(Un)decidability problems in first order logic

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Uses of logic

Formal definition of concepts and statements

- mathematics: definitions, theorems, . . .
- programming: data structures, program specification and verification, . . .
- hardware: specification and verification, . . .
- databases: query language and answering
- artificial intelligence: knowledge representation and reasoning

How many logics?

- Predicate calculus
  - propositional
  - first-order
  - second and higher order
- Multi-valued logics
- Modal and temporal logics
- Fuzzy and probabilistic logics
- Constructive or intuitionistic logics
- Description logic
- . . .

Table of contents

Propositional logic, or zero-order predicate calculus

Alphabet

(i) An infinitely enumerable set of propositional symbols $\mathcal{P}$, denoted by p, q, . . .

(ii) The symbols $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\equiv$ called connectives

(iii) The symbols T, F called truth symbols

(iv) The symbols (and)

Propositional well-formed formulas (pwffs)

(i) Every propositional symbol $p \in \mathcal{P}$ is a pwff

(ii) T and F are pwff

(iii) If A and B are pwffs then $A \wedge B$, $A \vee B$, $A \rightarrow B$ and $A \equiv B$ are pwffs

(iv) If A is a pwff then (A) and $\neg A$ are pwffs

Exercise

Translate from natural language to pwffs:

(i) It is false that both the alarm is swiched on and it is switched off

(ii) Alarm is switched on iff door is open and no card inserted. Therefore, if the door is closed then the alarm is switched off.

(iii) If Alice is elected class-president then either Betty is elected vice-president or Carol is elected treasurer. Betty is elected vice-president. Therefore, Alice is elected class-president.

Semantics of pwffs

A valuation v is a function from propositional symbols to the boolean set $\mathcal{B} = \{true, false\}$, i.e. $v : \mathcal{P} \rightarrow \mathcal{B}$

Valuations can be extended to pwffs:

- $v(T) = true$ and $v(F) = false$
- $v(A \wedge B) = true$ if $v(A) = v(B) = true$; $v(A \wedge B) = false$ otherwise
- $v(A \vee B) = false$ if $v(A) = v(B) = false$; $v(A \vee B) = true$ otherwise

- $v(A \rightarrow B) = false$ if $v(A) = true$ and $v(B) = false$; $v(A \rightarrow B) = true$ otherwise

- $v(A \equiv B) = true$ if $v(A) = v(B)$; $v(A \equiv B) = false$ otherwise

- $v(\neg A) = true$ if $v(A) = false$; $v(\neg A) = false$ if $v(A) = true$

- $v((A)) = v(A)$

Validity of pwffs

A pwff A is valid if for every valuation v we have $v(A) = true$

Exercise

(i) The translated statements of previous excercise are valid?

(ii) States whether the following are valid:

1. $p \vee \neg p \equiv T$ [Third excluded]
2. $\neg(p \vee q) \equiv \neg p \wedge \neg q$ [De Morgan]
3. $(p \wedge (p \rightarrow q)) \rightarrow q$ [Modus ponens]
4. $p \rightarrow q \equiv \neg p \vee q$ [$\rightarrow$ elimination]
5. $(p \equiv q) \equiv (p \rightarrow q) \wedge (q \rightarrow p)$ [$\equiv$ elimination]

Semantics and decidability

A pwff A is valid if for every valuation v we have $v(A) = true$ (and it is invalid otherwise)

Is validity decidable?

A pwff A is satisfiable if there is at least one valuation v such that $v(A) = true$ (and it is unsatisfiable otherwise)

Is satisfiability decidable?

Note. A is unsatisfiable iff $\neg A$ is valid.

Semantics and decidability

Two pwffs A and B are equivalent if for every valuation v we have

$$v(A) = v(B)$$

Is equivalence decidable?

Note. A and B are equivalent iff $A \equiv B$ is valid.

A pwff A is a logical consequence of a finite set of pwff $\mathcal{S}$ if for every valuation v such that $v(B) = true$ for every $B \in \mathcal{S}$ we have

$$v(A) = true$$

Is logical consequence decidable?

Note. A is a logical consequence of S iff $\wedge_{B \in \mathcal{S}} B \rightarrow A$ is valid.

Decidability of propositional calculus

Lemma. For a valuation v and a pwff A, we have:

$$v(A) = v|_A(A)$$

where $v|_A(p) = v(p)$ for p appearing in A,
and $v|_A(p) = false$ otherwise.

