The two-phase simplex method

Given an LP problem

$$\begin{aligned} \text{maximize} \quad & \sum_{j=1}^{n} c_j x_j \\ \text{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j \le b_i \quad (i = 1, 2, \ldots, m) \\ & x_j \ge 0 \quad (j = 1, 2, \ldots, n), \end{aligned}$$

we ask whether or not there is a feasible solution. The objective function $\sum_{j=1}^{n} c_j x_j$ is irrelevant to this question. Only the constraints matter here: our question is equivalent to asking whether or not there is a solution of the system

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i \;\; (i = 1, 2, \ldots, m), \quad x_j \ge 0 \;\; (j = 1, 2, \ldots, n). \tag{1}$$

To answer this question, we introduce an *artificial variable* w_i for each i such that $b_i < 0$ and, writing A for the set of subscripts of all these artificial variables, we consider the *auxiliary problem*

$$\begin{aligned} \text{minimize} \quad & \sum_{i \in A} w_i \\ \text{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j - w_i \le b_i \quad (i \in A) \\ & \sum_{j=1}^{n} a_{ij} x_j \le b_i \quad (i \notin A) \\ & x_j \ge 0 \;\; (j = 1, 2, \ldots, n), \;\; w_i \ge 0 \;\; (i \in A). \end{aligned}$$

System (1) has a solution if and only if the optimal value of the auxiliary problem is zero.

To avoid confusion between the objective function of the original problem and the objective function of the auxiliary problem, we will write

$$z = \sum_{j=1}^{n} c_j x_j \quad \text{and} \quad w = \sum_{i \in A} w_i.$$

The auxiliary problem has a feasible solution,

$$x_j = 0\ (j = 1, 2, \ldots, n),\ \ w_i = -b_i\ (i \in A).$$

This feasible solution is a basic feasible solution: its basic variables are the slack variables x_{n+i} such that $b_i \geq 0$ and all the artificial variables. The corresponding ditionary reads

$$\begin{array}{rclclclc}
x_{n+i} & = & b_i & - & \sum_{j=1}^n a_{ij}x_j & & & (i \notin A) \\
w_i & = & -b_i & + & \sum_{j=1}^n a_{ij}x_j & + & x_{n+i} & (i \in A) \\
\hline
w & = & -\sum_{i\in A} b_i & + & \sum_{j=1}^n (\sum_{i\in A} a_{ij})x_j & + & \sum_{i\in A} x_{n+i} &
\end{array}$$

Starting out from this dictionary, we pivot away until we reach an optimal dictionary for the auxiliary problem. There are three cases to consider:

Case 1: The optimal value of the auxiliary problem is strictly positive.
Case 2: The optimal value of the auxiliary problem is zero and,
in the optimal dictionary, all artificial variables are nonbasic.
Case 3: The optimal value of the auxiliary problem is zero and,
in the optimal dictionary, at least one artificial variable is basic.

In Case 1, system (1) has no solution. In Case 2, a feasible dictionary of the original problem can be obtained from the optimal dictionary of the auxiliary problem: we remove all artificial variables and we replace the equation for w by an equation for z, whose right-hand side expresses $\sum_{j=1}^n c_j x_j$ in terms of the current nonbasic variables. To illustrate the case, consider the problem

$$\begin{array}{lrcrcrcr}
\text{maximize} & x_1 & - & x_2 & + & x_3 & & \\
\text{subject to} & 2x_1 & - & x_2 & + & 2x_3 & \leq & 4 \\
& 2x_1 & - & 3x_2 & + & x_3 & \leq & -5 \\
& -x_1 & + & x_2 & - & 2x_3 & \leq & -1 \\
& & & & & x_1, x_2, x_3 & \geq & 0.
\end{array}$$

Here, the auxiliary problem reads

$$\begin{array}{lrcrcrcrcrcr}
\text{minimize} & & & & & & & w_2 & + & w_3 & & \\
\text{subject to} & 2x_1 & - & x_2 & + & 2x_3 & & & & & \leq & 4 \\
& 2x_1 & - & 3x_2 & + & x_3 & - & w_2 & & & \leq & -5 \\
& -x_1 & + & x_2 & - & 2x_3 & & & - & w_3 & \leq & -1 \\
& & & & & & & & x_1, x_2, x_3, & w_2, w_3 & \geq & 0.
\end{array}$$

$$\begin{array}{lrcrcrcrcrcl}
\text{minimize} & & & & & & & w_2 & + & w_3 & & \\
\text{subject to} & 2x_1 & - & x_2 & + & 2x_3 & & & & & \leq & 4 \\
& 2x_1 & - & 3x_2 & + & x_3 & - & w_2 & & & \leq & -5 \\
& -x_1 & + & x_2 & - & 2x_3 & & & - & w_3 & \leq & -1 \\
& & & & & & & \multicolumn{3}{r}{x_1, x_2, x_3, w_2, w_3} & \geq & 0.
\end{array}$$

