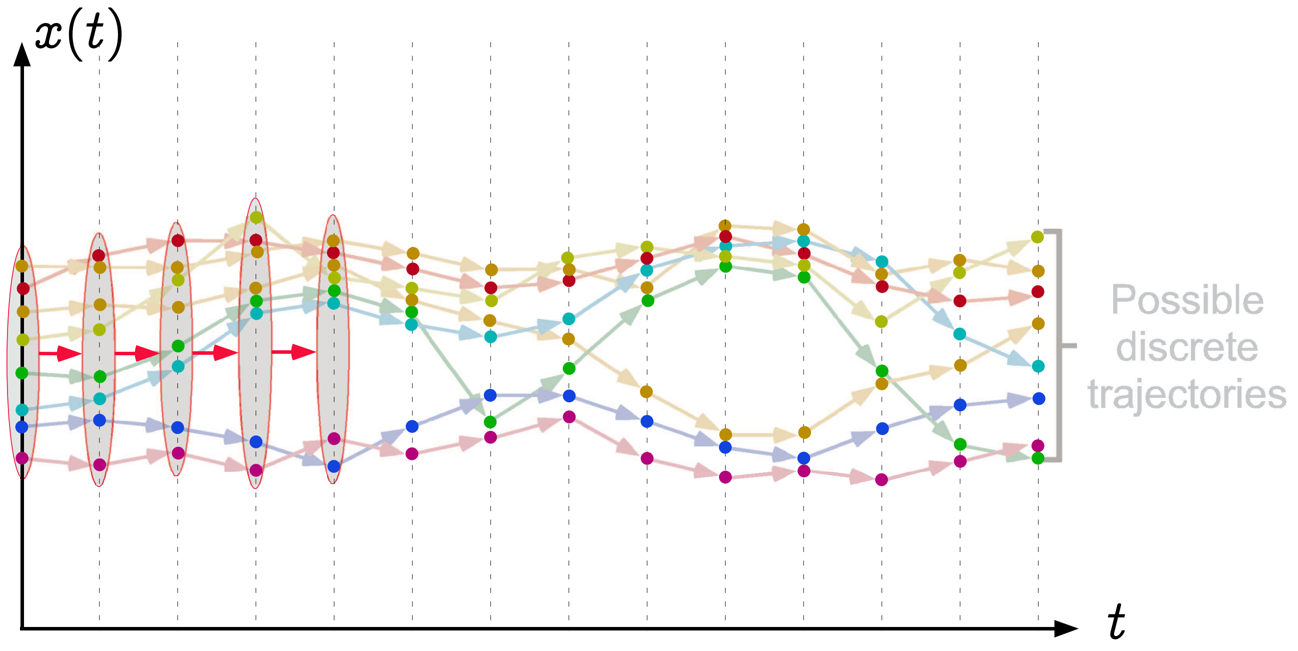

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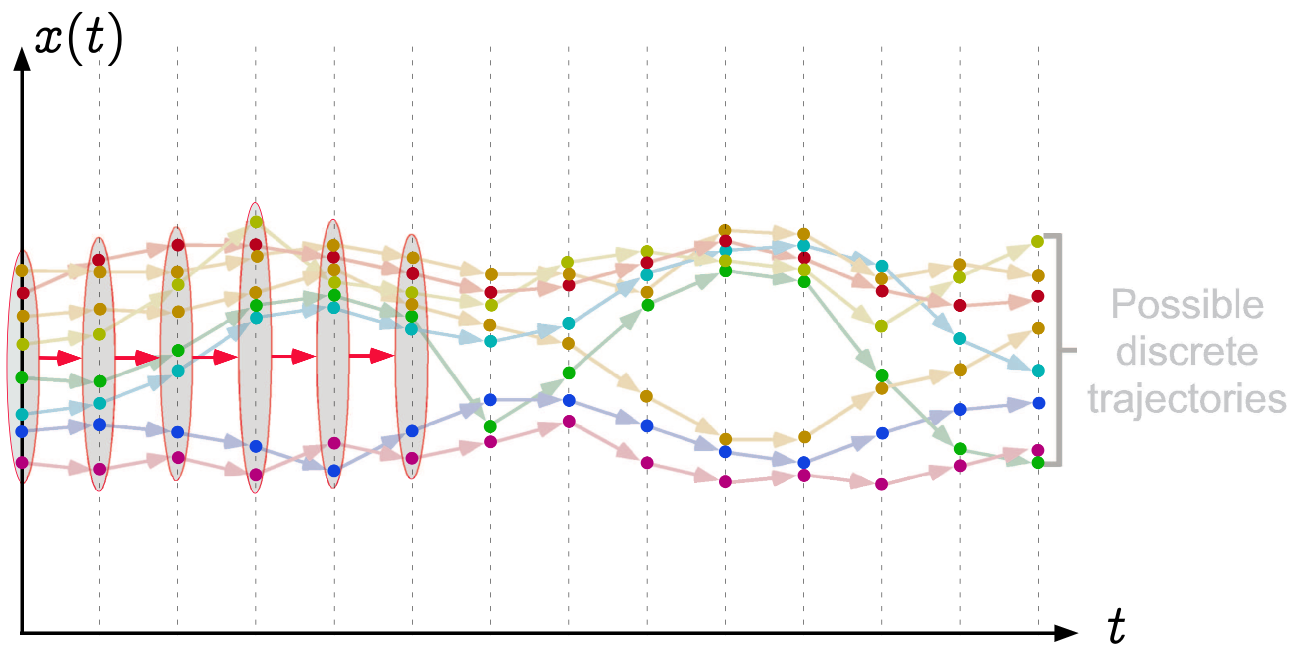

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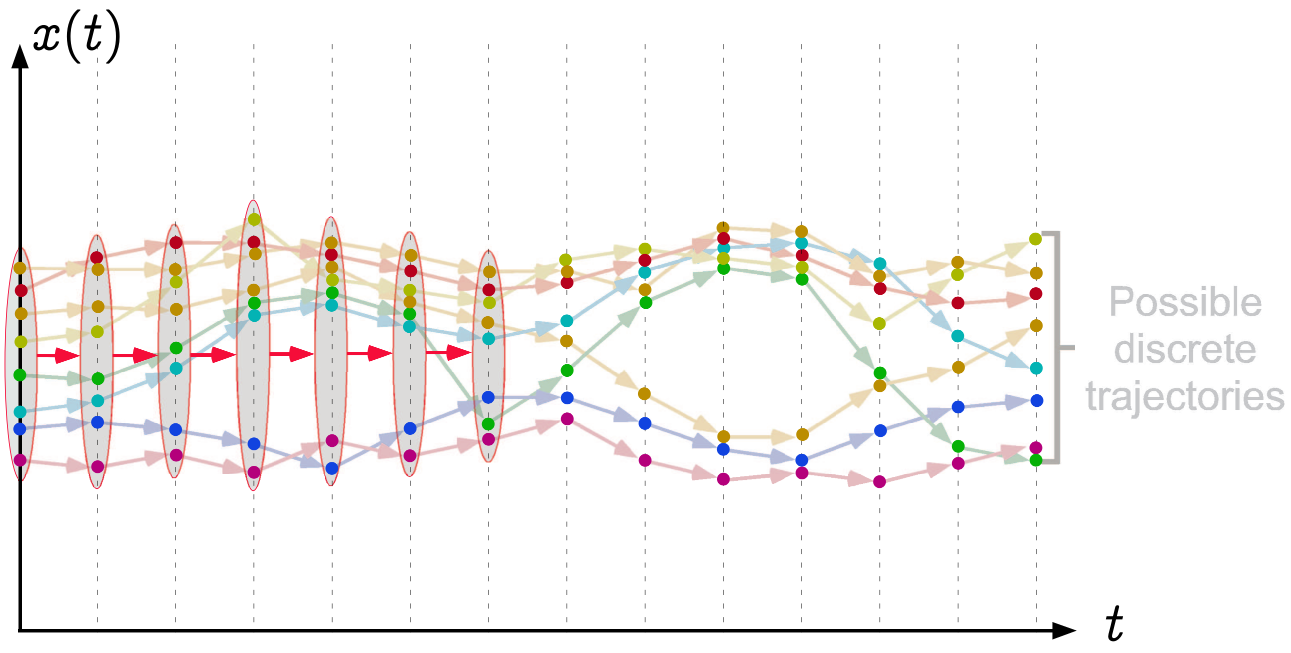
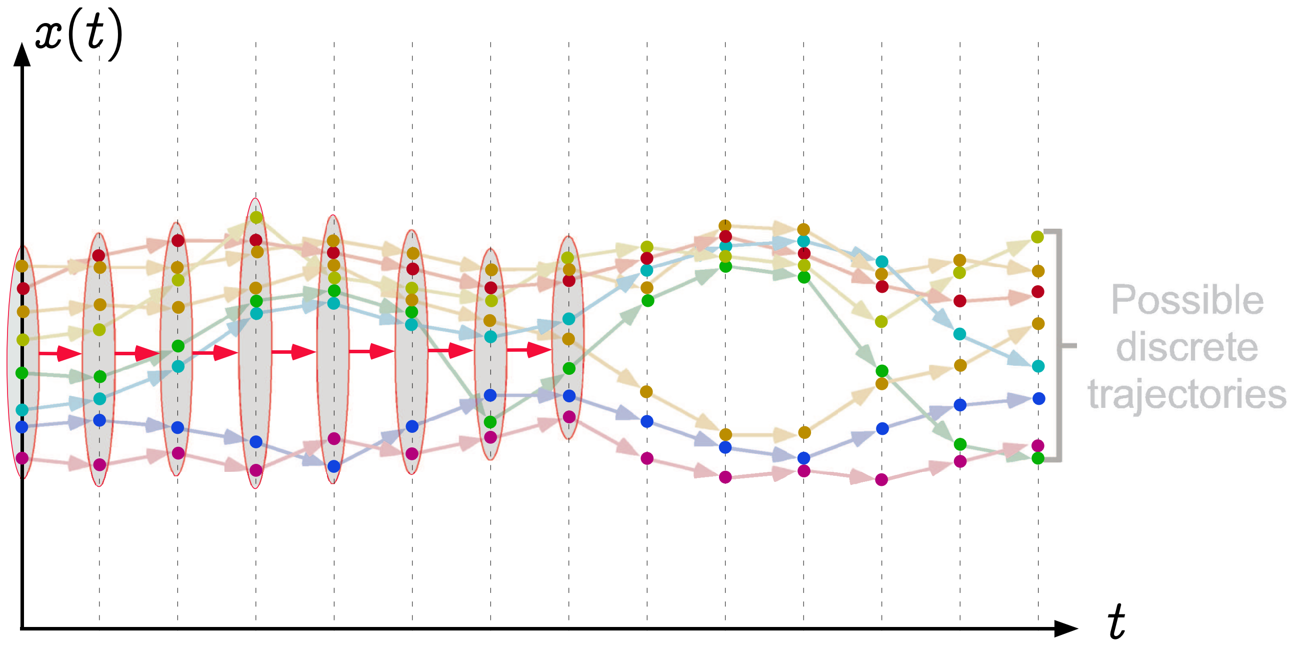
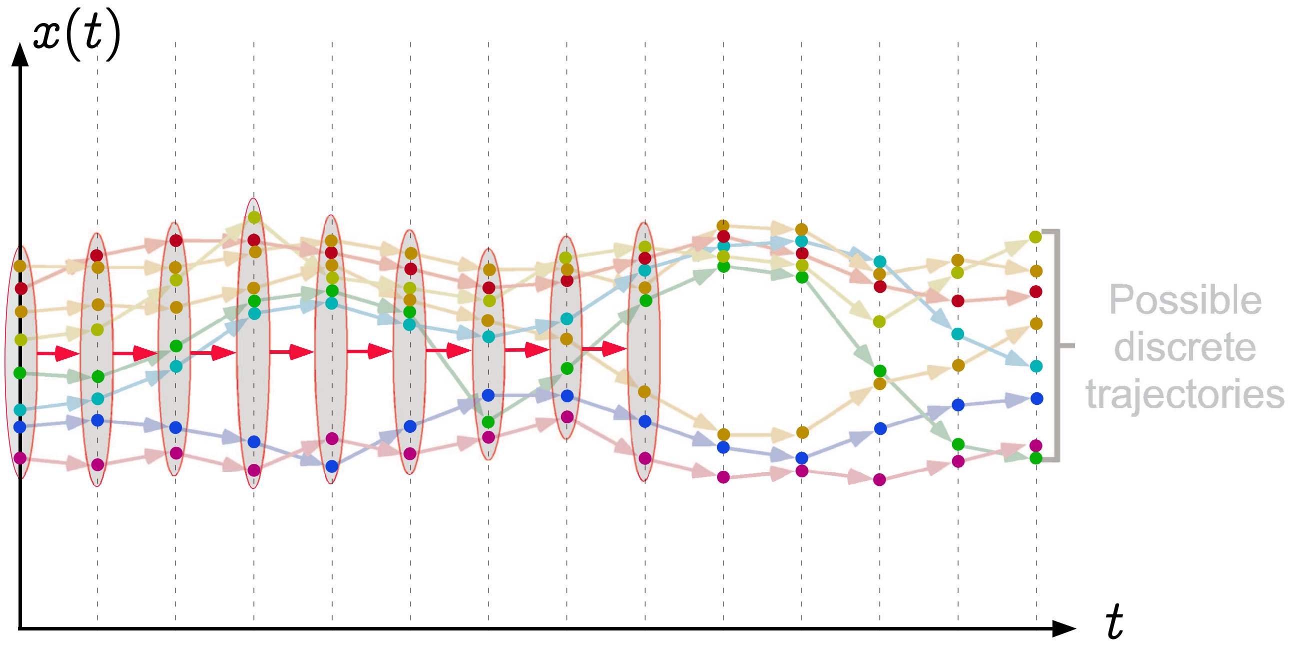


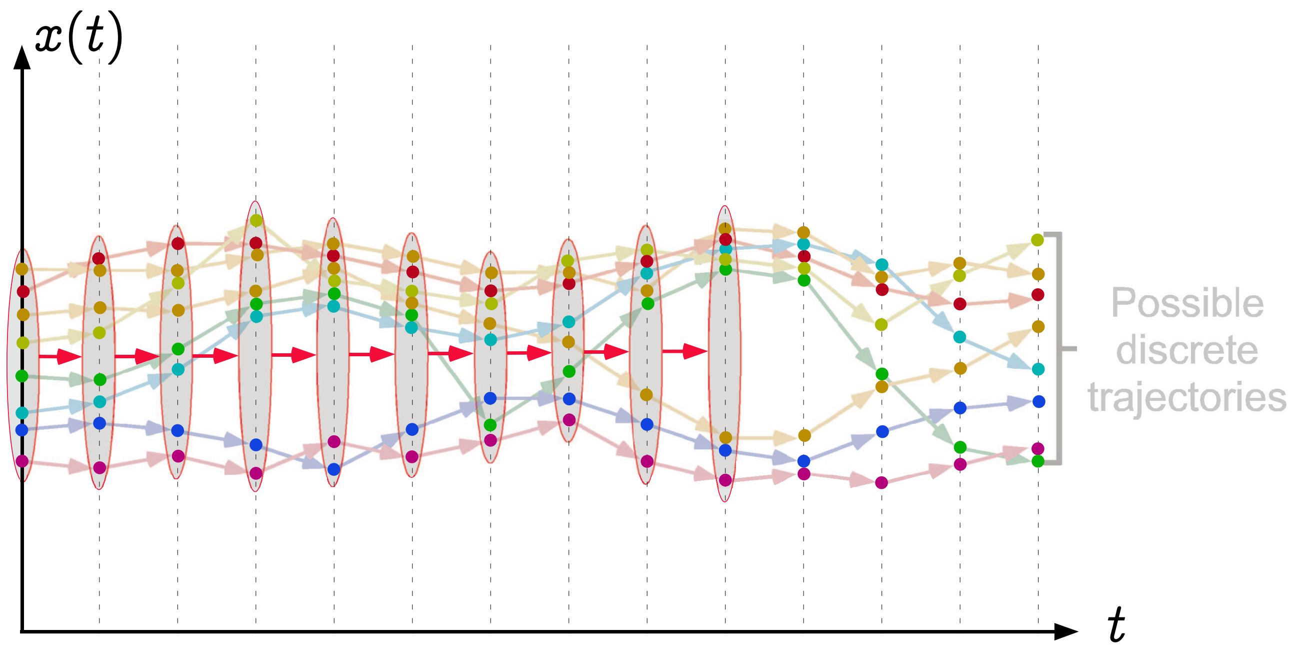
Graphic example: traces of sets of states in fixpoint form



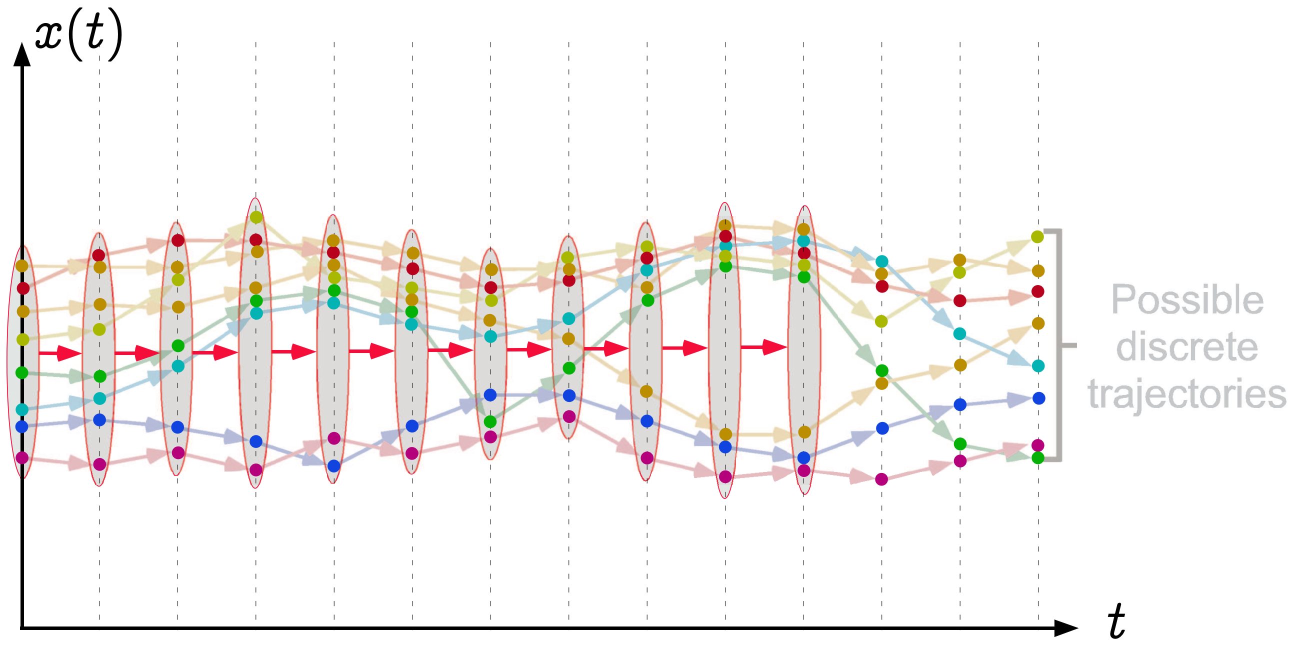
Graphic example: traces of sets of states in fixpoint form