Intuition: $v(A)$ depends only on the propositional symbols appearing in A.

Decidability of propositional calculus

If A contains k propositional symbols, there are 2^k of such $v|_A$, which can be enumerated to decide:

- validity, when all $v|_A(A) = true$
- consistency, when for some $v|_A(A) = true$
  - for A in conjunctive normal form, this is the SAT problem which is NP complete!

First order logic, or (first-order) predicate calculus

Alphabet

(i) A infinitely enumerable set of function symbols $\mathcal{F}$ each with associated an arity $n \geq 0$

- $f^n \in \mathcal{F}$ is called a constant when $n = 0$

(ii) A infinitely enumerable set of predicate symbols $\mathcal{P}$ each with associate an arity $n \geq 0$

- $p^n \in \mathcal{P}$ is called propositional when $n = 0$

Arities superscript is omitted when clear from the context

(iii) A infinitely enumerable set of variables $\mathcal{V}$, denoted by x, y, . . .

(iv) The symbols $\neg, \wedge$, $\vee$, $\rightarrow$, $\equiv$ called connectives

(v) The symbols $\forall$, $\exists$ called quantifiers

(vi) The symbols T, F called truth symbols

(vi) The symbols ( ) and ,

Terms

(i) Every variable x is a term

(ii) If $f^n \in \mathcal{F}$ and $t_1, \ldots, t_n$ are terms then $f^n(t_1, \ldots, t_n)$ is a term

- every constant is a term because $n = 0$

Atomic formulas

(i) If $p^n \in \mathcal{P}$ and $t_1, \ldots, t_n$ are terms then $p^n(t_1, \ldots, t_n)$ is an atomic formula

- every propositional symbol is a formula because $n = 0$

Well formed formulas

(i) Every atomic formula is a wff

(ii) If A and B are wffs then $A \wedge B$, $A \vee B$, $A \rightarrow B$ and $A \equiv B$ are wffs

(iii) If A is a wff and $x \in \mathcal{V}$ then $\forall x\ A$ and $\exists x\ A$ are wffs

(iv) If A is a wff then $\neg A$ and (A) are wffs

Exercise

Express the following statements as wffs:

- Students of informatics and mathematics are students of science. If Paul is a student of informatics then he is a student of science.
- A natural number is even iff it is zero or it is the successor of the successor of an even number. Then 4 is an even number.
- A list of values is either the empty list or a value (called head) followed by another list of values (called tail). The size of a list is the number of elements in the list. Then the size of the list $\langle c, b, a, q \rangle$ is 4.

Semantics: interpretation

An interpretation is any possible meaning of function and predicate symbols over a non-empty set.

Formally, an interpretation $\mathcal{I}$ consists of

(i) A non-empty set I, called the domain of the interpretation

(ii) For every function symbol f^n, of a total function $\tilde{f} : I^n \rightarrow I$

(ii) For every predicate symbol p^n, of a predicate $\tilde{p} : I^n \rightarrow \mathcal{B}$

Semantics: valuations

Given an interpretation the meaning of a wff is not yet defined, since variables are not taken into account.

A valuation v is a function from variables to the domain of the interpretation, i.e. $v : \mathcal{V} \to I$

Valuations can be extended to terms:

(i) $v(f(t_1, \ldots, t_n)) = \tilde{f}(v(t_1), \ldots, v(t_n))$

Semantics: valuations

Given an interpretation and a valuation, it is now possible to assign a boolean value to a wff, i.e. to extend $v : wffs \rightarrow \mathcal{B}$

- $v(p^n(t_1, \ldots, t_n)) = \tilde{p}(v(t_1), \ldots, v(t_n))$
- $v(T) = true$ and $v(F) = false$
- $v(A \wedge B) = true$ if $v(A) = v(B) = true$; $v(A \wedge B) = false$ otherwise
- $v(A \vee B) = false$ if $v(A) = v(B) = false$; $v(A \wedge B) = true$ otherwise

- $v(A \to B) = false$ if $v(A) = true$ and $v(B) = false$; $v(A \to B) = true$ otherwise

- $v(A \equiv B) = true$ if $v(A) = v(B)$; $v(A \equiv B) = false$ otherwise

- $v(\neg A) = true$ if $v(A) = false$; $v(\neg A) = false$ if $v(A) = true$

- $v(\forall x\ A) = true$ if for every $d \in I$, $v_x^d(A) = true$, where v_x^d is as v except that $v_x^d(x) = d$.

