Testing, Debugging, and Verification

Formal Verification, Part I

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[1] Lecture slides based on material from Wolfgang Aherndt,..

Recap: Loop Invariants and Variants

Loops are difficult to reason about.

- Don't know how many times we go around.
- But Dafny needs to consider all paths! How?

Solution: Loop Invariants

An invariant is an property which is true before entering loop and after each execution of loop body.

But what about termination?

Solution: Loop Variants

An variant is an expression which decrease with each iteration of the loop, and is bounded from below by 0.
Dafny can often guess variants automatically.

Formal Verification

Todays main topics:

- Dafny behind the scenes: How does it prove programs correct?
- Weakest Precondition Calculus

Formal Software Verification: Motivation

Limitations of Testing

- Testing ALL inputs is usually impossible.
- Even strongest coverage criteria cannot guarantee abcence of further defects.

Goal of Formal Verification

Given a formal specification S of the behaviour of a program P:
Give a mathematically rigorous proof that each run of P conforms to S

P is correct with respect to S

Formal Software Verification: Limitations

- No absolute notion of program correctness!
  - Correctness always relative to a given specification
- Hard and expensive to develop provable formal specifications
- Some properties may be difficult or impossible to specify.
- Requires lots of expertise and expenses (so far...)
- Even fully specified & verified programs can have runtime failures
  - Defects in the compiler
  - Defects in the runtime environment
  - Defects in the hardware

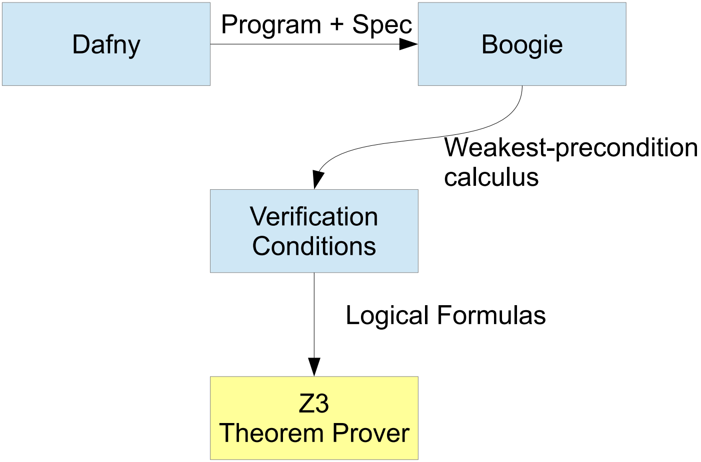
Possible & desirable:

Exclude defects in source code wrt. a given spec

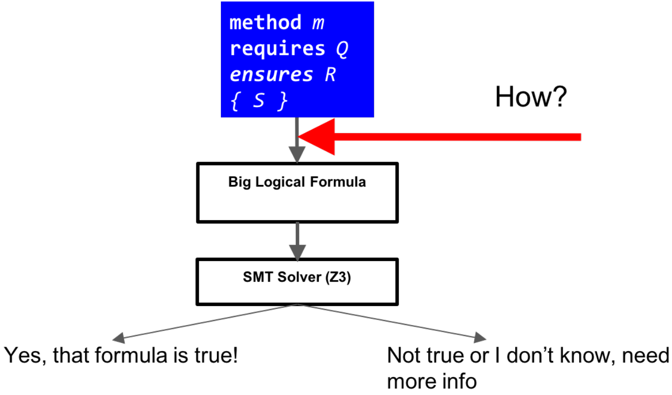
Dafny: Behind the Scenes

What happens when we ask Dafny to compile our program?
How does it prove that it is correct according to its specification?

Dafny: Behind the Scenes



Dafny: Behind the Scenes



- **Our focus:** How do we extract verification conditions (Big Logical Formula)?
- **This module**: Weakest precondition calculus.
- Won't deal with full Dafny/Boogie, but simplified subset involving assignments, if-statements, while loops.

What do we Need to Prove and How?

```
method MyMethod(. . .)
  requires Q
  ensures R
  {
   S: program statements
  }
```

In literature, often expressed as a Hoare Triple: {Q} S {R}

Hoare Triple: {Q} S {R}

If execution of program S starts in a state satisfying pre-condition Q, the is is guaranteed to terminate in a state satisfying the post-condition R.