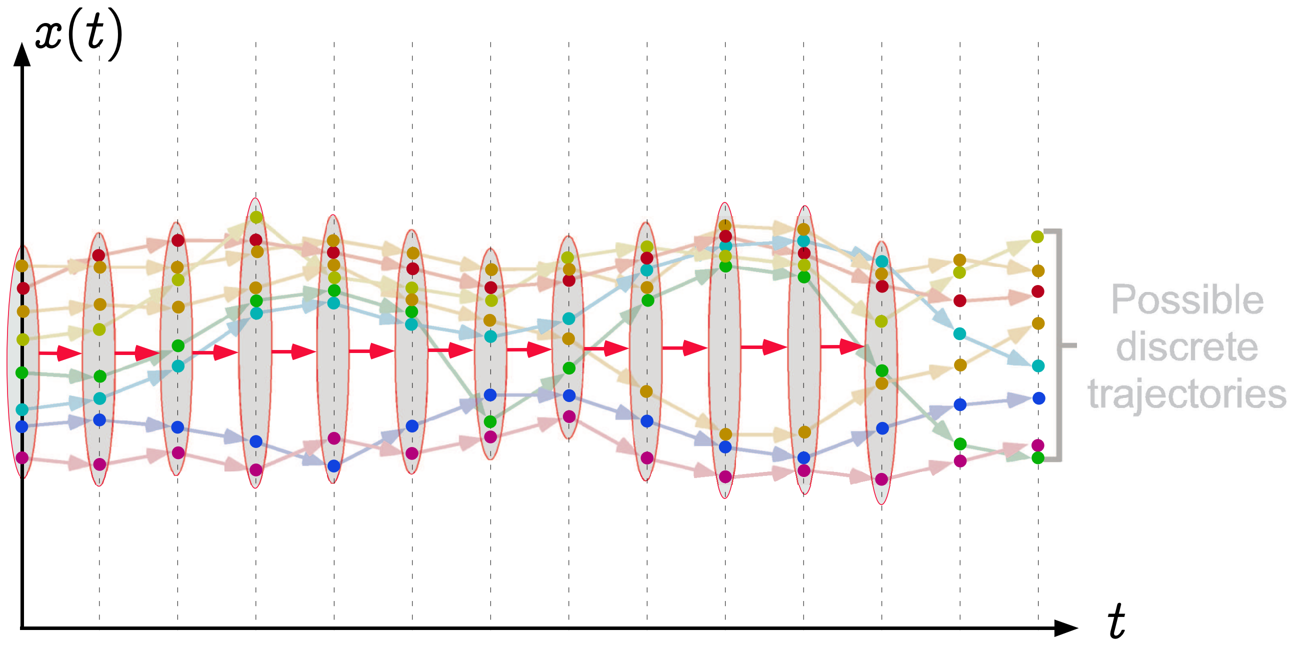
Graphic example: traces of sets of states in fixpoint form



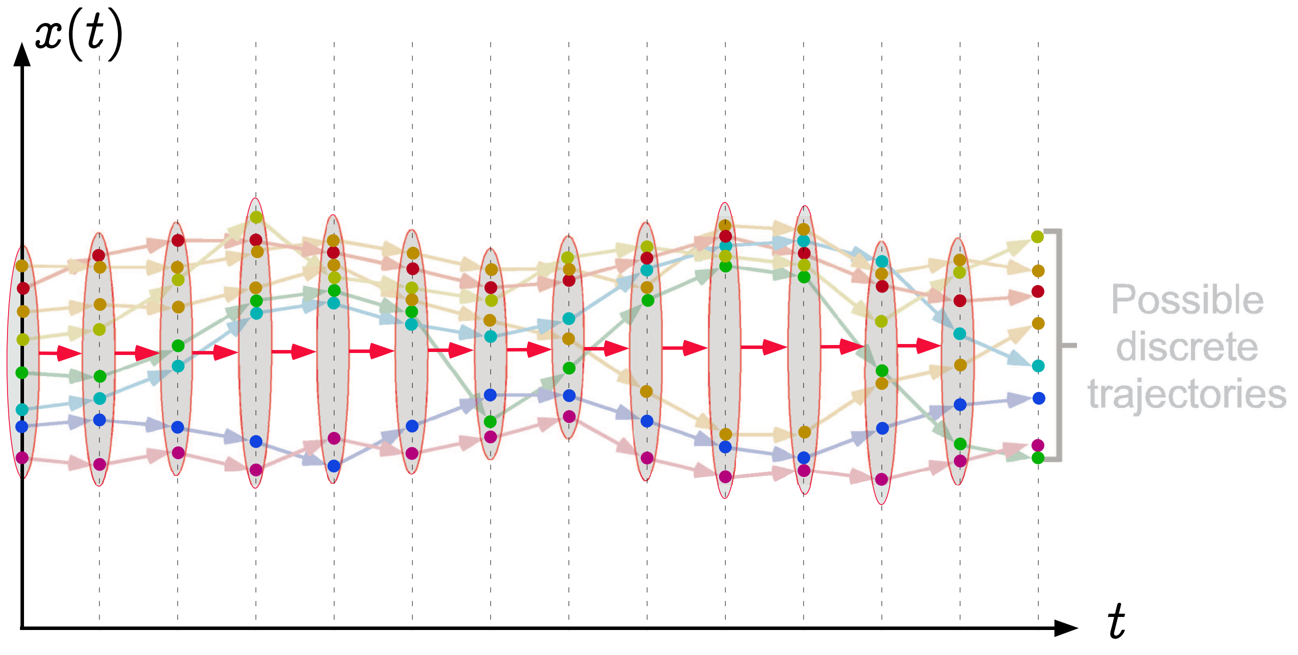
Graphic example: traces of sets of states in fixpoint form



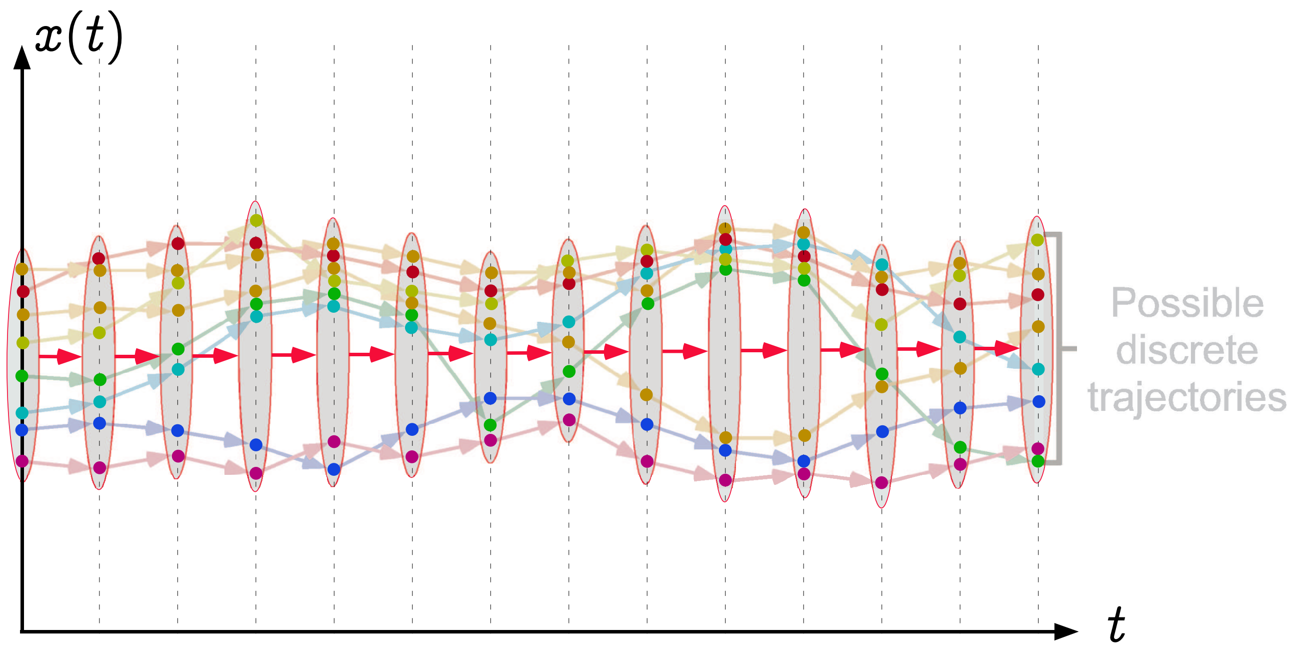
Graphic example: traces of sets of states in fixpoint form



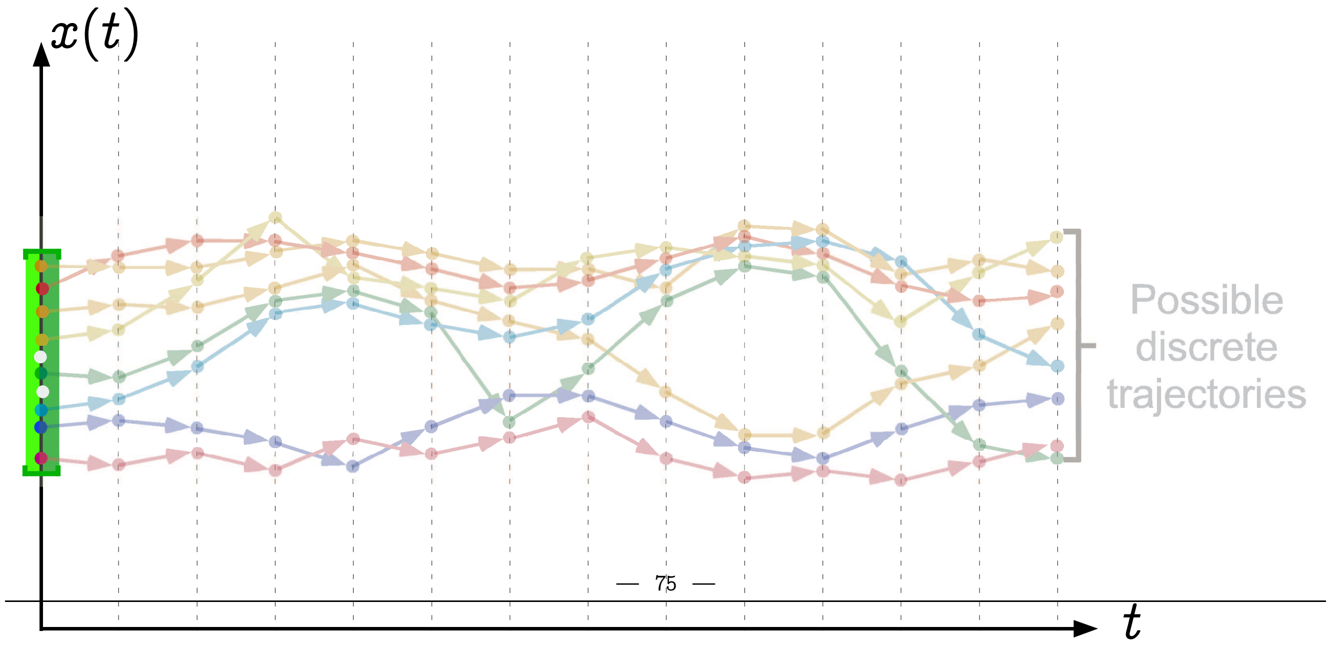
Graphic example: traces of sets of states in fixpoint form



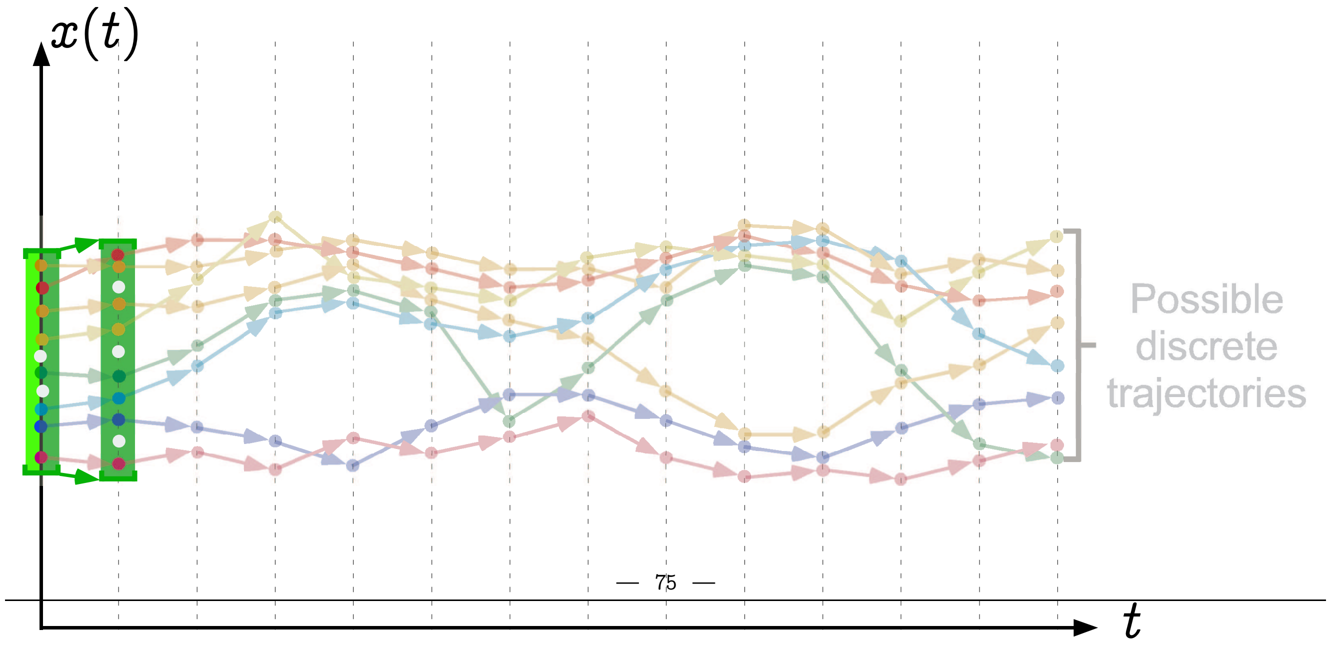
Graphic example: traces of sets of states in fixpoint form



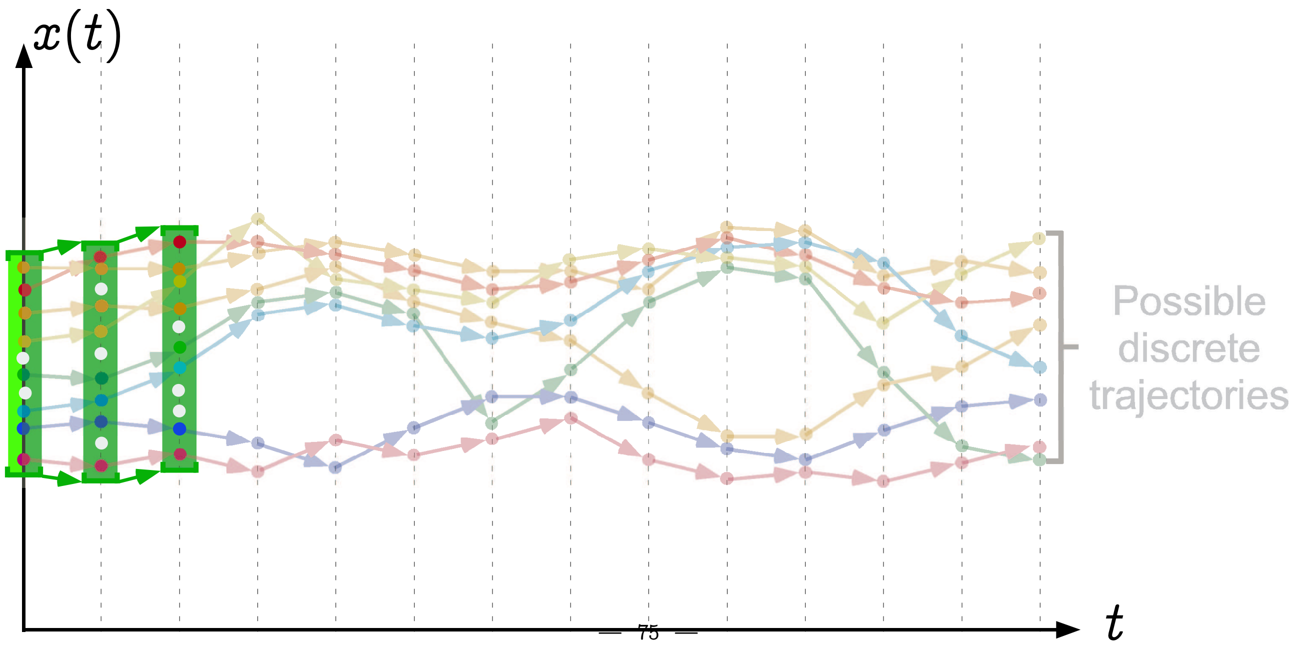
Graphic example: traces of intervals in fixpoint form