- $v(\exists x\ A) = true$ if for some $d \in I$, $v_x^d(A) = true$

Definition of valid wffs

An interpretation $\mathcal{I}$ is a model of a wff A if for every valuation v over $\mathcal{I}$ we have $v(A) = true$.

A wff A is valid if every interpretation is a model of A.

Exercise

(i) The translated statements of previous exercise are valid?

(ii) States whether the following are valid (A, B generic wffs):

1. $\neg \exists x \ A \equiv \forall x \ \neg A$ [De Morgan]

2. $\forall x \ \forall y \ A \equiv \forall y \ \forall x A$

3. $\forall x \ \exists y \ A \equiv \exists y \ \forall x \ A$

4. $\forall x \ A \land B \equiv (\forall x \ A) \land (\forall x \ B)$

Semantics and decidability

A wff A is valid if every interpretation is a model of A (and it is invalid otherwise)

Is validity decidable?

A wff A is satisfiable if there is at least an interpretation $\mathcal{I}$ and a valuation v over $\mathcal{I}$ such that $v(A) = true$ (and it is unsatisfiable otherwise)

Is satisfiability decidable?

Note. A is unsatisfiable iff $\neg A$ is valid.

Semantics and decidability

Two closed[*] wffs A and B are equivalent if every interpretation is a model of both or of none.

Is equivalence decidable?

Note. A and B are equivalent iff $A \equiv B$ is valid.

[*] A wwf is closed if every variable appearing in it is in the scope of a quantifier.

Semantics and decidability

A closed wff A is a logical consequence of a finite set of closed wff $\mathcal{S}$ if for every interpretation $\mathcal{I}$ such that $\mathcal{I}$ is a model of every $B \in \mathcal{S}$ we have that $\mathcal{I}$ is a model of A as well.

Is logical consequence decidable?

Note. A is a logical consequence of S iff $\wedge_{B \in \mathcal{S}} B \rightarrow A$ is valid.

Undecidability of FOL

Theorem. The validity problem of wffs is undecidable.

Proof. Next pages.

Substitution Lemma (weak version). Let A be a wff without quantifiers. If $v(\forall x\ A) = true$ then $v(A_x^t) = true$ for any term t, where A_x^t is obtained by literally substituting every occurrence of x in A with t.

Undecidability of FOL

For a Turing machine M, we map transitions into wffs:

$$(q, x, ay) \vdash_M (r, xb, y) \Rightarrow \forall x \ \forall y \, p_q(x, a(y)) \to p_r(b(x), y)$$

$$(q, xc, ay) \vdash_M (r, x, cby) \Rightarrow \forall x \ \forall y \, p_q(c(x), a(y)) \to p_r(x, c(b(y)))$$

$$(q, \epsilon, ay) \vdash_M (r, \epsilon, by) \Rightarrow \forall x \ \forall y \, p_q(\epsilon, a(y)) \to p_r(\epsilon, b(y))$$

$$(q, x, \epsilon) \vdash_M (r, xb, \epsilon) \Rightarrow \forall x \ \forall y \, p_q(x, \epsilon) \to p_r(b(x), \epsilon)$$

$$(q, xc, \epsilon) \vdash_M (r, x, cb\epsilon) \Rightarrow \forall x \ \forall y \, p_q(c(x), \epsilon) \to p_r(x, c(b(\epsilon)))$$

$$(q, \epsilon, \epsilon) \vdash_M (r, \epsilon, b\epsilon) \Rightarrow \forall x \ \forall y \, p_q(\epsilon, \epsilon) \to p_r(\epsilon, b(\epsilon))$$

Undecidability of FOL

Let T be the conjunction of all the formulaes above. Let q_0 be the initial state and q_a be the accepting state of M.

$$F(M, w) = (\ p_{q_0}(\epsilon, w) \wedge T\ ) \rightarrow \exists x\ \exists y\ p_{q_a}(x, y).$$

We wish to prove:

M accepts w iff $F(M, w)$ is valid

which reduces the universal language problem to the validity problem.