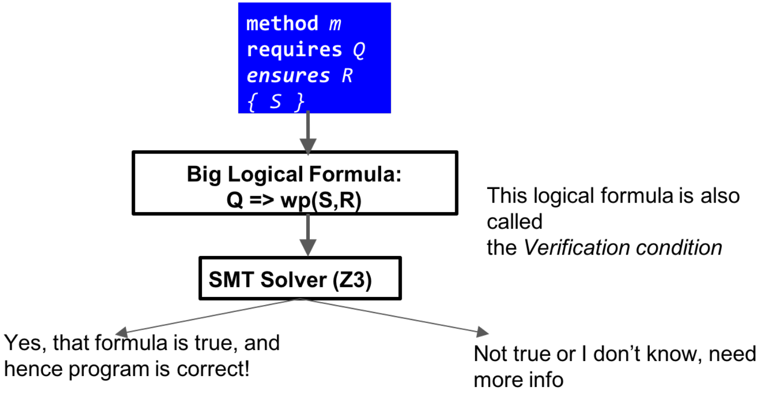
What do we Need to Prove and How?

```
method MyMethod(. . .)
  requires Q
  ensures R
  {
   S: program statements
  }
```

Weakest Precondition:

- Assuming that R holds after executing S,
- What is the least restricted (set of) state we could possibly begin from?
  - Weakest = Fewest restrictions on input state.
  - Formally: $wp(S, R)$
- Does Q satisfy at least these restrictions?
  - i.e. does Q imply the weakest pre-condition?
  - To prove: $Q \rightarrow wp(S, R)$
  - Proving Hoare triple {Q} S {R} amounts to showing that $Q \rightarrow wp(S, R)$.

What do we Need to Prove and How?



Weakest Precondition

Weakest Precondition: $wp(S, R)$

The weakest precondition of a program S and post-condition R represents the set of all states such that execution of S started in any of these is guaranteed to terminate in a state satisfying R.

First-Order Formulas and Program States

First-order formulas define sets of program states

What do we mean by $wp(S, R)$ defining a set of program states?

$wp(S, R)$ is a logical predicate F that is true in some states and not true in others.

Example

- `(i>j & j>=0)` is true in exactly those states S where $\texttt{i}^s > \texttt{j}^s$ and $\texttt{j}^s$ is non-negative.
- `exists i :: i == j`

  is true in any state S, because the value of `i` can be chosen to be $\texttt{j}^s$

Example

- Program statement S: `i := i + 1`
- Post-condition R: `i <= 1`

What is the weakest precondition, $wp(S, R)$?

- **Reason backwards**: $wp(i := i + 1, i <= 1) = i <= 0$
- Executing `i := i + 1` in any state satisfying `i <= 0` will end in a state satisfying `i <= 1`.
- **Note:** Taking Q: `i < -5` does also satisfy R. But overly restrictive, excludes initial states where `-5 <= i <=0`. Weakest precondition can help us find a suitable contract.

Mini Quiz: Guess the Weakest Precondition

Write down $wp(S, R)$ for the following S and R:

	S	R
a)	`i := i+1`	`i > 0`
b)	`i := i+2; j := j-2`	`i + j == 0`
c)	`a[i] := 1`	`a[i] == a[j]`
d)	`i := i+1; j := j-1`	`i * j == 0`

Solution:

a)	`i >= 0`
b)	`i + j == 0`
c)	`a[j] == 1`
d)	`i == -1 || j == 1`

Weakest Precondition Calculus

Our Verification Algorithm

- Have a program S, with precondition Q and postcondition R
- Compute $wp(S, R)$
- Prove that $Q \rightarrow wp(S, R)$

The rules of the weakest precondition calculus provide semantics, a logical meaning, for the statements in our programming language.

Weakest Precondition Calculus

We will prove validity of programs written in a slightly simplified subset of Dafny/Boogie featuring:

Assignment: `x := e`

Sequentials: `S1; S2`

Assertions: `assert B`

If-statements: `if B then S1 else S2`

While-loops: `while B S`

Semantics

We will define the weakest precondition for each of these program constructs.

Weakest Precondition Calculus: Assignment

Assignment

$wp(x := e, R) = R[x \mapsto e]$

Note: $R[x \mapsto e]$ means "R with all occurrences of x replaced by e".

Example

Let S:

```
i := i + 1;
```

Let R: $i > 0$

$wp(i := i + 1, i > 0) =$
(By Assignment rule)
$i + 1 > 0$

This program satisfies its postcondition if started in any state where $i \geq 0$.

Weakest Precondition Calculus: Sequential Composition

Sequential Composition

$wp(S1; S2, R) = wp(S1, wp(S2, R))$

Example

Let S:

```
x := i;
i := i + 1;
```

Let R: $x < i$

$wp(x := i; i := i + 1, x < i) =$
(By Sequential rule)
$wp(x := i, wp(i := i + 1, x < i)) =$
(By Assignment rule)
$wp(x := i, x < i + 1) =$
(By Assignment rule)
$i < i + 1$
(trivially true)

This program satisfies its postcondition in any initial state.