Graphic example: traces of intervals in fixpoint form



Graphic example: traces of intervals in fixpoint form



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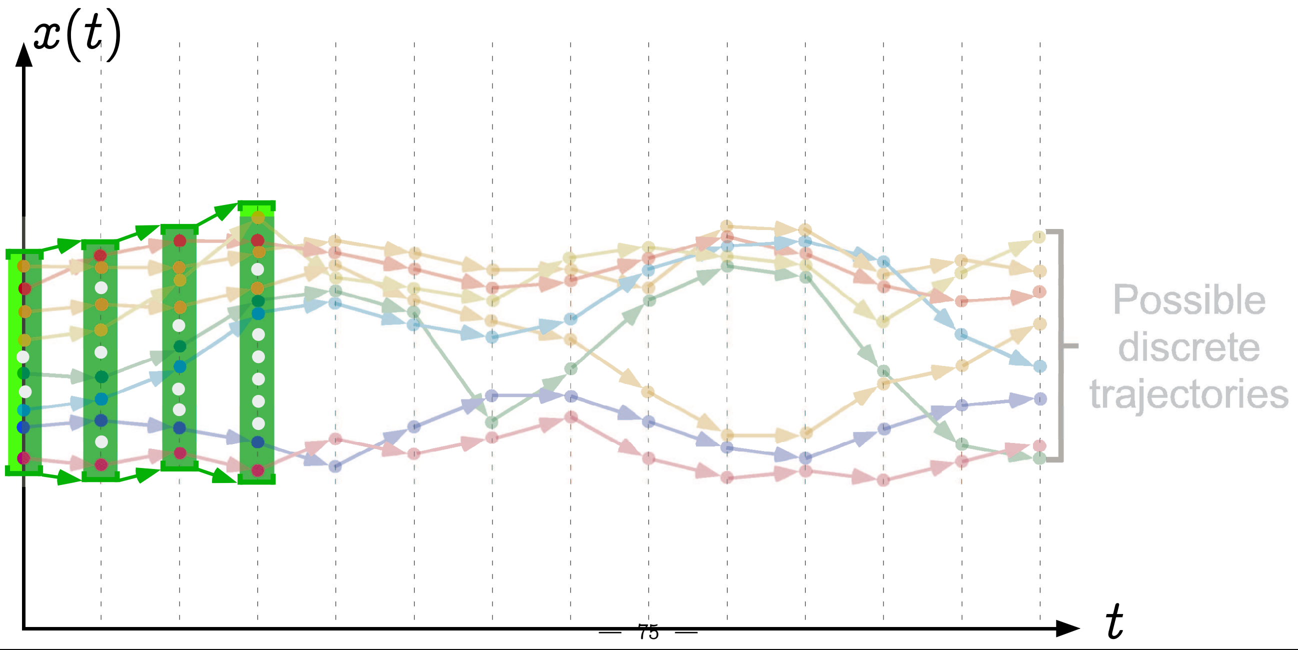
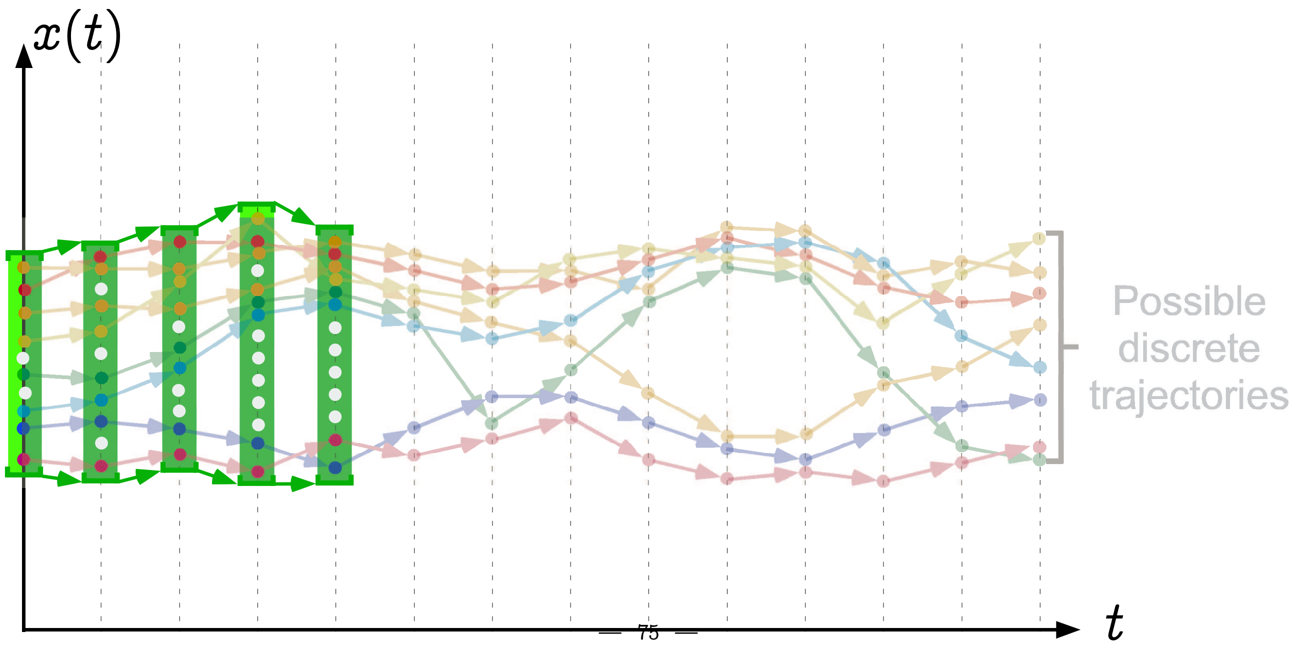
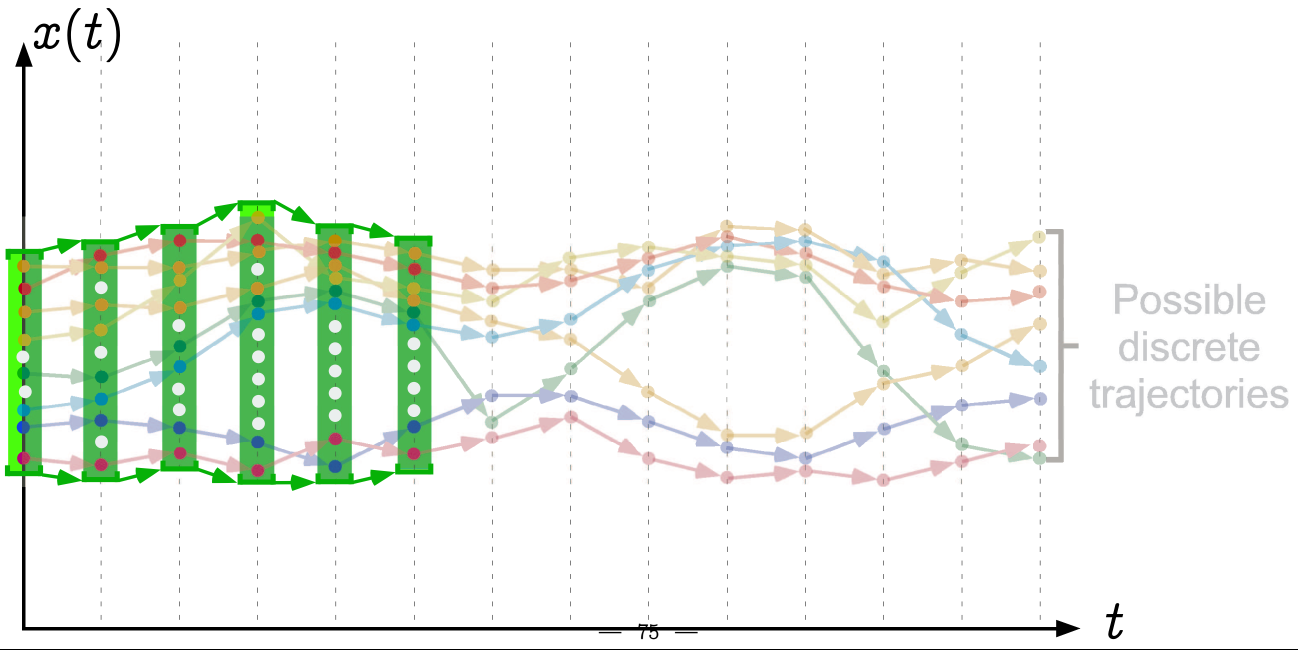


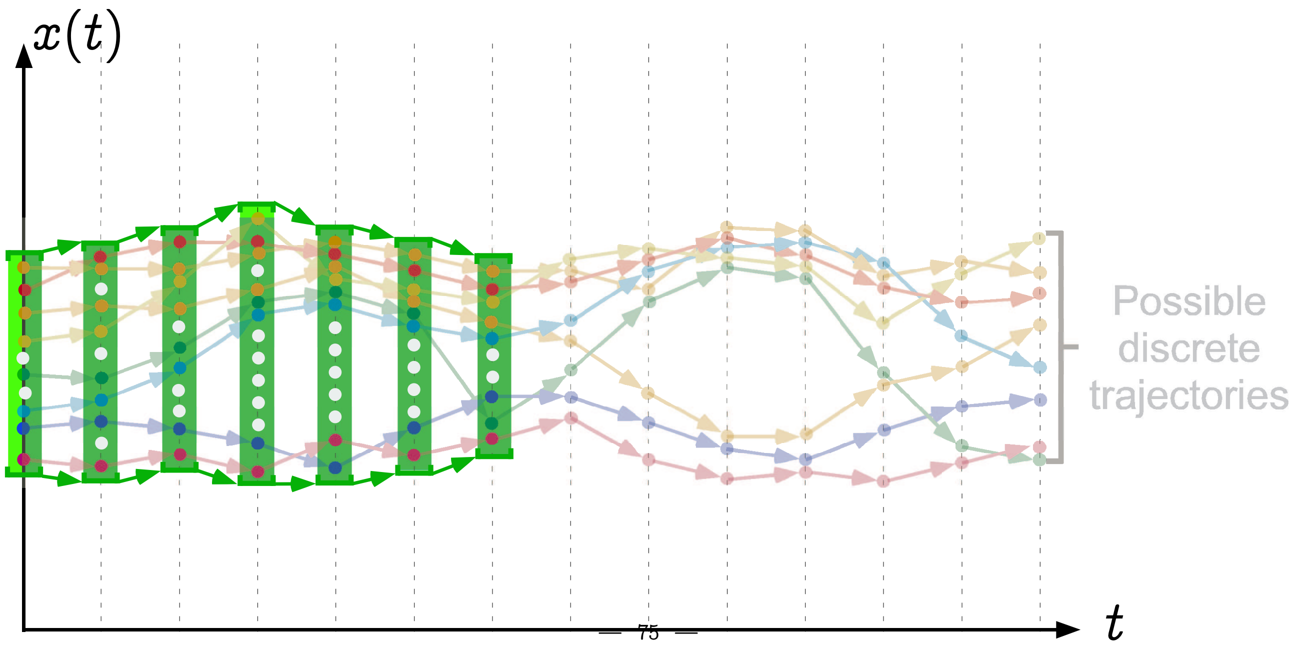
Graphic example: traces of intervals in fixpoint form



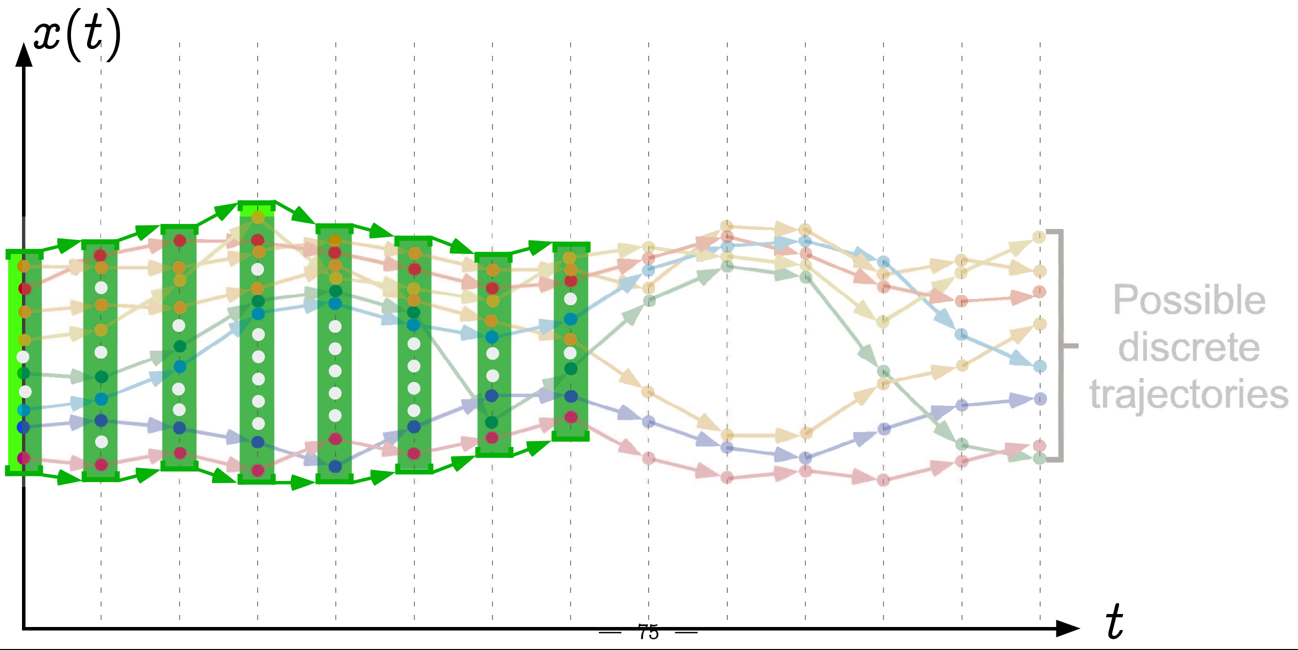
Graphic example: traces of intervals in fixpoint form



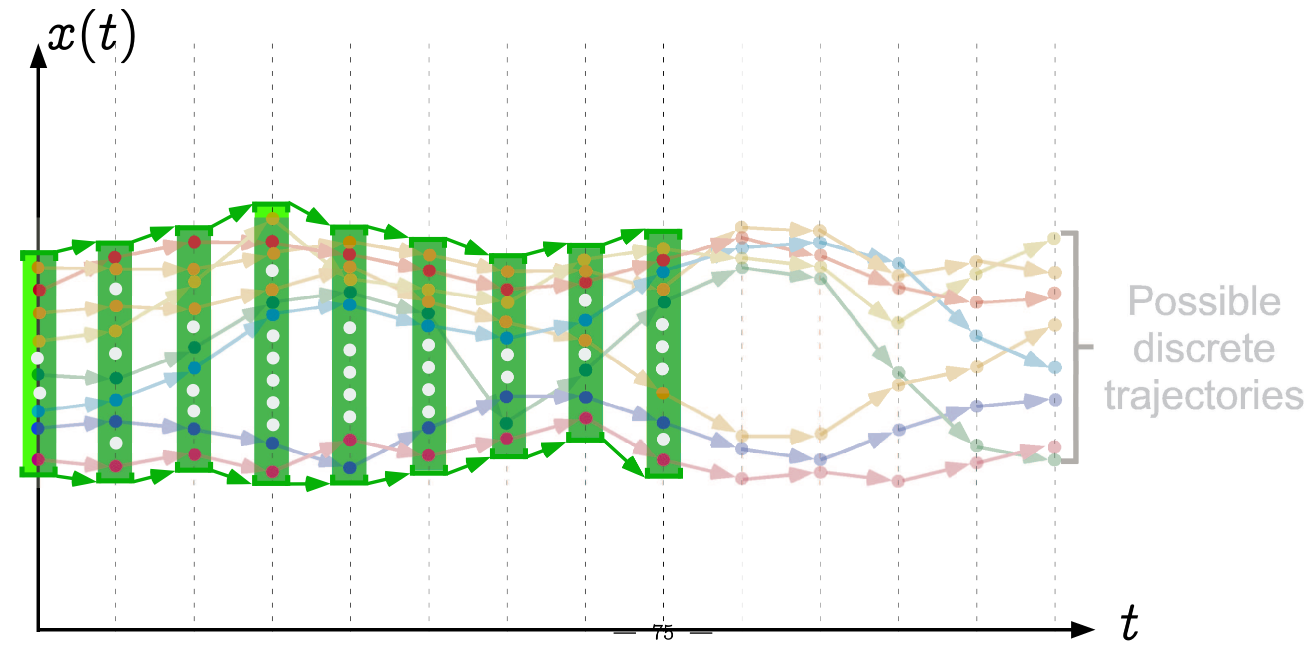
Graphic example: traces of intervals in fixpoint form



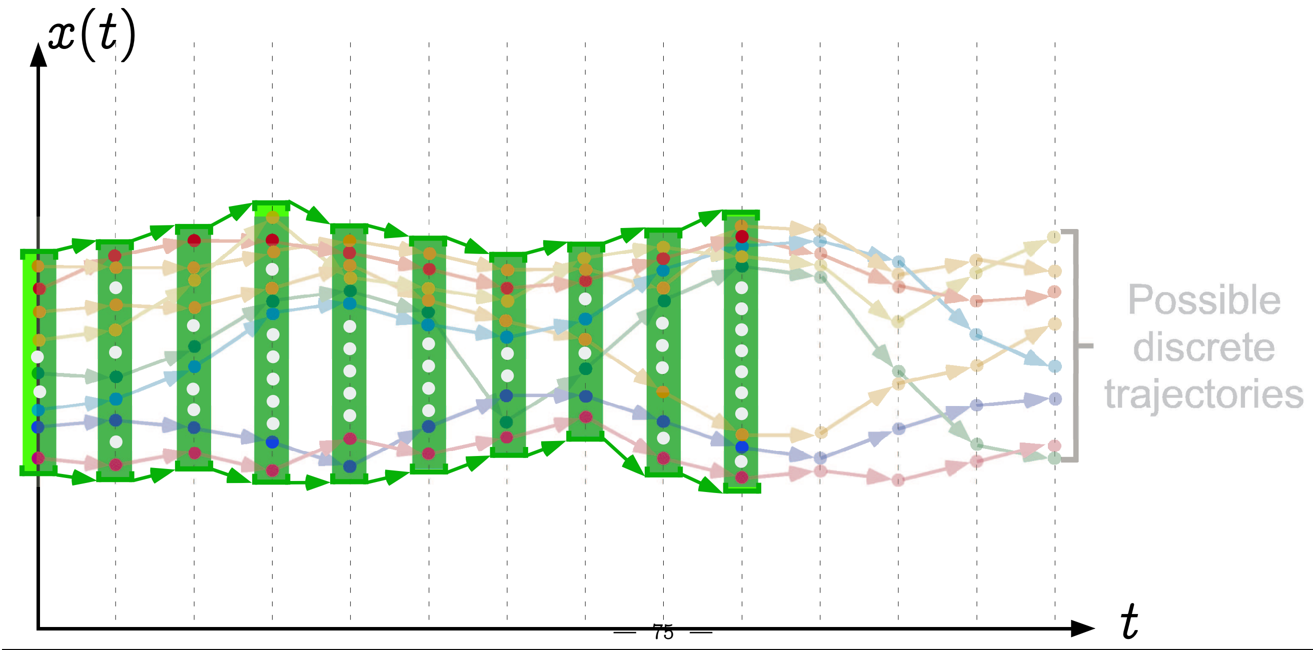
Graphic example: traces of intervals in fixpoint form



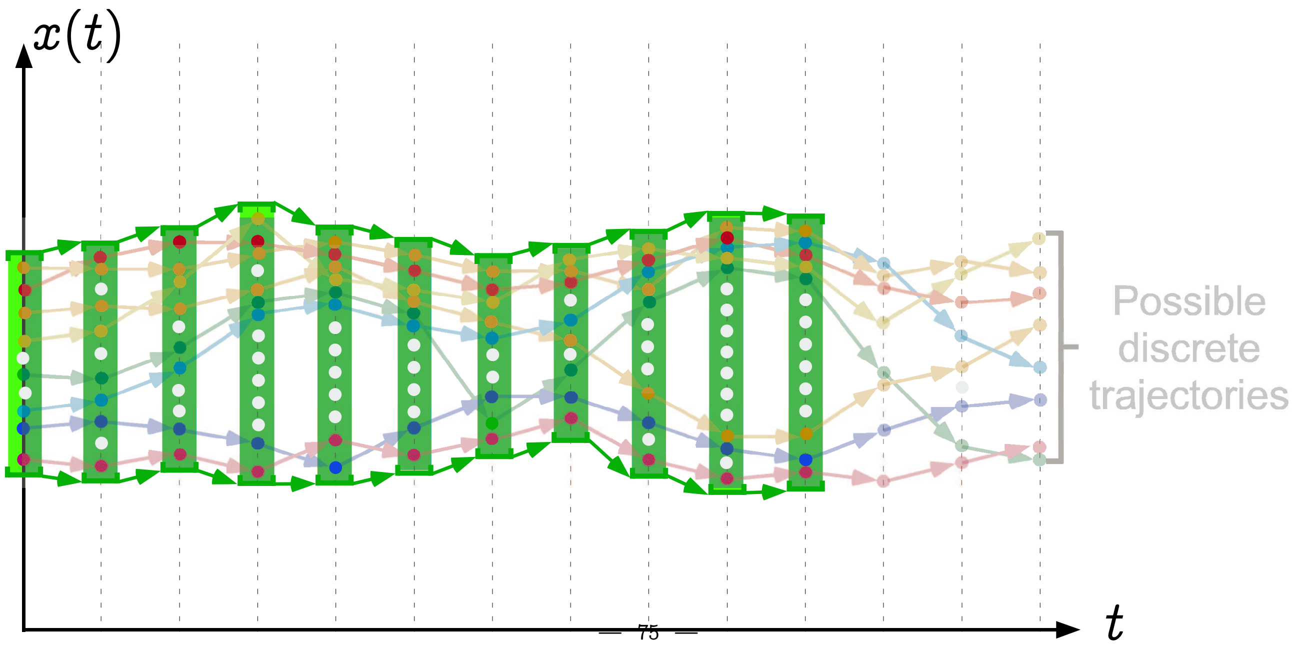
Graphic example: traces of intervals in fixpoint form