If-part

Let $F(M, w)$ be valid. We define the interpretation $\mathcal{T}$ such that:

- the domain is the set of strings on the alphabet of M
- $\tilde{\epsilon} = \epsilon$, i.e. the empty string;
- $\tilde{a}(s) = as$, i.e. $\tilde{a}$ concatenates a to its argument;
- $p_q(s, t) = true$ iff $(q_0, \epsilon, w) \vdash_M^* (q, \overline{s}, t)$, where $\overline{s}$ is s reversed.

If-part

For such an interpretation, and any valuation v:

- $v(p_{q_0}(\epsilon, w)) = true$ is clearly valid,
- $v(T) = true$ by construction of the conjuncts of T.

Since $\mathcal{T}$ is a model of $F(M, w)$, we have:

$$v(\exists x\ \exists y\ p_{q_a}(x, y)) = true$$

i.e. there exists s, t such that

$$v_{x\ y}^{t\ s}(p_{q_a}(x, y)) = true$$

i.e. $(q_o, \epsilon, w) \vdash_M^* (q_a, \overline{s}, t)$, i.e. M accepts w.

Only-if part

Let M accepts w. There exists a sequence:

$$(q_0, \epsilon, w) \vdash_M \ldots \vdash_M (q_i, \overline{s_i}, t_i) \vdash_M \ldots \vdash_M (q_n, \overline{s_n}, t_n)$$

where q_n is the accepting state q_a.

Consider any interpretation $\mathcal{I}$ and a valuation v such that

$$v(p_{q_0}(\epsilon, w) \wedge T) = true$$

If we could show that $v(q_i(s_i, t_i)) = true$ for all i's in the derivation above, we could conclude $v(q_a(s_n, t_n)) = true$ and then $v(\exists x\ \exists y\ p_{q_a}(x, y)) = true$, i.e. $F(M, w)$ is valid in $\mathcal{I}$.

Only-if part

$$(q_0, \epsilon, w) \vdash_M \ldots \vdash_M (q_i, \overline{s_i}, t_i) \vdash_M \ldots \vdash_M (q_n, \overline{s_n}, t_n)$$

Proof by induction on n:

- $v(p_{q_0}(\epsilon, w)) = true$ by assumption
- Assume $v(p_{q_i}(s_i, t_i)) = true$. For the moment, assume that $t_i = a(t)$ and let

$$(q_i, \overline{s_i}, at) \vdash_M (q_{i+1}, \overline{s_i}b, t)$$

Only-if part

be the transition in the sequence above. The formulae associated to that transition type is:

$$\forall x\ \forall y\, p_{q_i}(x, a(y)) \rightarrow p_{q_{i+1}}(b(x), y)$$

Since this formulae is a conjunct of T and $v(T) = true$, by the Substitution Lemma:

$$v(p_{q_i}(s_i, a(t)) \rightarrow p_{q_{i+1}}(b(s_i), t)) = true$$

and since the premise is true in v (inductive hypothesis), we conclude

$$v(p_{q_{i+1}}(b(s_i), t)) = v(p_{q_{i+1}}(s_{i+1}, t_{i+1})) = true$$

The proof is analogous for the other transition types.

An alternative proof based on PCP

A Post system over the alphabet $\Sigma = \{0, 1\}$ consists of k ordered pairs $(\alpha_1, \gamma_1), \ldots (\alpha_k, \gamma_k)$ where $\alpha_i, \gamma_i \in \Sigma^*$.

A solution is a sequence of integers $i_1, \ldots, i_n$ such that:

$$\alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_n} = \beta_{i_1}\beta_{i_2}\ldots\beta_{i_n}$$

The Post Correspondance Problem (PCP) consists of deciding whether there is a solution to the Post system.

Theorem. The PCP is undecidable.

Exercise

Show undecidability of validity of FOL by reducing PCP to it

Exercise - hint

Consider using a single constant a and two function symbols f_0 and f_1 of arity 1.

Consider using abbreviations like $f_{010}(x) = f_0(f_1(f_0(x)))$.

Express a wff with the following meaning:

"Given that a pair of strings is obtained as a single pair of the Post system or it is obtained by another pair of strings by appending a pair of the Post system, then there exists a pair of strings having the same string in both positions"

and show that the PCP is decidable iff such a wff is valid.

Semantics and semi-decidability

Is validity semi-decidable? i.e., the set of valid wffs is RE?

A wff A is satisfiable if there is at least an interpretation $\mathcal{I}$ and a valuation v over $\mathcal{I}$ such that $v(A) = true$ (and it is unsatisfiable otherwise)

Is satisfiability semi-decidable? i.e., satisfiable wffs is RE?