Weakest Precondition Calculus: Assertion

Assertion

$wp(assert\ B, R) = B \wedge R$

Example

Let S:

```
x := y;
assert x > 0;
```

Let R: $x < 20$

$wp(x := y; assert\ x > 0, x < 20) =$
(By Sequential rule)
$wp(x := y, wp(assert\ x > 0, x < 20)) =$
(By Assertion rule)
$wp(x := y, x > 0 \wedge x < 20) =$
(By Assignment rule)
$y > 0 \wedge y < 20$

This program satisfies its postcondition in those initial states where `y` is a number between 1 and 19 (inclusive).

Weakest Precondition Calculus: Conditional

Conditional

$wp(\mathit{if}\ B\ \mathit{then}\ S1\ \mathit{else}\ S2,\ R) =$
$\quad (B \rightarrow wp(S1, R)) \wedge (\neg B \rightarrow wp(S2, R))$

Example

Let S:

```
if (i >= 0) then
  x := i else x := -i
```

Abbreviate:

S1: `x := i`

S2: `x := -i`

Let R: $x \geq 0$

$wp(\mathit{if}\ (i \geq 0)\ \mathit{then}\ S1\ \mathit{else}\ S2, x \geq 0) =$

(By Conditional rule)

$i \geq 0 \rightarrow wp(x := i, x \geq 0) \wedge$
$\neg(i \geq 0) \rightarrow wp(x := -i, x \geq 0) =$

(By Assignment rule)

$(i \geq 0 \rightarrow i \geq 0) \wedge (\neg(i \geq 0) \rightarrow -i \geq 0) =$

true

This program satisfies its postcondition in any initial state.

Weakest Precondition Calculus: Conditional

Conditional, empty `else` branch

$wp(\textit{if } B \textit{ then } S1,\ R) = (B \rightarrow wp(S1, R)) \land (\neg B \rightarrow R)$

If `else` is empty, need to show that R follows just from negated guard.

Mini Quiz: Derive the weakest precondition

The Rules

$wp(x := e, R) = R[x \mapsto e]$
$wp(S1; S2, R) = wp(S1, wp(S2, R))$
$wp(assert\ B,\ R) = B \wedge R$
$wp(if\ B\ then\ S1\ else\ S2,\ R) =$
$\quad (B \rightarrow wp(S1, R)) \wedge (\neg B \rightarrow wp(S2, R))$

Derive the weakest precondition, stating which rules you use in each step.

	S	R
a)	`i := i+2; j := j-2`	`i + j == 0`
b)	`i := i+1; assert i > 0`	`i <= 0`
c)	`if isEven(x) then y:=x/2 else y:=(x-1)/2`	`isEven(y)`

Mini Quiz: Derive the weakest precondition

Solution:

a) `i + j == 0`
(apply seq. rule followed by assignment rule, simplify)

b) `i+1 > 0 && i+1 <= 0`
(apply seq rule, assert rule, assignment)
Simplifies to `i => 0 && i <= -1` which is false! No initial state can satisfy this postcondition.

c)
```
isEven(x) ==> isEven(x/2) && !isEven(x) ==>
isEven((x-1)/2)
```
(apply cond. rule, followed by assignment.)

Let's Prove ManyReturns Correct!

Recall

To prove correct a program S with precondition Q and postcondition R we need to show that $Q \to wp(S, R)$.

```
method ManyReturns(x:int, y:int) returns (more:int, less:
 int)
 requires 0 < y;
 ensures less < x < more;
 { more := x+y;
   less := x-y;
 }
```

Show that

$0 < y \to wp(more := x + y; less := x - y,\ less < x < more)$

Let's Prove ManyReturns Correct!

Show that

$0 < y \to wp(more := x + y; less := x - y,\ less < x < more)$

Seq. rule

$0 < y \to wp(more := x + y, wp(less := x - y,\ less < x < more))$

Assignment rule

$0 < y \to wp(more := x + y, x - y < x < more)$

Assignment rule

$0 < y \to (x - y < x < x + y)$

which follows from the precondition by simple arithmetic.

Hint

This level of detail is expected for your proofs in the lab and exam.