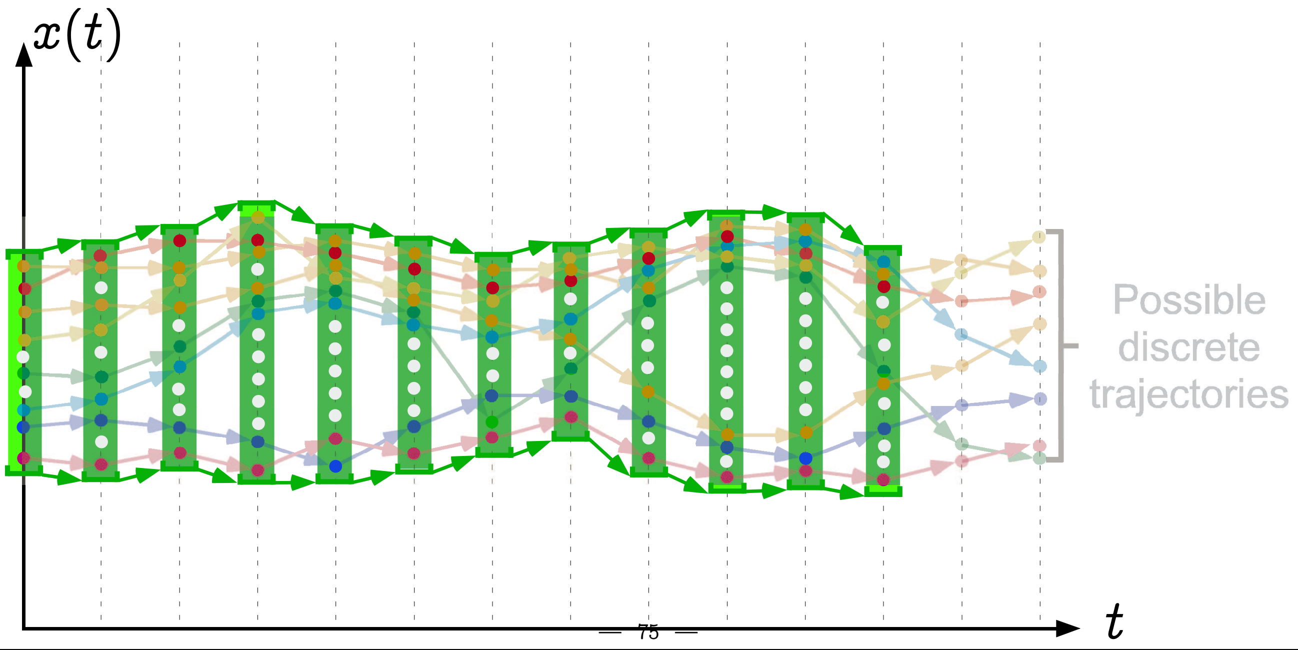
Graphic example: traces of intervals in fixpoint form



Graphic example: traces of intervals in fixpoint form



Graphic example: traces of intervals in fixpoint form



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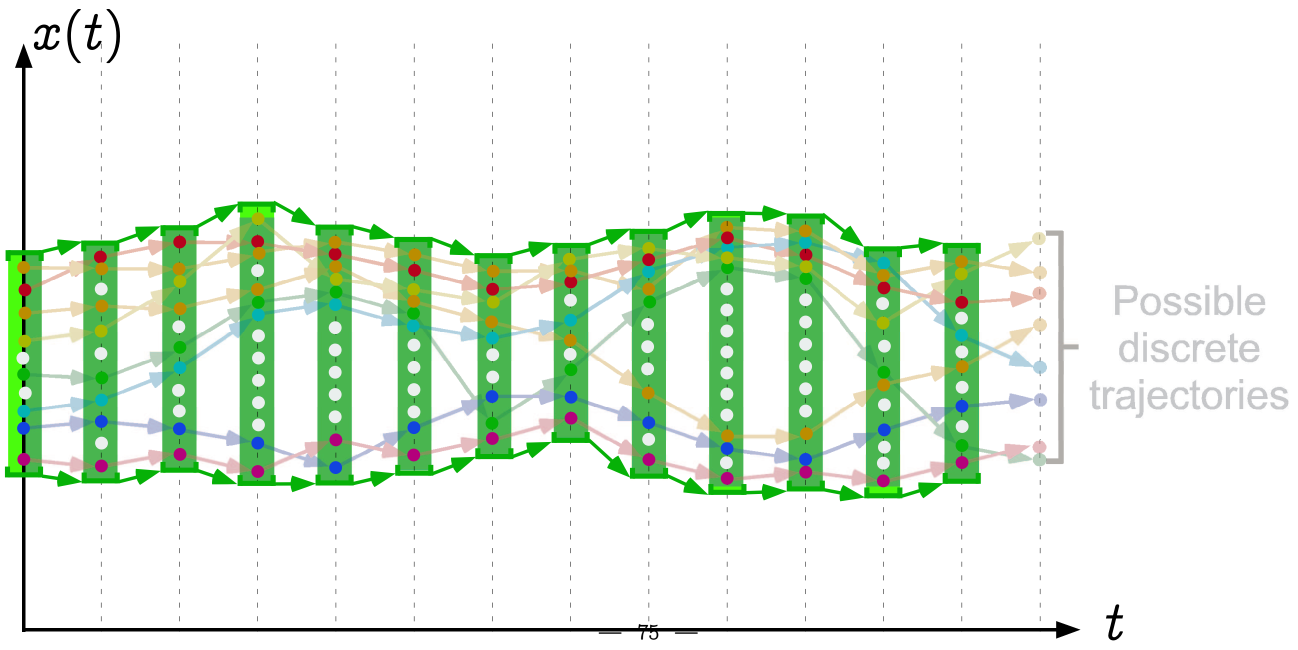
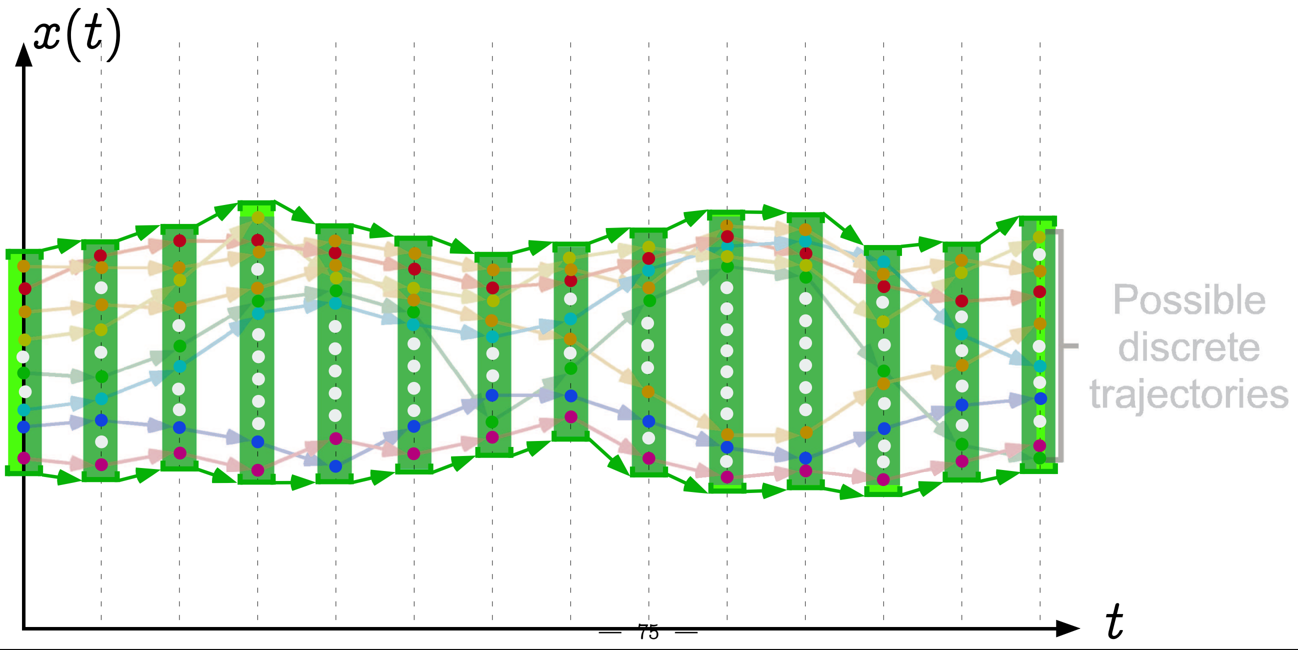
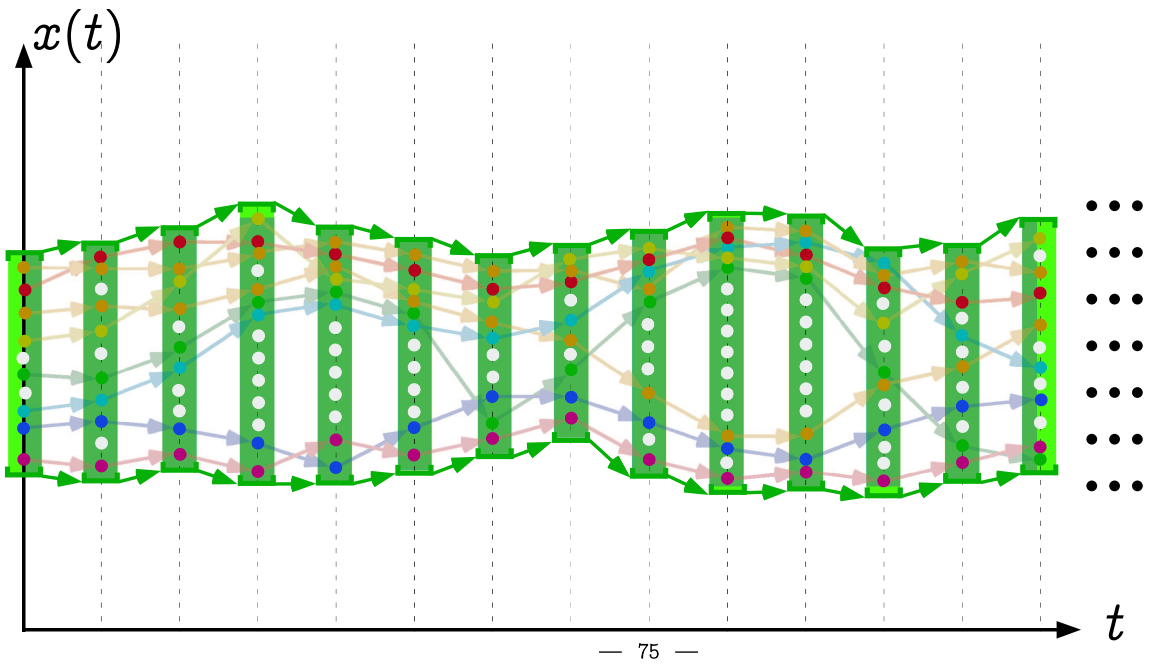


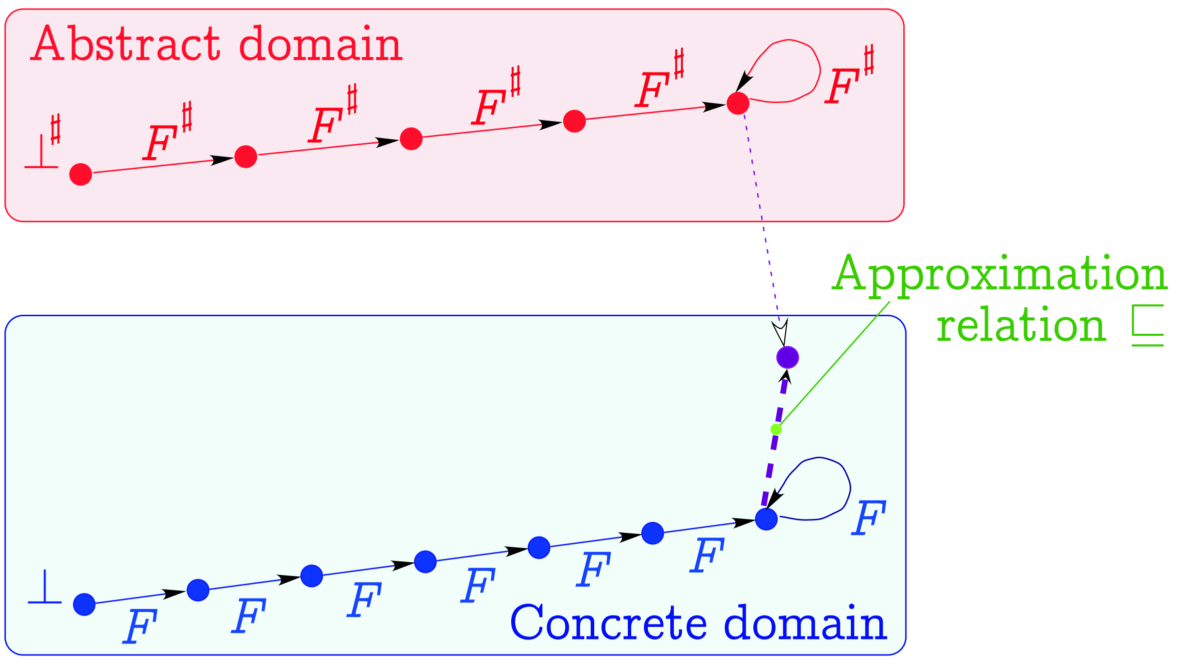
Graphic example: traces of intervals in fixpoint form



Graphic example: traces of intervals in fixpoint form



Approximate fixpoint abstraction



$\alpha(\mathsf{lfp}\ F) \sqsubseteq \mathsf{lfp}\ F^\sharp$

approximate/exact fixpoint abstraction

Exact Abstraction:

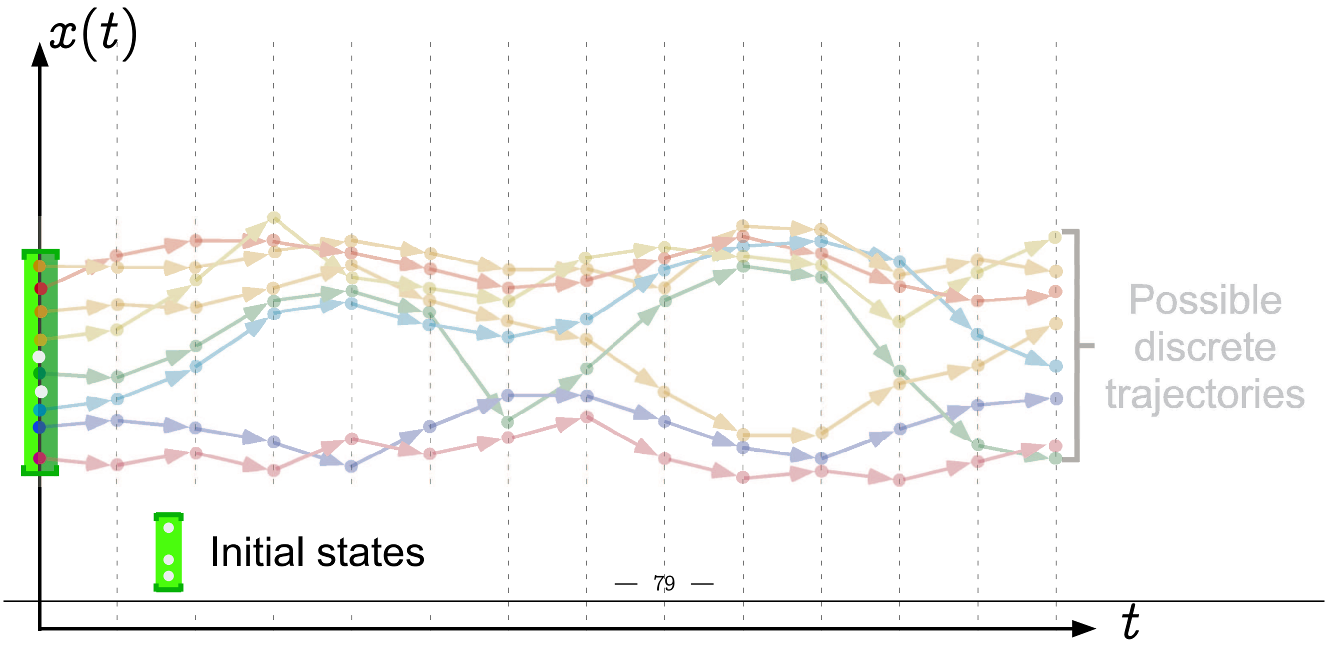
$$\alpha(\mathsf{lfp}\ F) = \mathsf{lfp}\ F^\sharp$$

Approximate Abstraction:

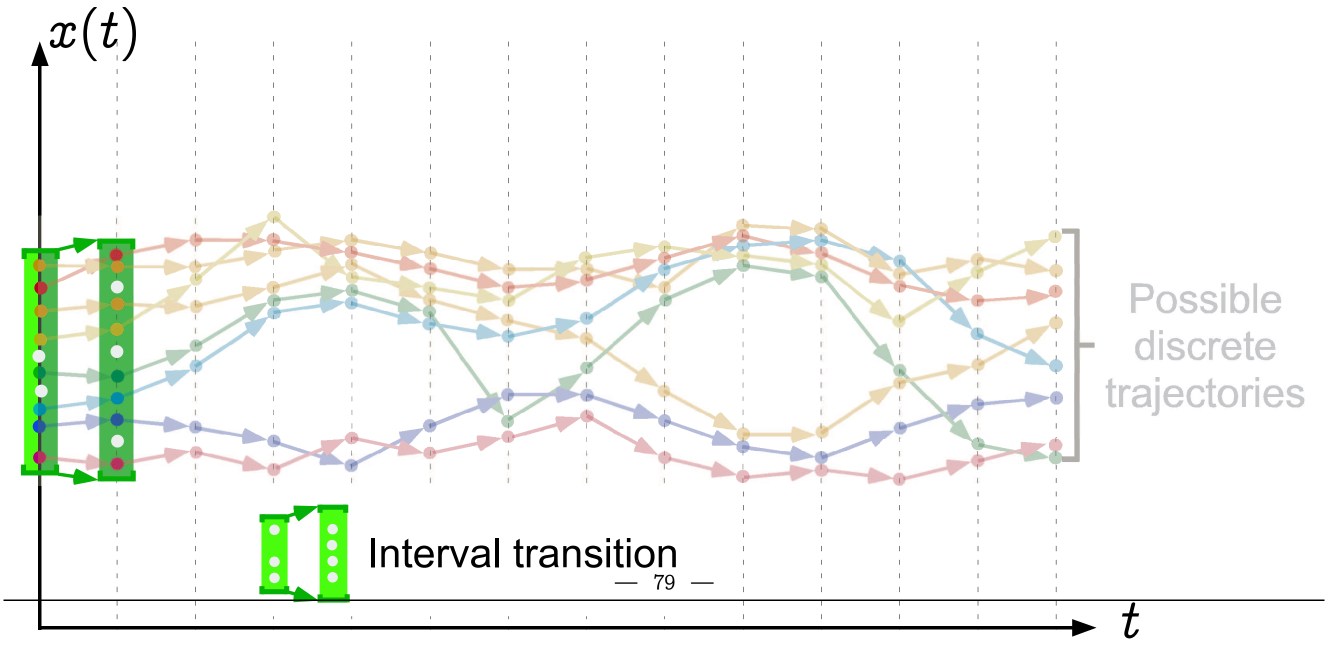
$$\alpha(\mathsf{lfp}\ F) \sqsubset^\sharp \mathsf{lfp}\ F^\sharp$$

