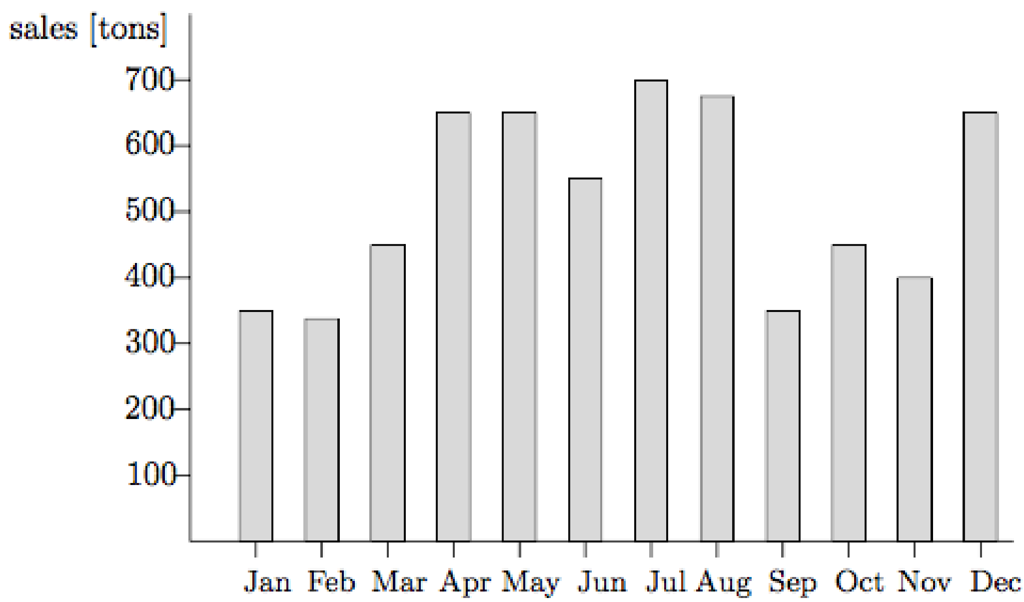
Traitement de la valeur absolue – Énoncé

Optimisation – L3 informatique

1 Gestion de production

The next application of linear programming again concerns food (which should not be surprising, given the importance of food in life and the difficulties in optimizing sleep or love). The ice cream manufacturer Icicle Works Ltd.[2] needs to set up a production plan for the next year. Based on history, extensive surveys, and bird observations, the marketing department has come up with the following prediction of monthly sales of ice cream in the next year:



Now Icicle Works Ltd. needs to set up a production schedule to meet these demands.

A simple solution would be to produce "just in time," meaning that all the ice cream needed in month i is also produced in month i, $i = 1, 2, \ldots, 12$. However, this means that the produced amount would vary greatly from month to month, and a change in the produced amount has significant costs: Temporary workers have to be hired or laid off, machines have to be adjusted, and so on. So it would be better to spread the production more evenly over the year: In months with low demand, the idle capacities of the factory could be used to build up a stock of ice cream for the months with high demand.

So another simple solution might be a completely "flat" production schedule, with the same amount produced every month. Some thought reveals that such a schedule need not be feasible if we want to end up with zero surplus at the end of the year. But even if it is feasible, it need not be ideal either, since storing ice cream incurs a nontrivial cost. It seems likely that the best production schedule should be somewhere between these two extremes (production following demand and constant production). We want a compromise minimizing the total cost resulting both from changes in production and from storage of surpluses.

To formalize this problem, let us denote the demand in month i by $d_i \geq 0$ (in tons). Then we introduce a nonnegative variable x_i for the production in month i and another nonnegative variable s_i for the total surplus in store at the end of month i. To meet the demand in month i, we may use the production in month i and the surplus at the end of month $i-1$:

$$x_i + s_{i-1} \geq d_i \quad \text{for } i = 1, 2, \ldots, 12.$$

The quantity $x_i + s_{i-1} - d_i$ is exactly the surplus after month i, and thus we have

$$x_i + s_{i-1} - s_i = d_i \quad \text{for } i = 1, 2, \ldots, 12.$$

Assuming that initially there is no surplus, we set $s_0 = 0$ (if we took the production history into account, s_0 would be the surplus at the end of the previous year). We also set $s_{12} = 0$, unless we want to plan for another year.