Another Example

```
method f ( x : int) returns (y : int)
requires x > 8
ensures  y > 10
        {
                y := x + 1;
                if (y mod 2 == 0) { y := 100; }
                else { y := y + 2; }
        }
```

Exercise: Prove f correct

Show that

$x > 8 \rightarrow wp(y := x + 1; \mathit{if} \cdots, y > 10)$.

Solution

First compute wp:

$wp(y := x + 1; \text{if } y\%2 == 0 \cdots, y > 10)$

Solution

First compute wp:

$wp(y := x + 1; \text{if } y\%2 == 0 \cdots, y > 10)$

Seq. rule

$= wp(y := x + 1;\ wp(\text{if } y\%2 == 0 \cdots, y > 10))$

Solution

First compute wp:

$wp(y := x + 1; \text{if } y\%2 == 0 \cdots, y > 10)$

Seq. rule

$= wp(y := x + 1; wp(\text{if } y\%2 == 0 \cdots, y > 10))$

Compute $wp(\text{if } y\%2 == 0 \cdots, y > 10)$

Solution

First compute wp:

$wp(y := x + 1; \text{if } y\%2 == 0 \cdots, y > 10)$

Seq. rule

$= wp(y := x + 1;\ wp(\text{if } y\%2 == 0 \cdots, y > 10))$

Compute $wp(\text{if } y\%2 == 0 \cdots, y > 10)$

If rule

$$= ((y\%2 == 0) \to wp(y := 100, y > 10))$$
$$\wedge(\neg(y\%2 == 0) \to wp(y := y + 2, y > 10))$$

Solution

First compute wp:

$wp(y := x + 1; \text{if } y\%2 == 0 \cdots, y > 10)$

Seq. rule

$= wp(y := x + 1;\ wp(\text{if } y\%2 == 0 \cdots, y > 10))$

Compute $wp(\text{if } y\%2 == 0 \cdots, y > 10)$

If rule

$= ((y\%2 == 0) \rightarrow wp(y := 100, y > 10))$

$\quad \wedge(\neg(y\%2 == 0) \rightarrow wp(y := y + 2, y > 10))$

Assignment rule (2x)

$= ((y\%2 == 0) \rightarrow 100 > 10) \wedge (\neg(y\%2 == 0) \rightarrow y + 2 > 10)$

Solution

First compute wp:

$wp(y := x + 1; \text{if } y\%2 == 0 \cdots, y > 10)$

Seq. rule

$= wp(y := x + 1;\ wp(\text{if } y\%2 == 0 \cdots, y > 10))$

Compute $wp(\text{if } y\%2 == 0 \cdots, y > 10)$

If rule

$= ((y\%2 == 0) \rightarrow wp(y := 100, y > 10))$
$\quad \wedge(\neg(y\%2 == 0) \rightarrow wp(y := y + 2, y > 10))$

Assignment rule (2x)

$= ((y\%2 == 0) \rightarrow 100 > 10) \wedge (\neg(y\%2 == 0) \rightarrow y + 2 > 10)$

Simplify

$= ((y\%2 == 0) \rightarrow true) \wedge (\neg(y\%2 == 0) \rightarrow y > 8))$

Solution

First compute wp:

$wp(y := x + 1; \text{if } y\%2 == 0 \cdots, y > 10)$

Seq. rule

$= wp(y := x + 1;\ wp(\text{if } y\%2 == 0 \cdots, y > 10))$

Compute $wp(\text{if } y\%2 == 0 \cdots, y > 10)$

If rule

$= ((y\%2 == 0) \rightarrow wp(y := 100, y > 10))$
$\quad \wedge(\neg(y\%2 == 0) \rightarrow wp(y := y + 2, y > 10))$

Assignment rule (2x)

$= ((y\%2 == 0) \rightarrow 100 > 10) \wedge (\neg(y\%2 == 0) \rightarrow y + 2 > 10)$

Simplify

$= ((y\%2 == 0) \rightarrow \mathit{true}) \wedge (\neg(y\%2 == 0) \rightarrow y > 8))$

By $a \rightarrow \mathit{true} = \mathit{true}$

$= \mathit{true} \wedge (\neg(y\%2 == 0) \rightarrow y > 8)$

Solution

First compute wp:

$wp(y := x + 1; \text{if } y\%2 == 0 \cdots, y > 10)$

Seq. rule

$= wp(y := x + 1;\ wp(\text{if } y\%2 == 0 \cdots, y > 10))$

Compute $wp(\text{if } y\%2 == 0 \cdots, y > 10)$

If rule

$= ((y\%2 == 0) \rightarrow wp(y := 100, y > 10))$
$\quad \wedge(\neg(y\%2 == 0) \rightarrow wp(y := y + 2, y > 10))$