Convergence acceleration by widening/narrowing

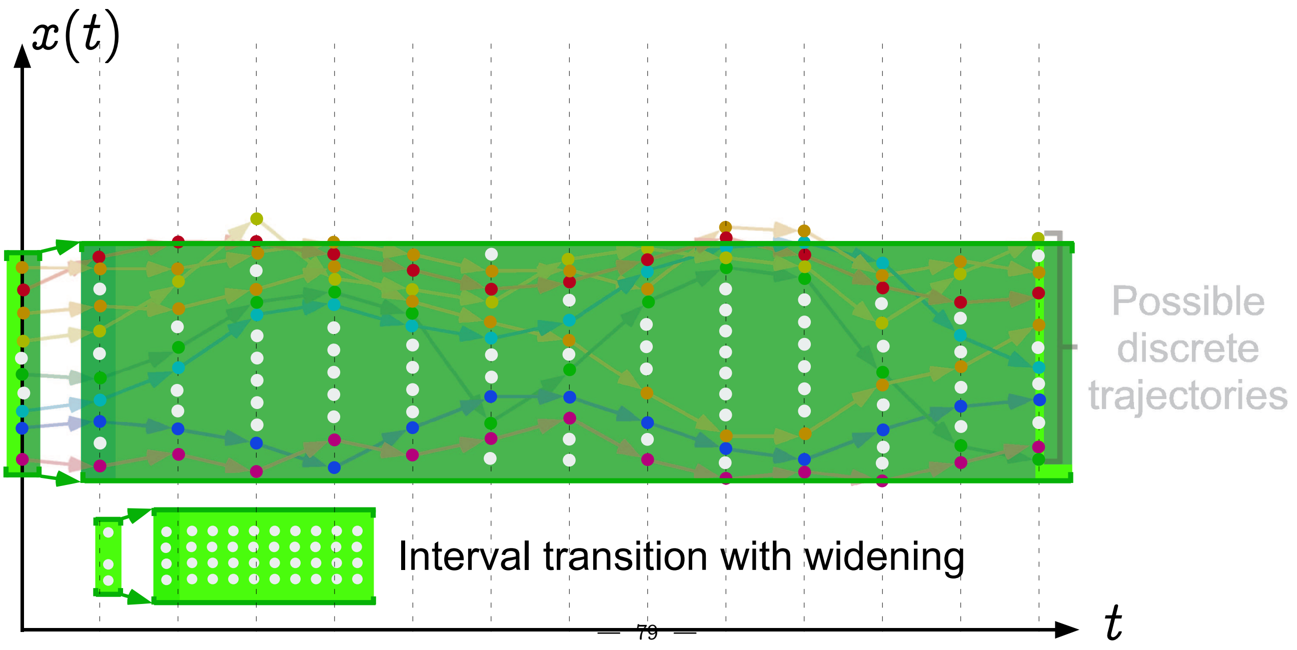
Graphic example: upward iteration with widening



Graphic example: upward iteration with widening

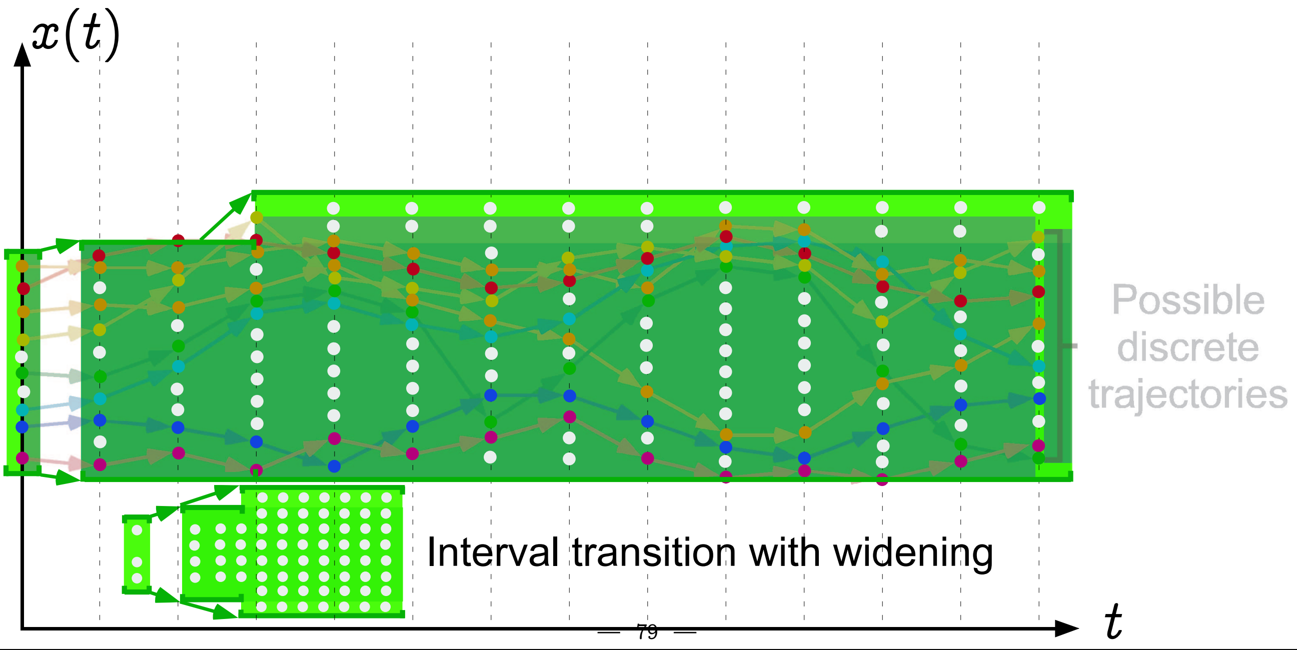


Graphic example: upward iteration with widening



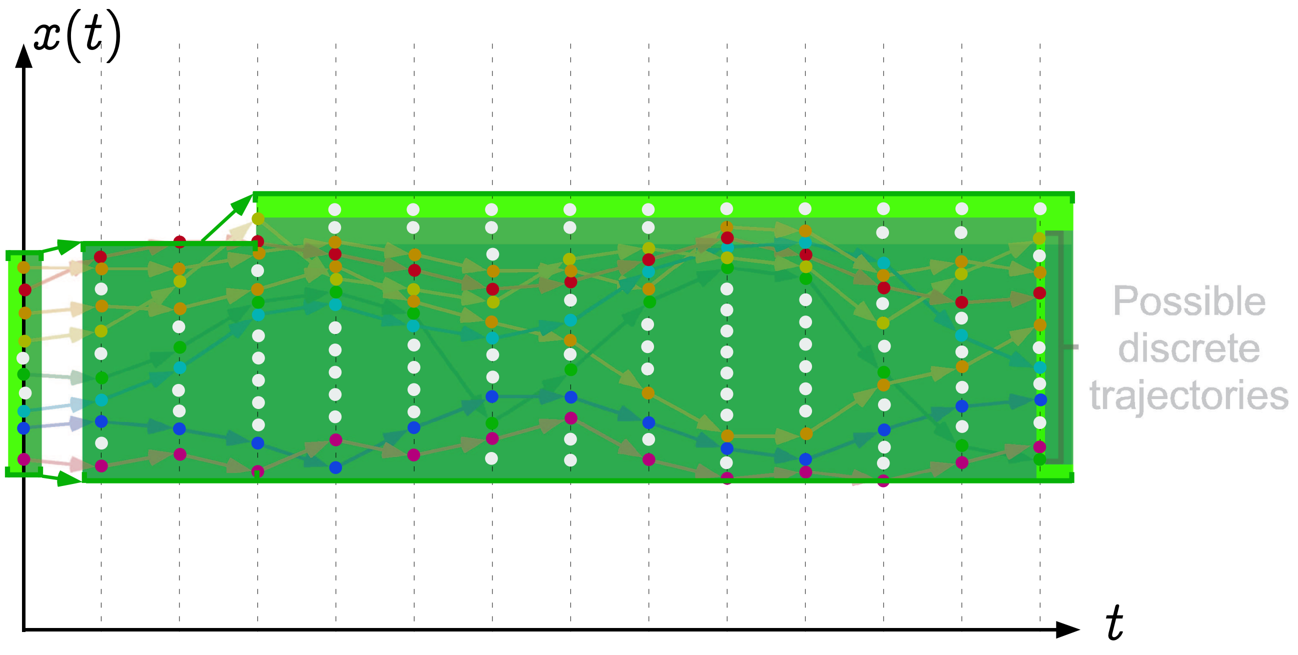
$x(t)$
Possible discrete trajectories
Interval transition with widening
t

Graphic example: upward iteration with widening



$x(t)$
Possible discrete trajectories
Interval transition with widening
t

Graphic example: stability of the upward iteration


$x(t)$
t
Possible discrete trajectories

Convergence acceleration with widening

Widening operator

A widening operator $\nabla \in \overline{L} \times \overline{L} \mapsto \overline{L}$ is such that:

– Correctness:

- $\forall x, y \in \overline{L} : \gamma(x) \sqsubseteq \gamma(x \nabla y)$
- $\forall x, y \in \overline{L} : \gamma(y) \sqsubseteq \gamma(x \nabla y)$

– Convergence:

- for all increasing chains $x^0 \sqsubseteq x^1 \sqsubseteq \ldots$, the increasing chain defined by $y^0 = x^0, \ldots, y^{i+1} = y^i \nabla x^{i+1}, \ldots$ is not strictly increasing.

Fixpoint approximation with widening

The upward iteration sequence with widening:

– $\hat{X}^0 = \overline{\bot}$ (infimum)

– $\hat{X}^{i+1} = \hat{X}^i$ if $\overline{F}(\hat{X}^i) \sqsubseteq \hat{X}^i$

$\quad = \hat{X}^i \,\nabla\, F(\hat{X}^i)$ otherwise

is ultimately stationary and its limit $\hat{A}$ is a sound upper approximation of $\mathsf{lfp}^{\overline{\bot}}\ \overline{F}$:

$$\mathsf{lfp}^{\overline{\bot}}\ \overline{F} \sqsubseteq \hat{A}$$

Interval widening

– $\overline{L} = \{\bot\} \cup \{[\ell, u] \mid \ell, u \in \mathbb{Z} \cup \{-\infty\} \wedge u \in \mathbb{Z} \cup \{\} \wedge \ell \leq u\}$

– The widening extrapolates unstable bounds to infinity:

$$\bot \,\nabla\, X = X$$
$$X \,\nabla\, \bot = X$$
$$[\ell_0,\ u_0] \,\nabla\, [\ell_1,\ u_1] = [\ \mathrm{f}\ \ell_1 < \ell_0 \text{ then } -\infty \text{ else } \ell_0,$$
$$\mathrm{f}\ \mathrm{u}_1 > \mathrm{u}_0 \text{ then } +\infty \text{ else } \mathrm{u}_0]$$

Not monotone. For example $[0,\ 1] \sqsubseteq [0,\ 2]$ but $[0,\ 1] \,\nabla\, [0,\ 2] = [0,\ +\infty] \not\sqsubseteq [0,\ 2] = [0,\ 2] \,\nabla\, [0,\ 2]$

Example: Interval analysis (1975)

Program to be analyzed:

```
   x := 1;
1:
   while x < 10000 do
2:
       x := x + 1
3:
   od;
4:
```

Example: Interval analysis (1975)

Equations (abstract interpretation of the semantics):

```
   x := 1;
1:
   while x < 10000 do
2:
       x := x + 1
3:
   od;
4:
```

$$\begin{cases} X_1 = [1,1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1,1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

Example: Interval analysis (1975)

Resolution by chaotic increasing iteration:

```
   x := 1;
1:
   while x < 10000 do
2:
       x := x + 1
3:
   od;
4:
```

$$\begin{cases} X_1 = [1,1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1,1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = \emptyset \\ X_2 = \emptyset \\ X_3 = \emptyset \\ X_4 = \emptyset \end{cases}$$

Example: Interval analysis (1975)

Increasing chaotic iteration:

```
   x := 1;
1:
   while x < 10000 do
2:
       x := x + 1
3:
   od;
4:
```

$$\begin{cases} X_1 = [1,1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1,1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

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Example: Interval analysis (1975)

Increasing chaotic iteration:

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Example: Interval analysis (1975)

Increasing chaotic iteration:

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Example: Interval analysis (1975)

Increasing chaotic iteration:

```
   x := 1;
1:
   while x < 10000 do
2:
       x := x + 1
3:
   od;
4:
```

$$\begin{cases} X_1 = [1,1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1,1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = [1,1] \\ X_2 = [1,2] \\ X_3 = [2,2] \\ X_4 = \emptyset \end{cases}$$

Example: Interval analysis (1975)

Increasing chaotic iteration: **convergence !**

```
   x := 1;
1:
   while x < 10000 do
2:
       x := x + 1
3:
   od;
4:
```

$$\begin{cases} X_1 = [1,1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1,1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = [1,1] \\ X_2 = [1,2] \\ X_3 = [2,3] \\ X_4 = \emptyset \end{cases}$$

Example: Interval analysis (1975)

Increasing chaotic iteration: **convergence !!**

```
   x := 1;
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$$\begin{cases} X_1 = [1,1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1,1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

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Example: Interval analysis (1975)

Increasing chaotic iteration: **convergence !!!**

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```

$$\begin{cases} X_1 = [1,1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1,1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = [1,1] \\ X_2 = [1,3] \\ X_3 = [2,4] \\ X_4 = \emptyset \end{cases}$$

Example: Interval analysis (1975)

Increasing chaotic iteration: **convergence !!!!**

```
   x := 1;
1:
   while x < 10000 do
2:
       x := x + 1
3:
   od;
4:
```

$$\begin{cases} X_1 = [1,1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1,1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

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       x := x + 1
3:
   od;
4:
```

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Example: Interval analysis (1975)

Increasing chaotic iteration: **convergence !!!!!!**

```
   x := 1;
1:
   while x < 10000 do
2:
       x := x + 1
3:
   od;
4:
```

$$\begin{cases} X_1 = [1,1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1,1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = [1,1] \\ X_2 = [1,5] \\ X_3 = [2,5] \\ X_4 = \emptyset \end{cases}$$

Example: Interval analysis (1975)

Increasing chaotic iteration: **convergence !!!!!!!**

```
   x := 1;
1:
   while x < 10000 do
2:
       x := x + 1
3:
   od;
4:
```

$$\begin{cases} X_1 = [1,1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1,1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = [1,1] \\ X_2 = [1,5] \\ X_3 = [2,6] \\ X_4 = \emptyset \end{cases}$$

Example: Interval analysis (1975)

Convergence speed-up by widening:

```
   x := 1;
1:
   while x < 10000 do
2:
       x := x + 1
3:
   od;
4:
```

$$\begin{cases} X_1 = [1,1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1,1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = [1,1] \\ X_2 = [1,+\infty] \quad \Leftarrow \text{widening} \\ X_3 = [2,6] \\ X_4 = \emptyset \end{cases}$$

Example: Interval analysis (1975)

Decreasing chaotic iteration:

```
   x := 1;
1:
   while x < 10000 do
2:
       x := x + 1
3:
   od;
4:
```

$$\begin{cases} X_1 = [1,1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1,1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = [1,1] \\ X_2 = [1,+\infty] \\ X_3 = [2,+\infty] \\ X_4 = \emptyset \end{cases}$$

Example: Interval analysis (1975)

Decreasing chaotic iteration:

```
   x := 1;
1:
   while x < 10000 do
2:
       x := x + 1
3:
   od;
4:
```

$$\begin{cases} X_1 = [1, 1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1, 1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = [1, 1] \\ X_2 = [1, 9999] \\ X_3 = [2, +\infty] \\ X_4 = \emptyset \end{cases}$$

Example: Interval analysis (1975)

Decreasing chaotic iteration:

```
   x := 1;
1:
   while x < 10000 do
2:
          x := x + 1
3:
   od;
4:
```

$$\begin{cases} X_1 = [1, 1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1, 1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = [1, 1] \\ X_2 = [1, 9999] \\ X_3 = [2, +10000] \\ X_4 = \emptyset \end{cases}$$

Example: Interval analysis (1975)

Final solution:

```
   x := 1;
1:
   while x < 10000 do
2:
       x := x + 1
3:
   od;
4:
```

$$\begin{cases} X_1 = [1,1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1,1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = [1,1] \\ X_2 = [1, 9999] \\ X_3 = [2, +10000] \\ X_4 = [+10000, +10000] \end{cases}$$

Example: Interval analysis (1975)

Result of the interval analysis:

```
   x := 1;
1: {x = 1}
   while x < 10000 do
2: {x ∈ [1, 9999]}
       x := x + 1
3: {x ∈ [2, +10000]}
   od;
4: {x = 10000}
```

$$\begin{cases} X_1 = [1,1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1,1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = [1,1] \\ X_2 = [1, 9999] \\ X_3 = [2, +10000] \\ X_4 = [+10000, +10000] \end{cases}$$

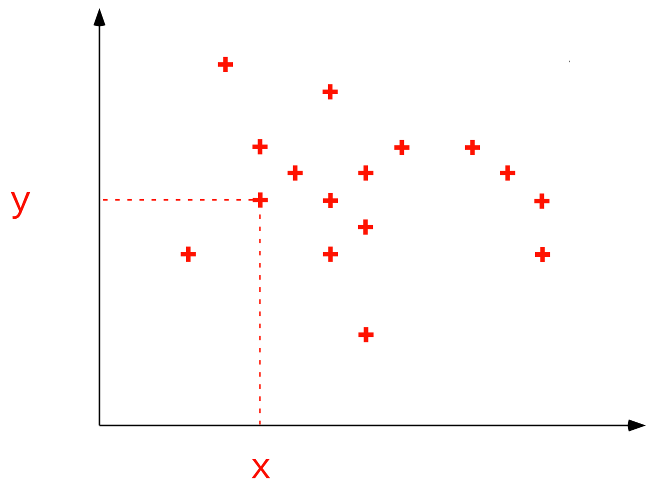
Example: Interval analysis (1975)

Checking absence of runtime errors with interval analysis:

```
   x := 1;
1: {x = 1}
   while x < 10000 do
2: {x ∈ [1, 9999]}            ⟵ no overflow
       x := x + 1
3: {x ∈ [2, +10000]}
   od;
4: {x = 10000}
```

Refinement of abstractions

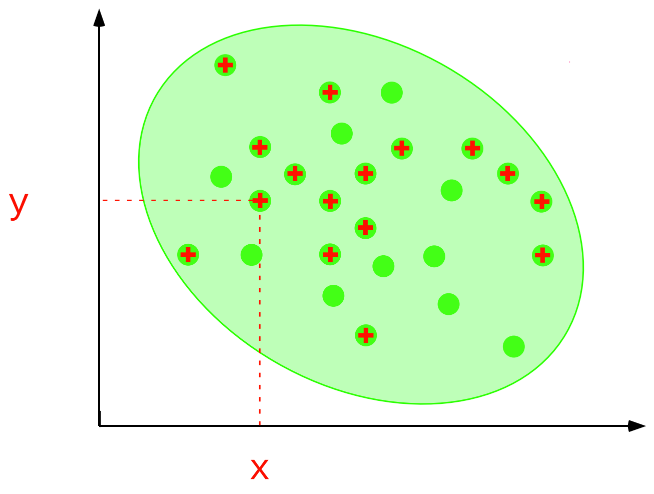
Approximations of an [in]finite set of points:



$\{\ldots, \langle 19,\ 77\rangle, \ldots,$
$\langle 20,\ 03\rangle, \ldots\}$

Approximations of an [in]finite set of points:

from above



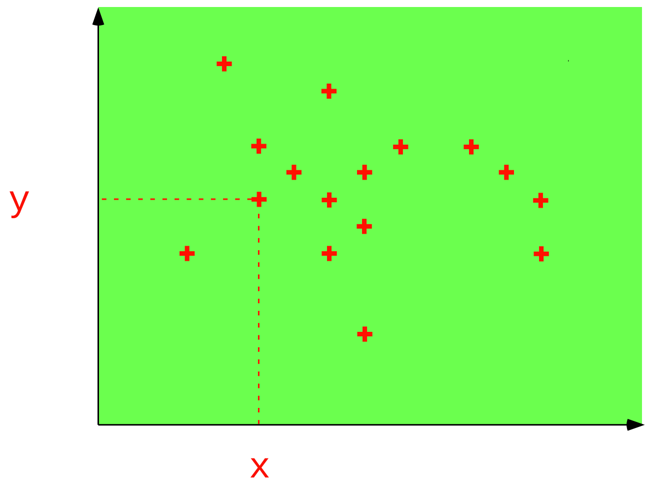
$\{\ldots, \langle 19,\ 77\rangle, \ldots,$

$\langle 20,\ 03\rangle, \langle ?,\ ?\rangle, \ldots\}$

From Below: dual [3] + combinations.