Its initial feasible dictionary is

$$\begin{array}{rcrcrcrcrcrcr}
x_4 & = & 4 & - & 2x_1 & + & x_2 & - & 2x_3 & & & & \\
w_2 & = & 5 & + & 2x_1 & - & 3x_2 & + & x_3 & + & x_5 & & \\
w_3 & = & 1 & - & x_1 & + & x_2 & - & 2x_3 & & & + & x_6 \\
\hline
w & = & 6 & + & x_1 & - & 2x_2 & - & x_3 & + & x_5 & + & x_6
\end{array}$$

and its optimal dictionary is

$$\begin{array}{rcrcrcrcrcrcr}
x_2 & = & 2.2 & + & 0.6x_1 & + & 0.4x_5 & + & 0.2x_6 & - & 0.4w_2 & - & 0.2w_3 \\
x_3 & = & 1.6 & - & 0.2x_1 & + & 0.2x_5 & + & 0.6x_6 & - & 0.2w_2 & - & 0.6w_3 \\
x_4 & = & 3 & - & x_1 & & & - & x_6 & & & & \\
\hline
w & = & & & & & & & & & w_2 & + & w_3
\end{array}$$

To convert this dictionary into a feasible dictionary of the original problem, we remove all artificial variables, which gives us

$$\begin{array}{rcrcrcrcr}
x_2 & = & 2.2 & + & 0.6x_1 & + & 0.4x_5 & + & 0.2x_6 \\
x_3 & = & 1.6 & - & 0.2x_1 & + & 0.2x_5 & + & 0.6x_6 \\
x_4 & = & 3 & - & x_1 & & & - & x_6
\end{array}$$

and then we express z in terms of the current nonbasic variables: substituting for x_2 and x_3 in the equation $z = x_1 - x_2 + x_3$, we get

$$\begin{aligned}
z &= x_1 - (2.2 + 0.6x_1 + 0.4x_5 + 0.2x_6) + (1.6 - 0.2x_1 + 0.2x_5 + 0.6x_6) \\
&= -0.6 + 0.2x_1 - 0.2x_5 + 0.4x_6.
\end{aligned}$$

The resulting feasible dictionary of the original problem is

$$\begin{array}{rcrcrcrcr}
x_2 & = & 2.2 & + & 0.6x_1 & + & 0.4x_5 & + & 0.2x_6 \\
x_3 & = & 1.6 & - & 0.2x_1 & + & 0.2x_5 & + & 0.6x_6 \\
x_4 & = & 3 & - & x_1 & & & - & x_6 \\
\hline
z & = & -0.6 & + & 0.2x_1 & - & 0.2x_5 & + & 0.4x_6
\end{array}$$

We are going to show that Case 3 can be reduced to Case 2 by a sequence of degenerate pivots:

if an artificial variable w_k is basic,
then the slack variable x_{n+k} is nonbasic
and a degenerate pivot makes w_k leave the basis.

Our argument is based on the following observation: If equations

$$\begin{array}{lclclcll} \sum_{j=1}^{n} a_{ij}x_j & + & x_{n+i} & & & = & b_i & (i \notin A), \\ \sum_{j=1}^{n} a_{ij}x_j & + & x_{n+i} & - & w_i & = & b_i & (i \in A), \end{array}$$

are satisfied by numbers x_j $(j = 1, 2, \ldots, n+m)$ and w_i $(i \in A)$, then these equations remain satisfied when the values of x_{n+k} and w_k are incremented by the same amount and the values of the remaining variables are left unchanged. In particular, the values of x_{n+k} and w_k are not determined by the values of the remaining variables, and so x_{n+k}, w_k can never be basic in the same dictionary. Furthermore, if w_k is basic and x_{n+k} is nonbasic, then changing the value of x_{n+k} and keeping the values of the remaining nonbasic variables unchanged has the effect of changing the value of w_k and keeping the values of the remaining basic variables unchanged. Therefore x_{n+k} has a nonzero coefficient in the row of the dictionary that expresses w_k in terms of nonbasic variables. In particular, when the value of w_k is zero, a degenerate pivot brings x_{n+k} into and w_k out of the basis.