Assignment rule (2x)

$= ((y\%2 == 0) \rightarrow 100 > 10) \wedge (\neg(y\%2 == 0) \rightarrow y + 2 > 10)$

Simplify

$= ((y\%2 == 0) \rightarrow \mathit{true}) \wedge (\neg(y\%2 == 0) \rightarrow y > 8))$

By $a \rightarrow \mathit{true} = \mathit{true}$

$= \mathit{true} \wedge (\neg(y\%2 == 0) \rightarrow y > 8)$

By $\mathit{true} \wedge a = a$

$= (\neg(y\%2 == 0) \rightarrow y > 8)$

Another Example

$wp(y := x + 1;\ wp(\text{if } y\%2 == 0 \cdots, y > 10))$
By $wp(\text{if } y\%2 == 0 \cdots, y > 10) = (\neg(y\%2 == 0) \rightarrow y > 8)$
$= wp(y := x + 1; (\neg(y\%2 == 0) \rightarrow y > 8))$

Another Example

$wp(y := x + 1;\ wp(\text{if } y\%2 == 0 \cdots, y > 10))$

By $wp(\text{if } y\%2 == 0 \cdots, y > 10) = (\neg(y\%2 == 0) \to y > 8)$

$= wp(y := x + 1; (\neg(y\%2 == 0) \to y > 8))$

By Assignment Rule

$= (\neg((x + 1)\%2 == 0) \to x + 1 > 8)$

Another Example

$wp(y := x + 1;\ wp(\text{if } y\%2 == 0 \cdots, y > 10))$
By $wp(\text{if } y\%2 == 0 \cdots, y > 10) = (\neg(y\%2 == 0) \rightarrow y > 8)$
$= wp(y := x + 1; (\neg(y\%2 == 0) \rightarrow y > 8))$
By Assignment Rule
$= (\neg((x + 1)\%2 == 0) \rightarrow x + 1 > 8)$
Simplify
$= (\neg((x + 1)\%2 == 0) \rightarrow x > 7)$

Another Example

$wp(y := x + 1;\ wp(\text{if } y\%2 == 0 \cdots, y > 10))$

By $wp(\text{if } y\%2 == 0 \cdots, y > 10) = (\neg(y\%2 == 0) \rightarrow y > 8)$

$= wp(y := x + 1; (\neg(y\%2 == 0) \rightarrow y > 8))$

By Assignment Rule

$= (\neg((x + 1)\%2 == 0) \rightarrow x + 1 > 8)$

Simplify

$= (\neg((x + 1)\%2 == 0) \rightarrow x > 7)$

To prove: $x > 8 \rightarrow wp(y := x + 1; \mathit{if} \cdots, y > 10)$

$x > 8 \rightarrow (\neg((x + 1)\%2 == 0) \rightarrow x > 7)$

Another Example

$wp(y := x + 1;\ wp(\text{if } y\%2 == 0 \cdots, y > 10))$
By $wp(\text{if } y\%2 == 0 \cdots, y > 10) = (\neg(y\%2 == 0) \rightarrow y > 8)$
$= wp(y := x + 1; (\neg(y\%2 == 0) \rightarrow y > 8))$
By Assignment Rule
$= (\neg((x + 1)\%2 == 0) \rightarrow x + 1 > 8)$
Simplify
$= (\neg((x + 1)\%2 == 0) \rightarrow x > 7)$
To prove: $x > 8 \rightarrow wp(y := x + 1; if \cdots, y > 10)$
$x > 8 \rightarrow (\neg((x + 1)\%2 == 0) \rightarrow x > 7)$
Simplify using $x > 8$ in RHS
$= x > 8 \rightarrow (\neg((x + 1)\%2 == 0) \rightarrow true)$
By $a \rightarrow true = true$
$= x > 8 \rightarrow true$
By $a \rightarrow true = true$
$=$ true

What Next?

While loops!

Difficulties of While Loops

- Need to “unwind” loop body one by one
- In general, no fixed loop bound known (depends on input)
- How the loop invariants and variants are used in proofs.

Summary

- Testing cannot replace verification
- Formal verification can prove properties for all runs,
  ... but has inherent limitations, too.
- Dafny is compile to intermediate language Boogie.
- Verification conditions (VCs) extracted, using weakest precondition calculus rule.
- VCs are logical formulas, which can be passed to a theorem prover.
- Prove that precondition imply *wp*.

Reading: *The Science of Programming* by David Gries. Chapters 6-10, bearing in mind that the notation and language differ slightly from ours. Available as E-book from Chalmers library.