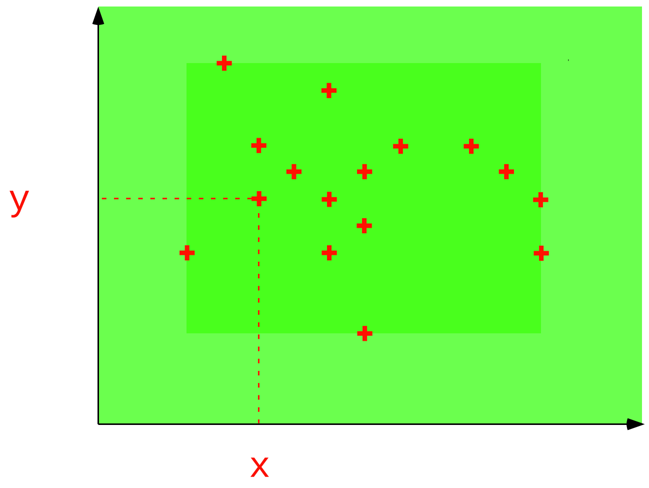
[3] Trivial for finite states (liveness model-checking), more difficult for infinite states (variant functions).

Effective computable approximations of an [in]finite set of points; Signs [4]



$$\begin{cases} x \geq 0 \\ y \geq 0 \end{cases}$$

Effective computable approximations of an [in]finite set of points; Intervals [5]

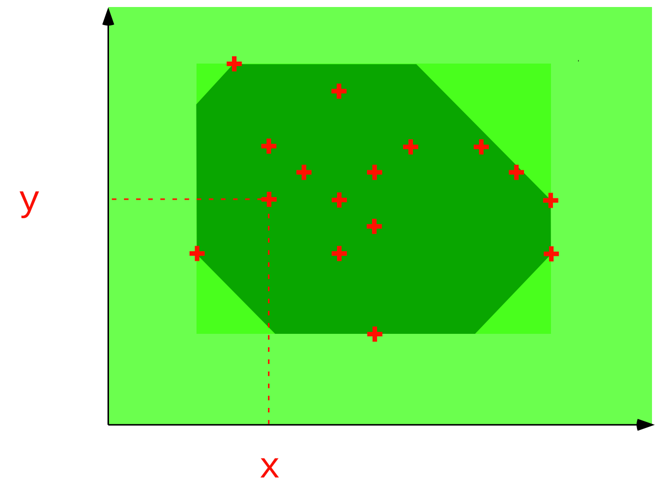


$$\begin{cases} x \in [19, \ 77] \\ y \in [20, \ 03] \end{cases}$$

[4] P. Cousot & R. Cousot. *Systematic design of program analysis frameworks.* ACM POPL'79, pp. 269–282, 1979.

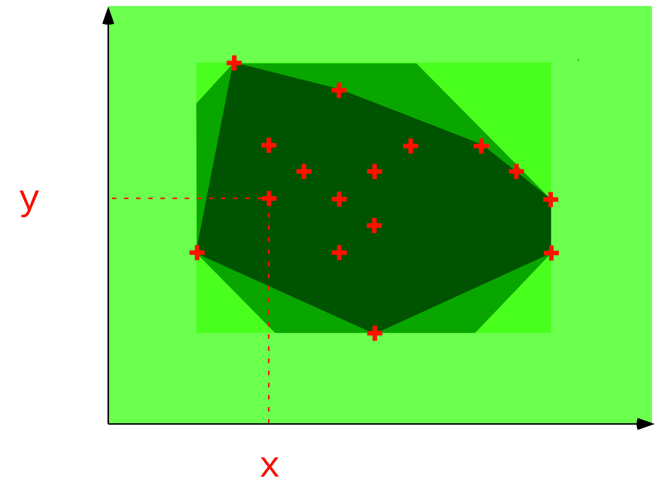
[5] P. Cousot & R. Cousot. *Static determination of dynamic properties of programs.* Proc. 2nd Int. Symp. on Programming, Dunod, 1976.

Effective computable approximations of an [in]finite set of points; Octagons [6]



$$\begin{cases} 1 \le x \le 9 \\ x + y \le 77 \\ 1 \le y \le 9 \\ x - y \le 99 \end{cases}$$

Effective computable approximations of an [in]finite set of points; Polyhedra [7]

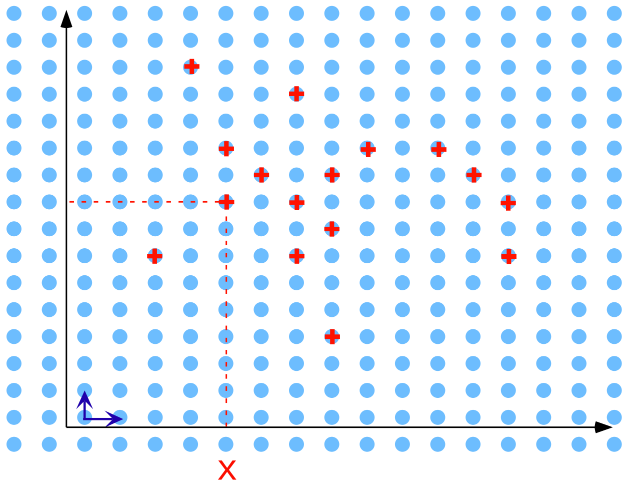


$$\begin{cases} 19x + 77y \le 2004 \\ 20x + 03y \ge 0 \end{cases}$$

[6] A. Miné. *A New Numerical Abstract Domain Based on Difference-Bound Matrices*. PADO '2001. LNCS 2053, pp. 155–172. Springer 2001. See the *The Octagon Abstract Domain Library* on `http://www.di.ens.fr/~mine/oct/`

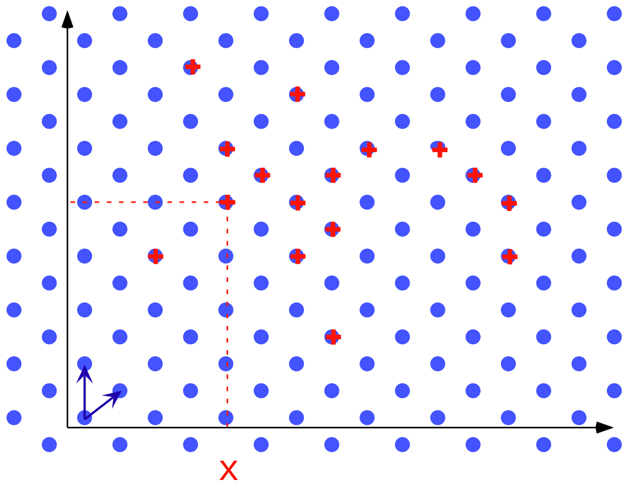
[7] P. Cousot & N. Halbwachs. *Automatic discovery of linear restraints among variables of a program*. ACM POPL, 1978, pp. 84–97.

Effective computable approximations of an [in]finite set of points; Simple congruences [8]



$$\begin{cases} x = 19 \bmod 77 \\ y = 20 \bmod 99 \end{cases}$$

Effective computable approximations of an [in]finite set of points; Linear congruences [9]

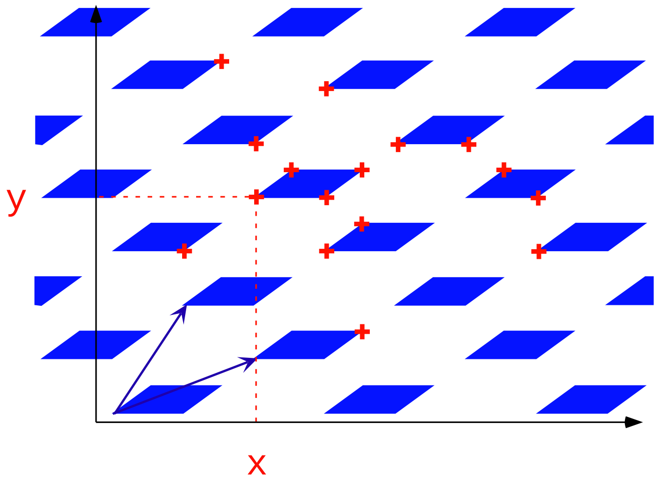


$$\begin{cases} 1x + 9y = 7 \bmod 8 \\ 2x - 1y = 9 \bmod 9 \end{cases}$$

[8] Ph. Granger. *Static Analysis of Arithmetical Congruences*. Int. J. Comput. Math. 30, 1989, pp. 165–190.

[9] Ph. Granger. *Static Analysis of Linear Congruence Equalities among Variables of a Program*. TAPSOFT '91, pp. 169–192. LNCS 493, Springer, 1991.

Effective computable approximations of an [in]finite set of points; Trapezoidal linear congruences [10]

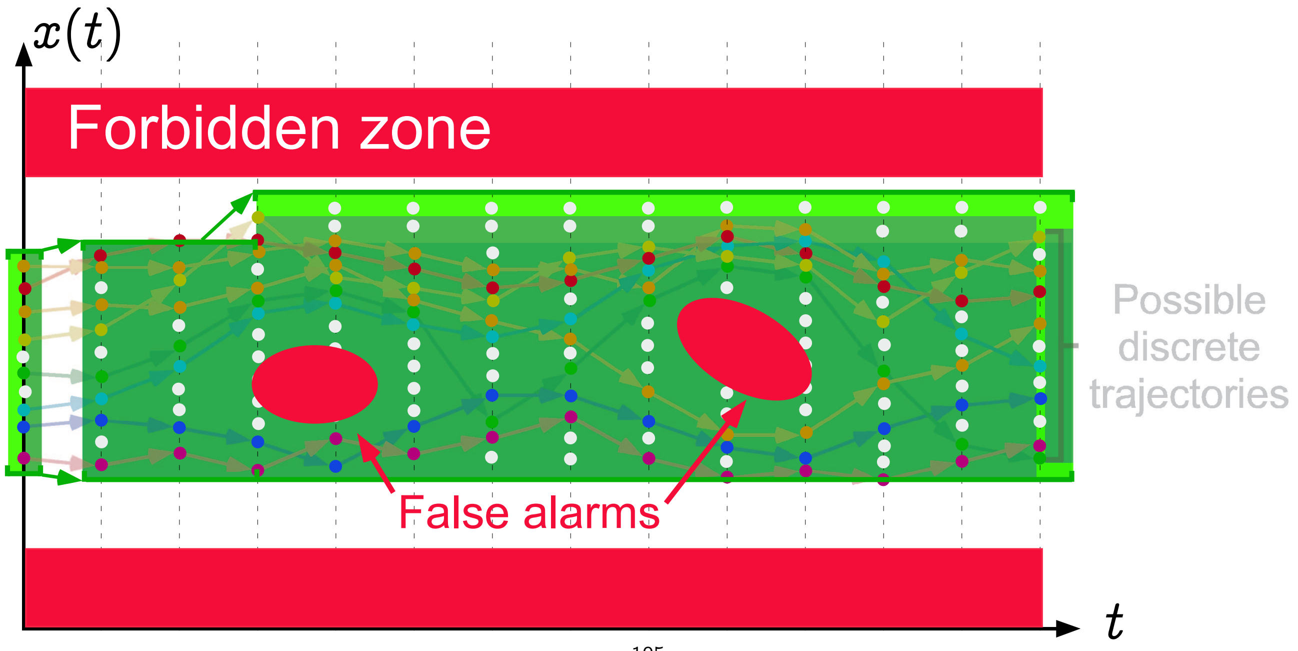


$$\begin{cases} 1x + 9y \in [0, 77] \bmod 10 \\ 2x - 1y \in [0, 99] \bmod 11 \end{cases}$$

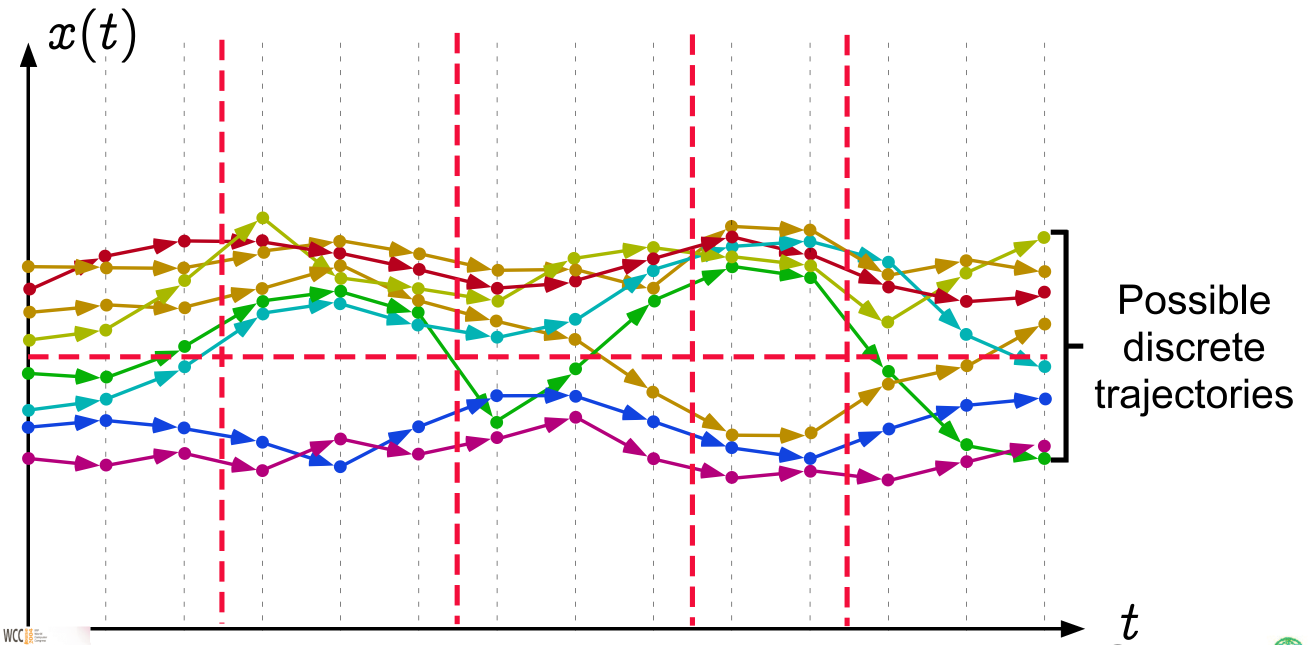
Refinement of iterates

10 F. Masdupuy. *Array Operations Abstraction Using Semantic Analysis of Trapezoid Congruences.* ACM ICS '92.

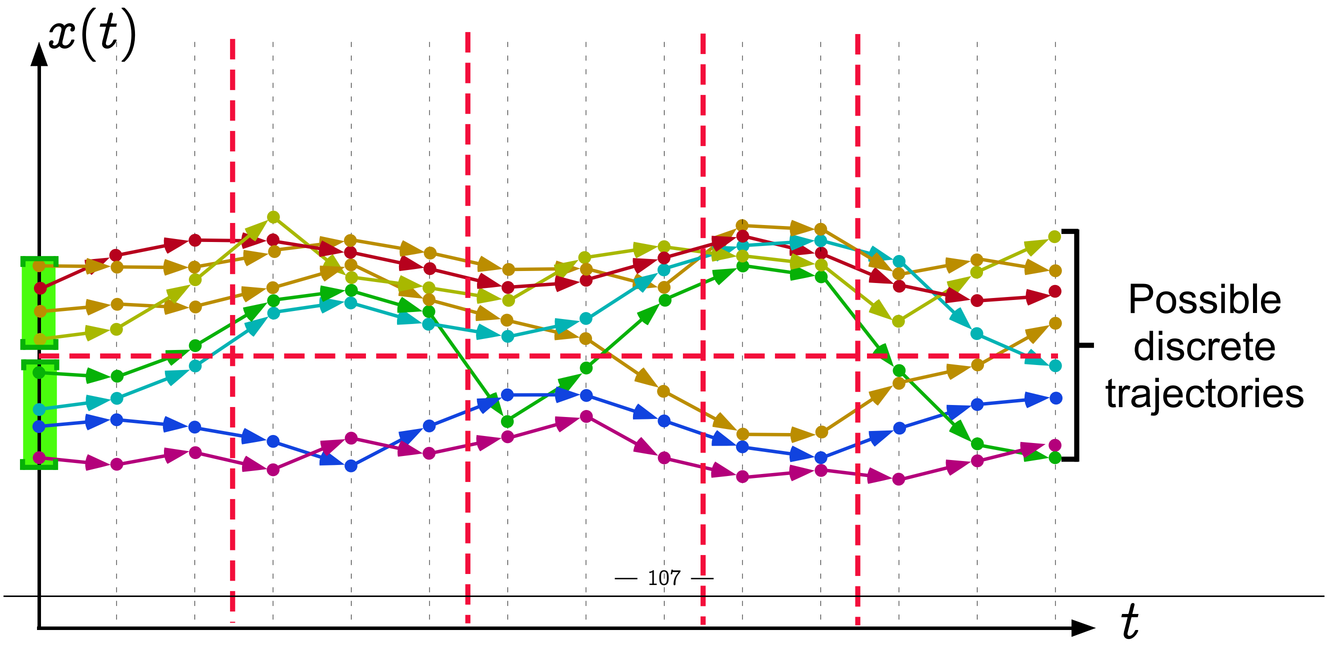
Graphic example: Refinement required by false alarms