Among all nonnegative solutions to these equations, we are looking for one that minimizes the total cost. Let us assume that changing the production by 1 ton from month $i-1$ to month i costs €50, and that storage facilities for 1 ton of ice cream cost €20 per month. Then the total cost is expressed by the function

$$50 \sum_{i=1}^{12} |x_i - x_{i-1}| + 20 \sum_{i=1}^{12} s_i,$$

where we set $x_0 = 0$ (again, history can easily be taken into account).

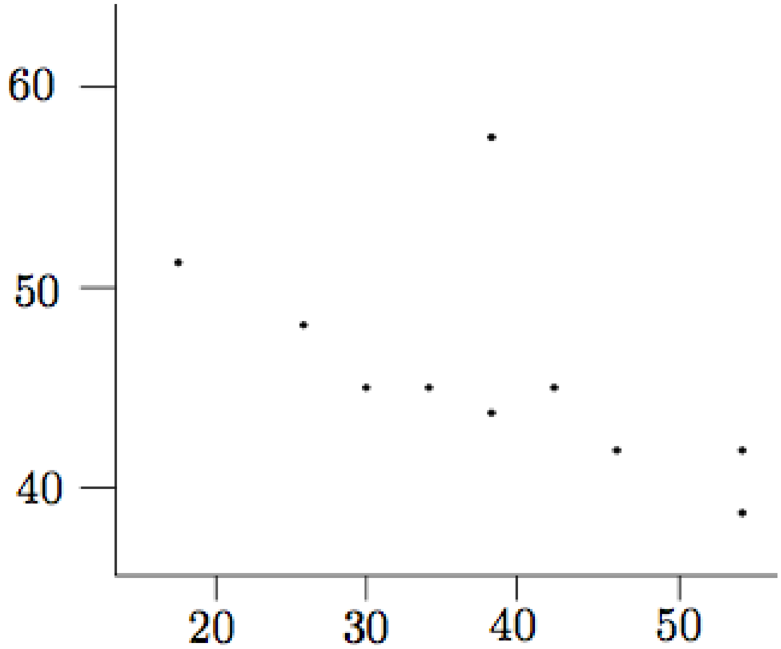
Unfortunately, this cost function is not linear. Fortunately, there is a simple but important trick that allows us to make it linear, at the price of introducing extra variables.

The change in production is either an increase or a decrease. Let us introduce a nonnegative variable y_i for the increase from month $i-1$ to month i, and a nonnegative variable z_i for the decrease. Then

$$x_i - x_{i-1} = y_i - z_i \text{ and } |x_i - x_{i-1}| = y_i + z_i.$$

Prennez des exemples ! Formulez le problème initial en programmation linéaire. Que peut on dire de l'optimum ? Résolvez numériquement.

2 Régression linaire et optimisation linéaire



How can one formulate mathematically that a given line "best fits" the points? There is no unique way, and several different criteria are commonly used for line fitting in practice.

The most popular one is the method of *least squares*, which for given points $(x_1, y_1), \ldots, (x_n, y_n)$ seeks a line with equation $y = ax + b$ minimizing the expression

$$\sum_{i=1}^{n} (ax_i + b - y_i)^2. \tag{2.1}$$

In words, for every point we take its vertical distance from the line, square it, and sum these "squares of errors."

This method need not always be the most suitable. For instance, if a few exceptional points are measured with very large error, they can influence the resulting line a great deal. An alternative method, less sensitive to a small number of "outliers," is to minimize the sum of absolute values of all errors:

$$\sum_{i=1}^{n} |ax_i + b - y_i|. \tag{2.2}$$

Soit x un rationnel. Si on minimise e sous les contraintes $e \geq x, e \geq -x$, quel est le lien entre e et x ? Concernant le problème initial, quelles sont les données et les inconnues ? Formulez le problème initial en programmation linéaire. Résolvez numériquement.