The Curry-Howard correspondence:

- **propositions as types**
- **proofs as programs**
- **simplification of proofs as evaluation of programs**

Figure 1. Gerhard Gentzen (1935)—Natural Deduction.

$$\frac{A \qquad B}{A \,\&\, B}\,\&\text{-I} \qquad \frac{A \,\&\, B}{A}\,\&\text{-E}_1 \qquad \frac{A \,\&\, B}{B}\,\&\text{-E}_2$$

$$\frac{\begin{matrix}[A]^x \\ \vdots \\ B\end{matrix}}{A \supset B}\supset\text{-I}^x \qquad \frac{A \supset B \qquad A}{B}\supset\text{-E}$$

Figure 5. Alonzo Church (1935)—Lambda Calculus.

$$\frac{M : A \qquad N : B}{\langle M, N\rangle : A \times B}\times\text{-I} \qquad \frac{L : A \times B}{\pi_1\, L : A}\times\text{-E}_1 \qquad \frac{L : A \times B}{\pi_2\, L : B}\times\text{-E}_2$$

$$\frac{\begin{matrix}[x : A]^x \\ \vdots \\ N : B\end{matrix}}{\lambda x. N : A \to B}\to\text{-I}^x \qquad \frac{L : A \to B \qquad M : A}{L\, M : B}\to\text{-E}$$

Figure 2. A proof.

$$\dfrac{\dfrac{\dfrac{[B \,\&\, A]^z}{A}\,\&\text{-E}_2 \qquad \dfrac{[B \,\&\, A]^z}{B}\,\&\text{-E}_1}{A \,\&\, B}\,\&\text{-I}}{(B \,\&\, A) \supset (A \,\&\, B)}\supset\text{-I}^z$$

Figure 6. A program.

$$\dfrac{\dfrac{\dfrac{[z : B \times A]^z}{\pi_2\, z : A}\times\text{-E}_2 \qquad \dfrac{[z : B \times A]^z}{\pi_1\, z : B}\times\text{-E}_1}{\langle \pi_2\, z, \pi_1\, z\rangle : A \times B}\times\text{-I}}{\lambda z\,.\,\langle \pi_2\, z, \pi_1\, z\rangle : (B \times A) \to (A \times B)}\to\text{-I}^z$$

Figure 3. Simplifying proofs.

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ A\end{matrix}\qquad\begin{matrix}\vdots\\ B\end{matrix}}{A \,\&\, B}\,\&\text{-I}}{A}\,\&\text{-E}_1 \quad\Longrightarrow\quad \begin{matrix}\vdots\\ A\end{matrix}$$

$$\dfrac{\dfrac{\begin{matrix}[A]^x\\ \vdots\\ B\end{matrix}}{A\supset B}\,\supset\text{-I}^x \qquad \begin{matrix}\vdots\\ A\end{matrix}}{B}\,\supset\text{-E} \quad\Longrightarrow\quad \begin{matrix}\vdots\\ A\\ \vdots\\ B\end{matrix}$$

Figure 4. Simplifying a proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{[B \,\&\, A]^z}{A}\,\&\text{-E}_2 \qquad \dfrac{[B \,\&\, A]^z}{B}\,\&\text{-E}_1}{A \,\&\, B}\,\&\text{-I}}{(B \,\&\, A)\supset(A \,\&\, B)}\,\supset\text{-I}^z \qquad \dfrac{B\qquad A}{B \,\&\, A}\,\&\text{-I}}{A \,\&\, B}\,\supset\text{-E}$$

$$\Downarrow$$

$$\dfrac{\dfrac{\dfrac{B\qquad A}{B \,\&\, A}\,\&\text{-I}}{A}\,\&\text{-E}_2\qquad \dfrac{\dfrac{B\qquad A}{B \,\&\, A}\,\&\text{-I}}{B}\,\&\text{-E}_1}{A \,\&\, B}\,\&\text{-I}$$

$$\Downarrow$$

$$\dfrac{A\qquad B}{A \,\&\, B}\,\&\text{-I}$$

Figure 7. Evaluating programs.

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ M:A\end{matrix}\qquad\begin{matrix}\vdots\\ N:B\end{matrix}}{\langle M,N\rangle : A\times B}\,\times\text{-I}}{\pi_1\,\langle M,N\rangle : A}\,\times\text{-E}_1 \quad\Longrightarrow\quad \begin{matrix}\vdots\\ M:A\end{matrix}$$

$$\dfrac{\dfrac{\begin{matrix}[x:A]^x\\ \vdots\\ N:B\end{matrix}}{\lambda x.\,N : A\to B}\,\to\text{-I}^x \qquad \begin{matrix}\vdots\\ M:A\end{matrix}}{(\lambda x.\,N)\;M : B}\,\to\text{-E} \quad\Longrightarrow\quad \begin{matrix}\vdots\\ M : A\\ \vdots\\ N[M/x]:B\end{matrix}$$

Figure 8. Evaluating a program.

$$\dfrac{\dfrac{\dfrac{\dfrac{[z:B\times A]^z}{\pi_2\,z:A}\,\times\text{-E}_2\qquad\dfrac{[z:B\times A]^z}{\pi_1\,z:B}\,\times\text{-E}_1}{\langle\pi_2\,z,\pi_1\,z\rangle:A\times B}\,\times\text{-I}}{\lambda z.\,\langle\pi_2\,z,\pi_1\,z\rangle:(B\times A)\to(A\times B)}\,\to\text{-I}^z\qquad\dfrac{y:B\qquad x:A}{\langle y,x\rangle:B\times A}\,\times\text{-I}}{(\lambda z.\,\langle\pi_2\,z,\pi_1\,z\rangle)\;\langle y,x\rangle:A\times B}\,\to\text{-E}$$

$$\Downarrow$$

$$\dfrac{\dfrac{\dfrac{y:B\qquad x:A}{\langle y,x\rangle:B\times A}\,\times\text{-I}}{\pi_2\,\langle y,x\rangle:A}\,\times\text{-E}_2\qquad\dfrac{\dfrac{y:B\qquad x:A}{\langle y,x\rangle:B\times A}\,\times\text{-I}}{\pi_1\,\langle y,x\rangle:B}\,\times\text{-E}_1}{\langle\pi_2\,\langle y,x\rangle,\pi_1\,\langle y,x\rangle\rangle:A\times B}\,\times\text{-I}$$

$$\Downarrow$$

$$\dfrac{x:A\qquad y:B}{\langle x,y\rangle:A\times B}\,\times\text{-I}$$