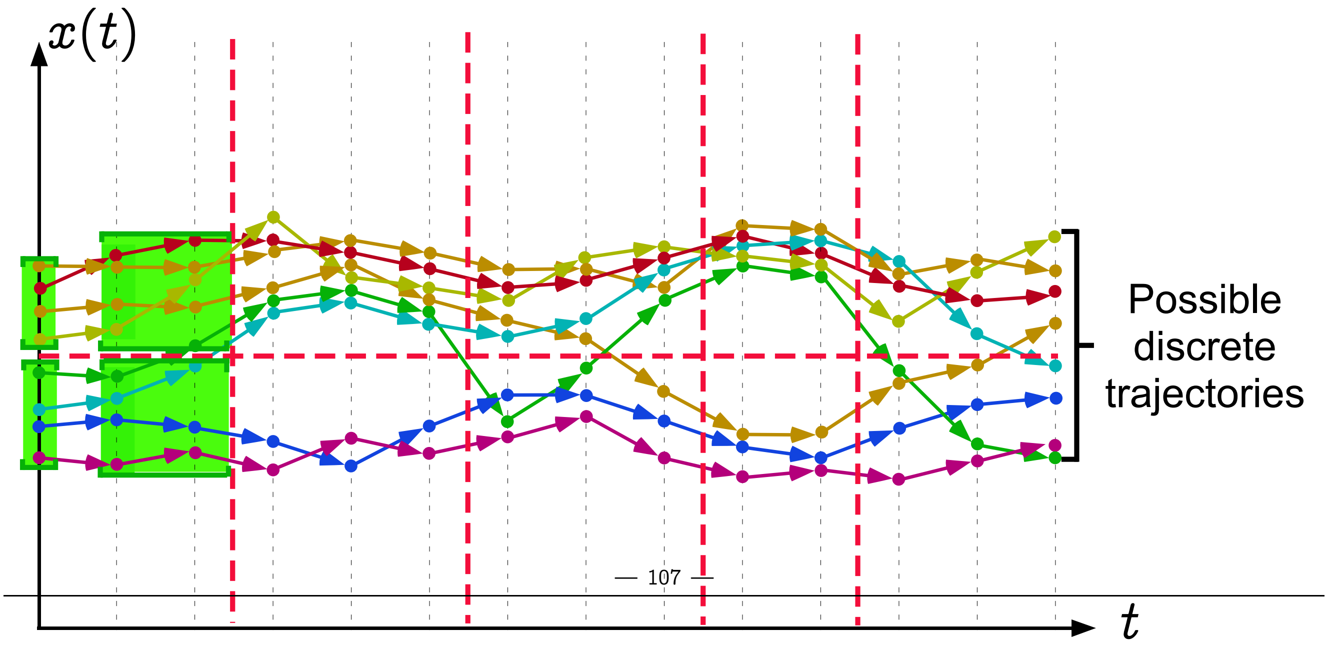
Graphic example: Partitionning



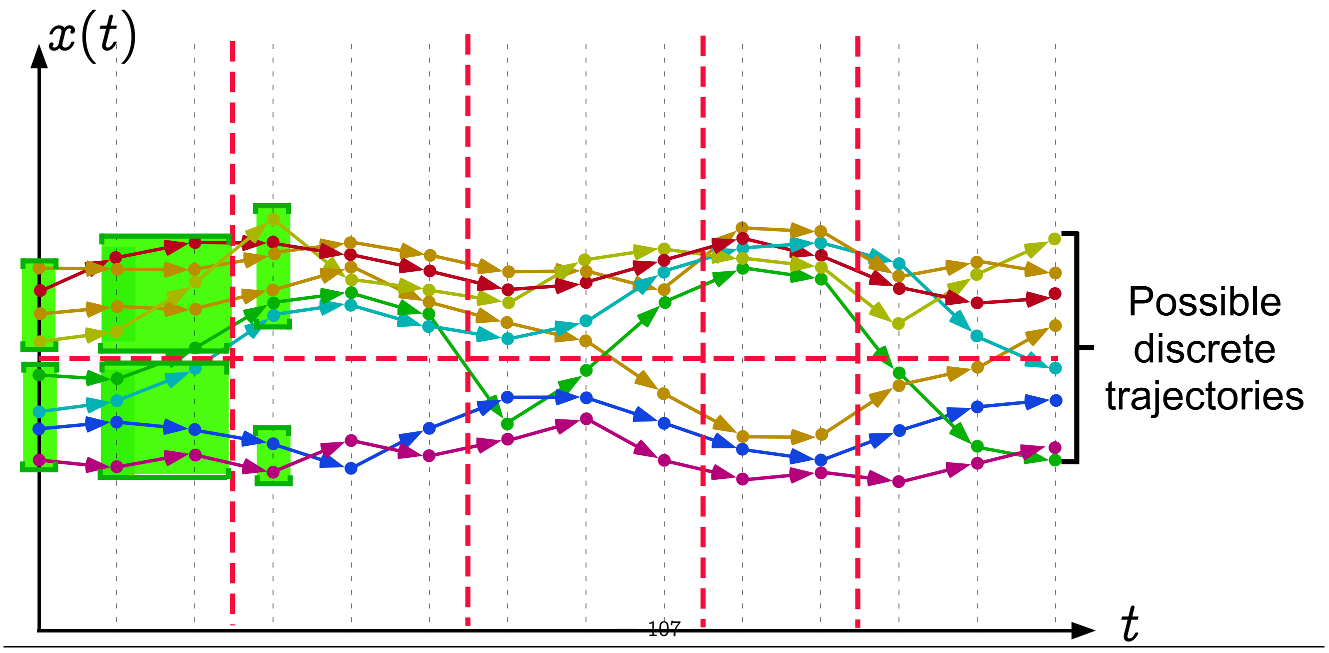
Graphic example: partitionned upward iteration with widening



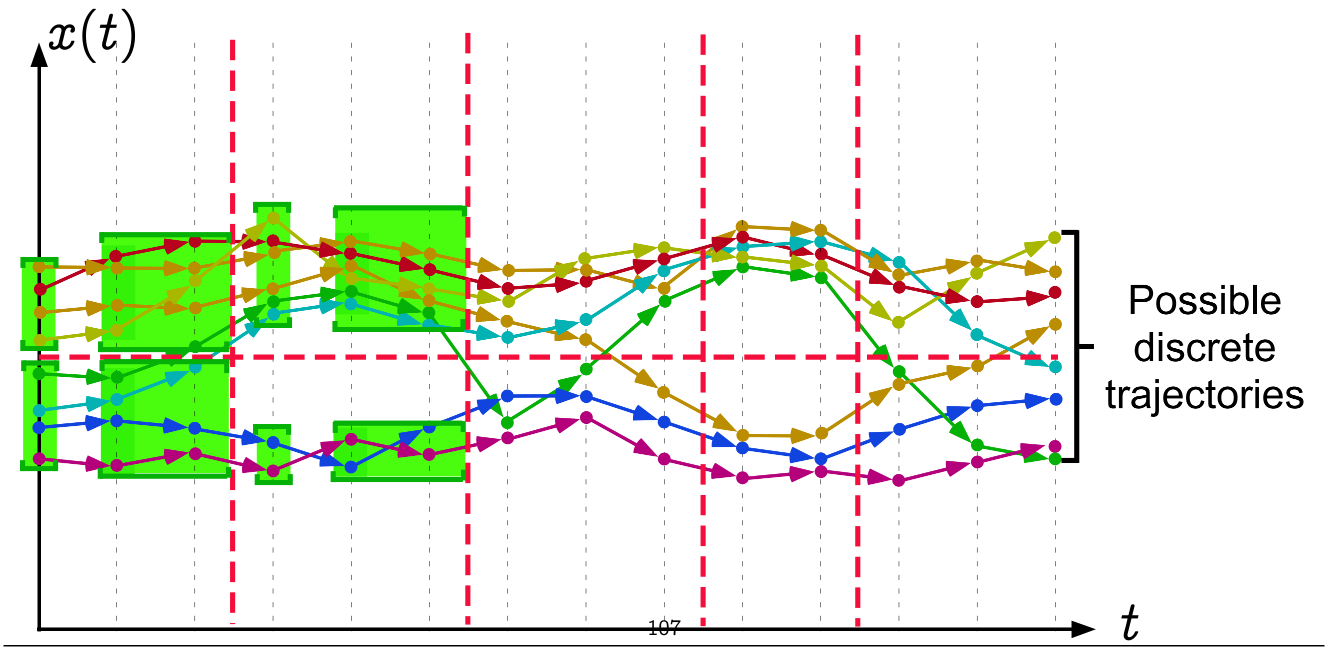
Graphic example: partitionned upward iteration with widening



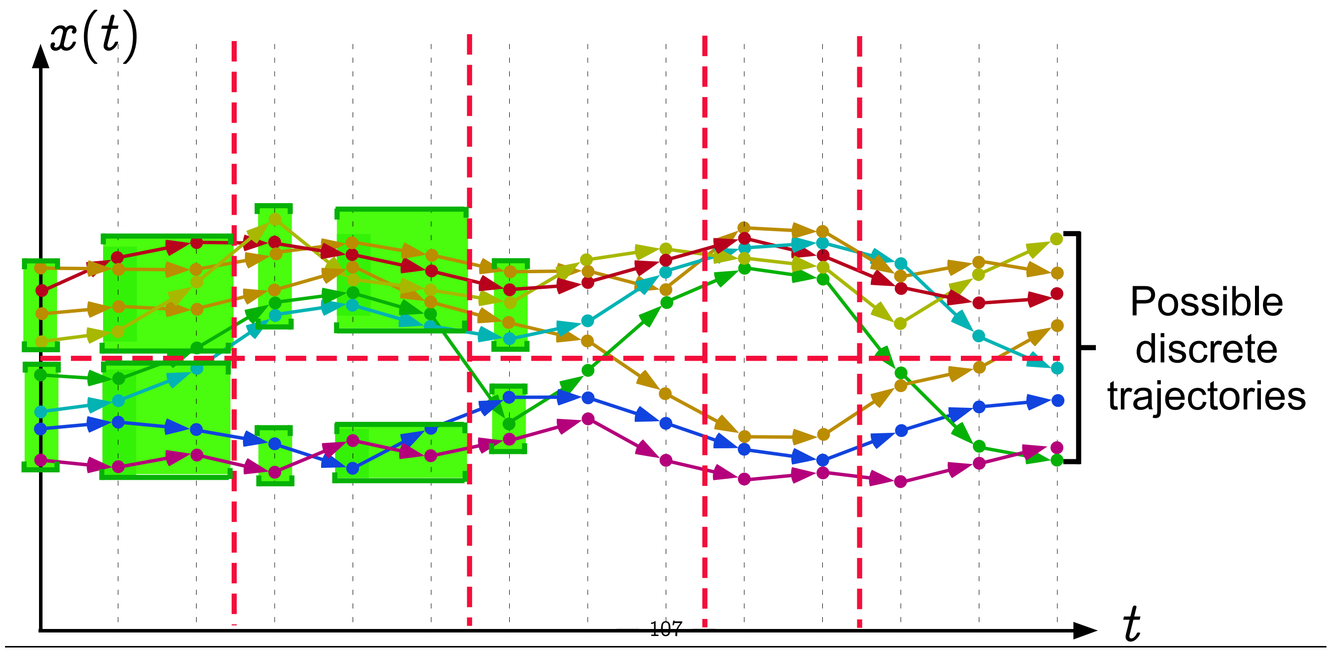
Graphic example: partitionned upward iteration with widening



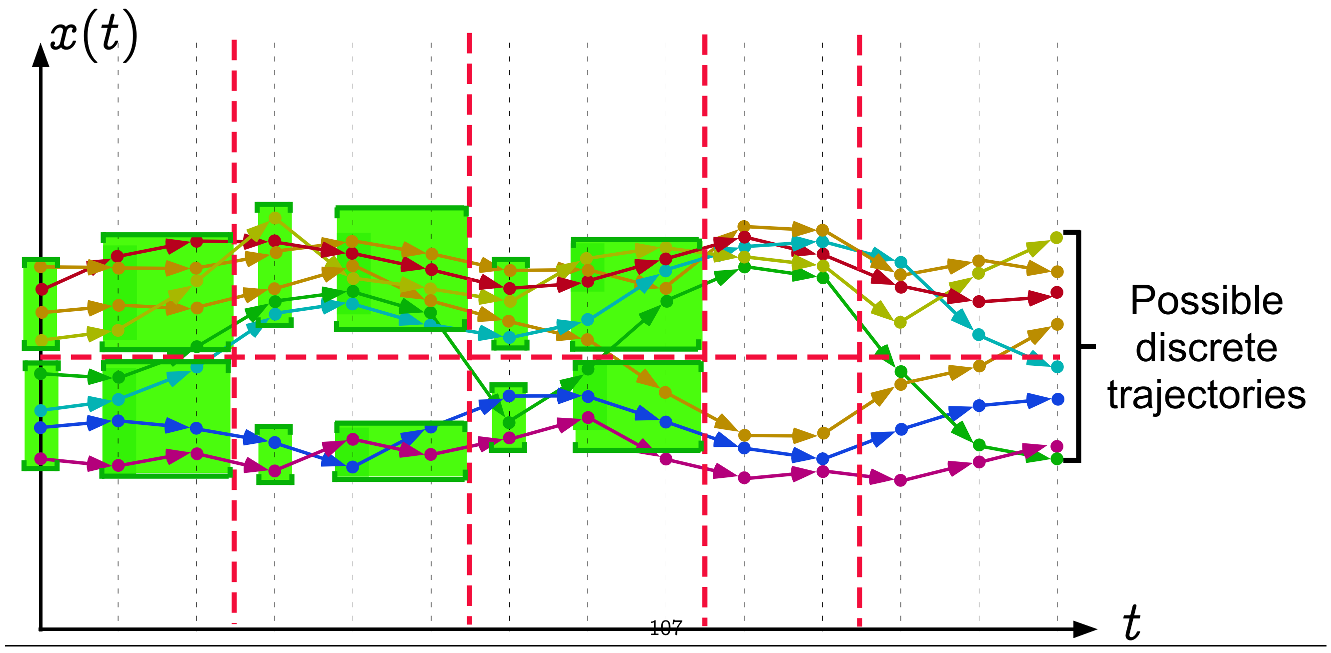
Graphic example: partitionned upward iteration with widening



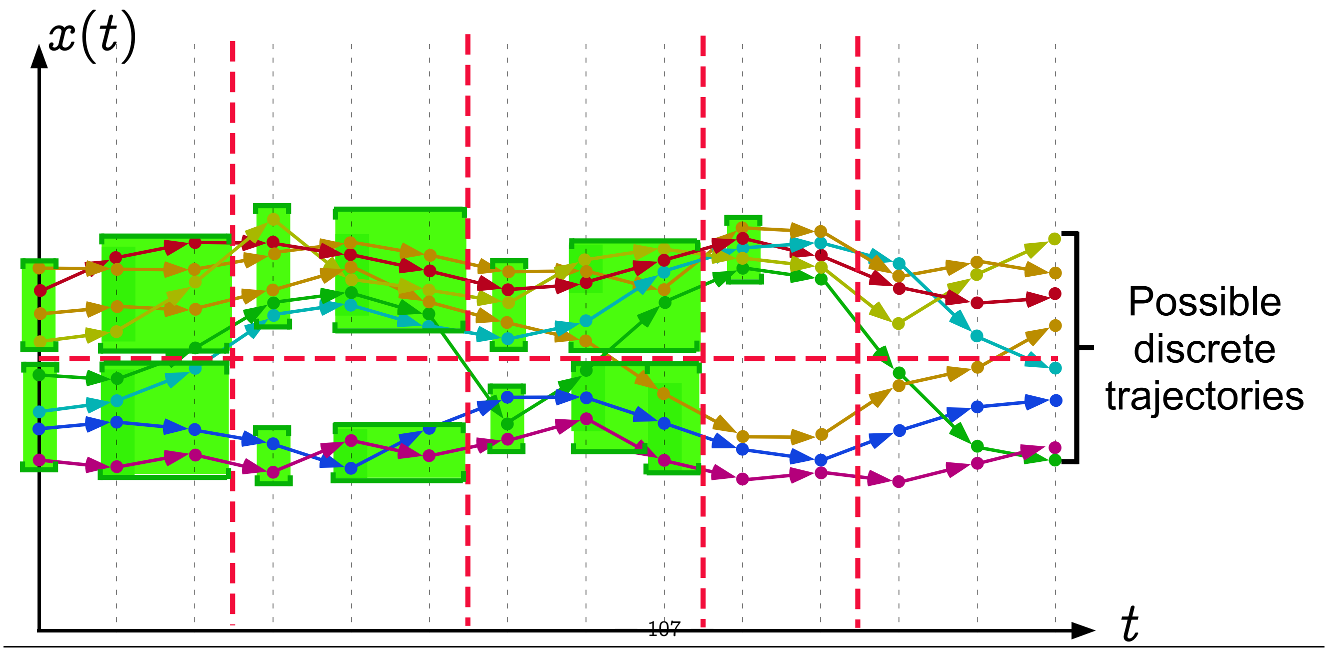
Graphic example: partitionned upward iteration with widening



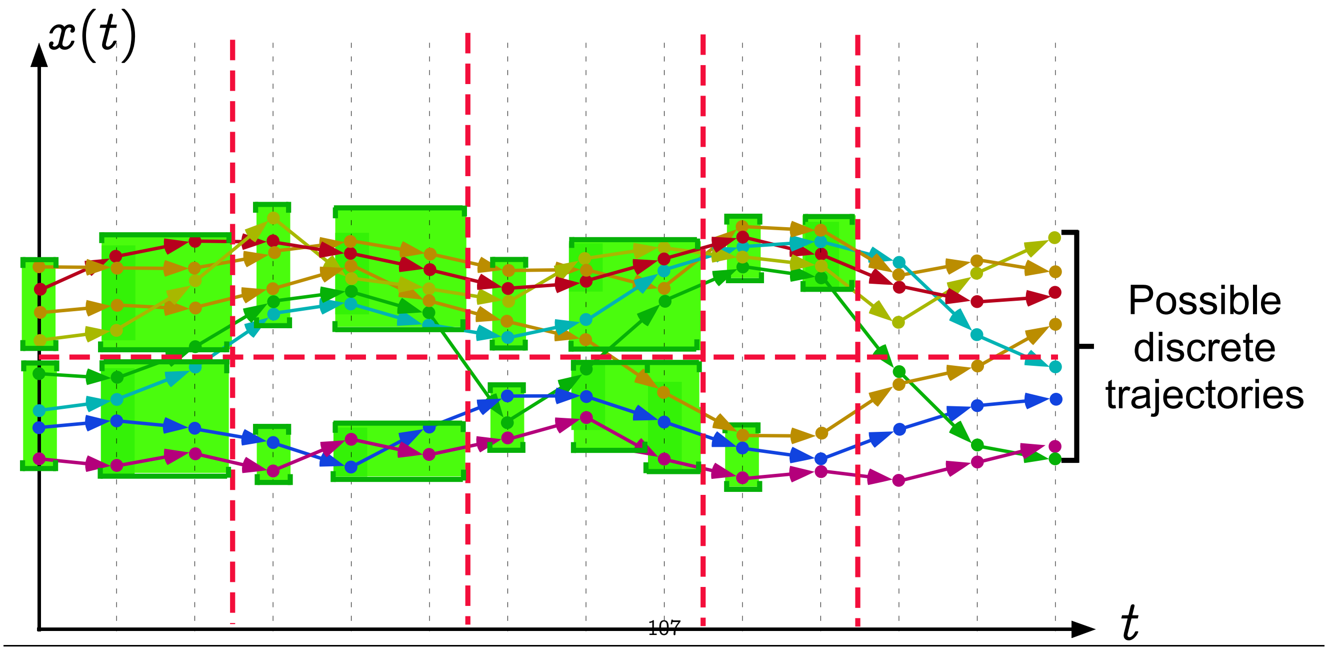
Graphic example: partitionned upward iteration with widening



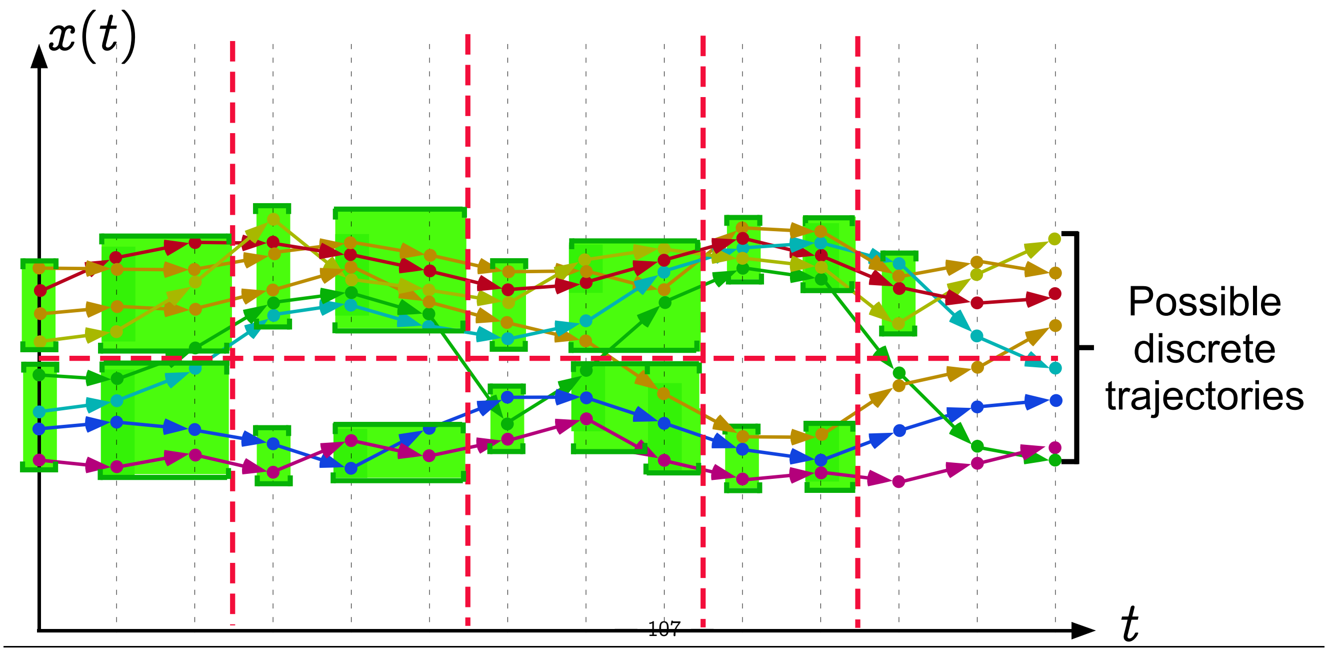
Graphic example: partitionned upward iteration with widening



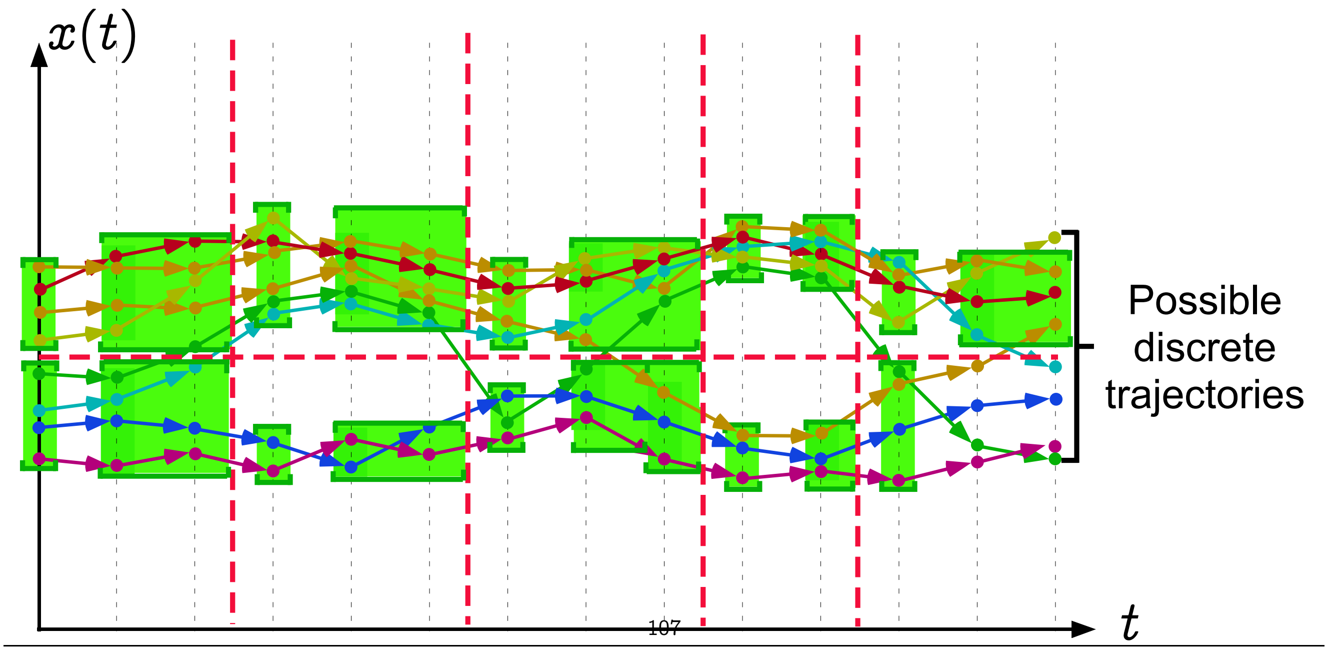
Graphic example: partitionned upward iteration with widening



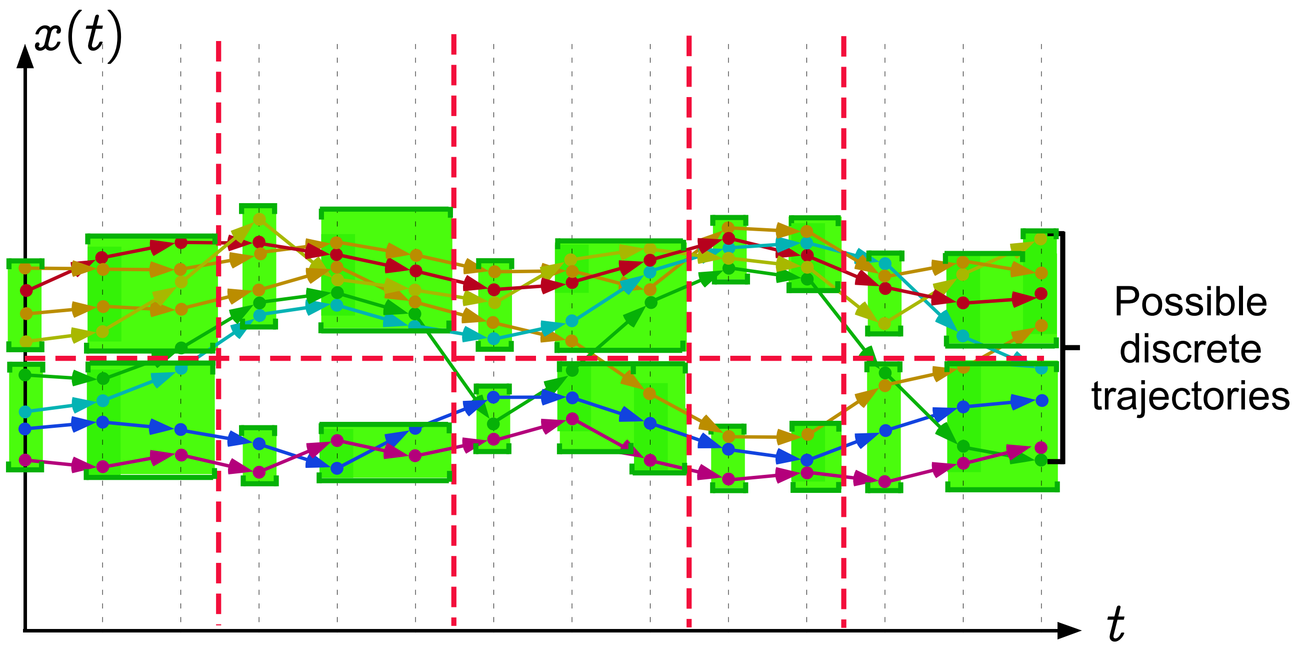
Graphic example: partitionned upward iteration with widening



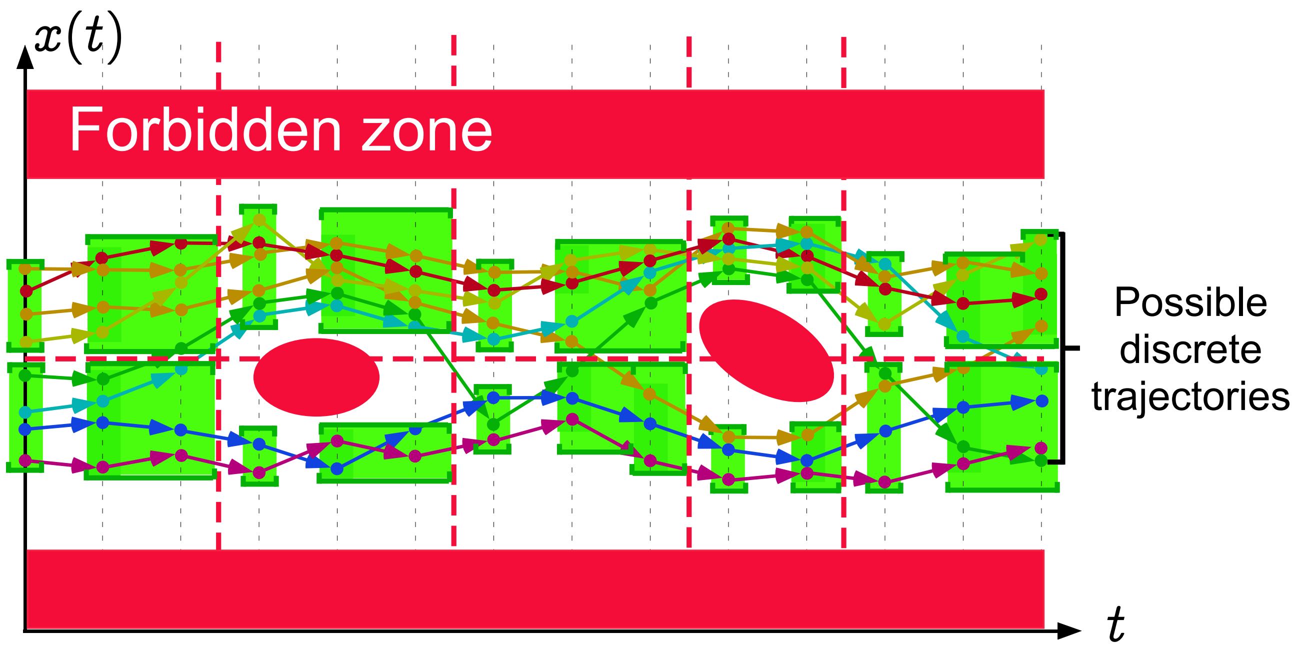
Graphic example: partitionned upward iteration with widening



Graphic example: partitionned upward iteration with widening



Graphic example: safety verification



Examples of partitionnings

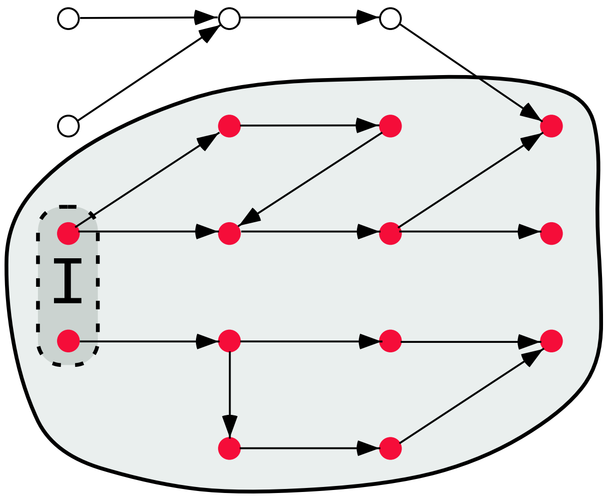
– sets of control states: attach local information to program points instead of global information for the whole program/procedure/loop
– sets of data states:
  - case analysis (test, switches)
– fixpoint iterates:
  - widening with threshold set

Interval widening with threshold set

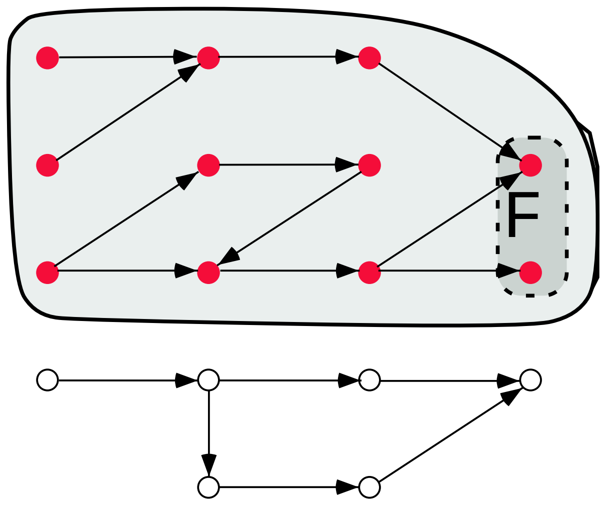
– The threshold set T is a finite set of numbers (plus $+\infty$ and $-\infty$),
– $$[a, b] \nabla_T [a', b'] = [\mathit{if}\ a' < a\ \mathit{then}\ \max\{\ell \in T \mid \ell \leq a'\}\ \mathit{else}\ a,\ \mathit{if}\ b' > b\ \mathit{then}\ \mathrm{m\ n}\{h \in T \mid h \geq b'\}\ \mathit{else}\ b]\ .$$
– Examples (intervals):
  - sign analysis: $T = \{-\infty, 0, +\infty\}$;
  - strict sign analysis: $T = \{-\infty, -1, 0, +1, +\infty\}$;
– T is a parameter of the analysis.

Combinations of abstractions

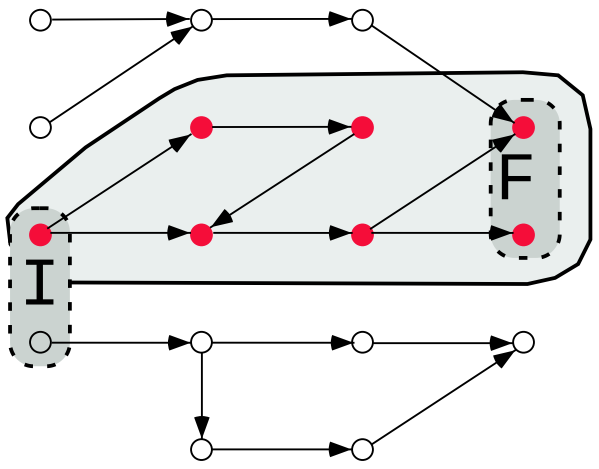
Forward/reachability analysis



Backward/ancestry analysis



Iterated forward/backward analysis



Example of iterated forward/backward analysis

Arithmetical mean of two integers x and y:

```
{x>=y}
  while (x <> y) do
    {x>=y+2}
       x := x - 1;
    {x>=y+1}
       y := y + 1
    {x>=y}
  od
{x=y}
```

Necessarily $x \geq y$ for proper termination

Example of iterated forward/backward analysis

Adding an auxiliary counter k decremented in the loop body and asserted to be null on loop exit:

```
{x=y+2k,x>=y}
  while (x <> y) do
    {x=y+2k,x>=y+2}
      k := k - 1;
    {x=y+2k+2,x>=y+2}
      x := x - 1;
    {x=y+2k+1,x>=y+1}
      y := y + 1
    {x=y+2k,x>=y}
  od
{x=y,k=0}
  assume (k = 0)
{x=y,k=0}
```

Moreover the difference of x and y must be even for proper termination

Bibliography

Seminal papers

– Patrick Cousot & Radhia Cousot. Abstract interpretation: a unified lattice model for static analysis of programs by construction or approximation of fixpoints. In 4th Symp. on Principles of Programming Languages, pages 238—252. ACM Press, 1977.

– Patrick Cousot & Nicolas Halbwachs. Automatic discovery of linear restraints among variables of a program. In 5th Symp. on Principles of Programming Languages, pages 84—97. ACM Press, 1978.

– Patrick Cousot & Radhia Cousot. Systematic design of program analysis frameworks. In 6th Symp. on Principles of Programming Languages pages 269—282. ACM Press, 1979.

Recent surveys

– Patrick Cousot. Interprétation abstraite. Technique et Science Informatique, Vol. 19, Nb 1-2-3. Janvier 2000, Hermès, Paris, France. pp. 155-164.
– Patrick Cousot. Abstract Interpretation Based Formal Methods and Future Challenges. In Informatics, 10 Years Back — 10 Years Ahead, R. Wilhelm (Ed.), LNCS 2000, pp. 138-156, 2001.
– Patrick Cousot & Radhia Cousot. Abstract Interpretation Based Verification of Embedded Software: Problems and Perspectives. In Proc. 1st Int. Workshop on Embedded Software, EMSOFT 2001, T.A. Henzinger & C.M. Kirsch (Eds.), LNCS 2211, pp. 97–113. Springer, 2001.

Conclusion

Theoretical applications of abstract interpretation

– **Static Program Analysis** [POPL '77,78,79] inluding **Data-flow Analysis** [POPL '79,00], **Set-based Analysis** [FPCA '95], etc
– **Syntax Analysis** [TCS 290(1) 2002]
– **Hierarchies of Semantics (including Proofs)** [POPL '92, TCS 277(1–2) 2002]
– **Typing** [POPL '97]
– **Model Checking** [POPL '00]
– **Program Transformation** [POPL '02]
– **Software watermarking** [POPL '04]

Practical applications of abstract interpretation

– **Program analysis and manipulation**: a small rate of false alarms is acceptable
  - AiT: worst case execution time – Christian Ferdinand
– **Program verification**: no false alarms is acceptable
  - TVLA: A system for generating abstract interpreters – Mooly Sagiv
  - Astrée: verification of absence of run-time errors – Laurent Mauborgne

Industrial applications of abstract interpretation

- Both to Program analysis and verification
- Experience with the industrial use of abstract interpretation-based static analysis tools – Jean Souyris (Airbus France)

THE END

More references at URL www.di.ens.fr/~cousot.