Note 1. A is unsatisfiable iff $\neg A$ is valid.

Note 2. A is satisfiable iff $\neg A$ is invalid.

Semantics and semi-decidability

Two closed* wffs A and B are equivalent if every interpretation is a model of both or of none.

Is equivalence semi-decidable?

Note. A and B are equivalent iff $A \equiv B$ is valid.

*A wff is closed if every variable appearing in it is in the scope of a quantifier.

Semantics and semi-decidability

A closed wff A is a logical consequence of a finite set of closed wff $\mathcal{S}$ if for every interpretation $\mathcal{I}$ such that $\mathcal{I}$ is a model of every $B \in \mathcal{S}$ we have that $\mathcal{I}$ is a model of A as well.

Is logical consequence semi-decidable?

Note. A is a logical consequence of S iff $\wedge_{B \in \mathcal{S}} B \rightarrow A$ is valid.

Proof Systems

A proof system $\mathcal{P}$ consists of a recursive set of rules $\frac{A_1 \ldots A_n}{A}$. When $n = 0$, A is called an axiom.

A proof trace for a wff A is a sequence $A_1, \ldots, A_k$ such that A_k is A and for every $i \in [1, k]$:

- A_i is an axiom, or:
- there exists a rule $\frac{A_{h_1}, \ldots, A_{h_n}}{A_i}$ with $h_1, \ldots, h_n < i$.

A theorem is any wff that has a proof trace.

Proof Systems

Proof systems allow to derive formulas on a mechanic and syntactic way as deduction of formulas from axioms and from already shown theorems.

Lemma. The set of theorems is RE.

Proof (sketch)

1. It is decidable whether a sequence $A_1, \ldots, A_n$ is a proof trace:

   - finitely many candidate rules with premises/conclusion appearing in the sequence,

   - it is decidable whether a candidate is a rule (since proof rules is a recursive set).

2. The set of sequences of formulas is enumerable .

Proof System and Semantics

Theorem There exists proof system that are sound and complete with respect to validity,
i.e. A is a theorem iff A is valid.

E.g., Hilbert calculus, Natural deduction, Gentzen Sequent calculus, Tableaux, Resolution.

Theorem prover tools:
http://www-formal.stanford.edu/clt/ARS/systems.html

Corollary. The set of valid wffs is RE.

Decidable subsets of first order logic

Undecidability of validity

The validity problem is undecidable . . .

(LU reduction) . . . even for wffs with only predicate symbols of arity $= 2$ and function symbols of arity ≤ 1.

(PCP reduction) . . . even for wffs with only ONE predicate symbol of arity $= 2$, ONE function symbol of arity 0, and TWO function symbols of arity 1.

(Kalmar-Surányi 1950) . . . even for wffs with only ONE predicate symbol of arity $= 2$, and NO function symbols.

Monadic predicate calculus

(i) A infinitely enumerable set of function symbols $\mathcal{F}$ of arity 0, i.e. only constants

(ii) A infinitely enumerable set of predicate symbols $\mathcal{P}$ of arity $n \leq 1$, i.e. only unary or propositional predicates

The finite model property

Monadic wffs have the *finite model property*,

Theorem. Let A be a monadic wff. There exists a finite set of interpretations $\mathcal{S}$ whose domain is finite and computable such that:

A is valid iff every $\mathcal{I} \in \mathcal{S}$ is a model of A.

Corollary. Validity of monadic wffs is decidable.

Since $\mathcal{S}$ is finite, we can enumerate each interpretation $\mathcal{I}$. Since the domain of $\mathcal{I}$ is finite, there are finitely many valuations $v|_A$ to check $v|_A(A) = true$.

Proof of the finite model property

Assume A has k monadic predicates $p_0, \ldots, p_{k-1}$. The set $\mathcal{S}$ is defined as the set of interpretations $\mathcal{I}$ such that:

- the domain I is a non-empty subset of $\{0, \ldots, 2^k - 1\}$

- if a is a constant not appearing in A, then $\tilde{a}$ is some fixed value (e.g. $min(I)$)

- if p is a unary predicate symbol not appearing in A, then $\tilde{p}(k) = false$ for every $k \in I$

- if p is a 0-ary predicate symbol not appearing in A, then $\tilde{p} = false$.

Proof of the finite model property

$\mathcal{S}$ is finite since:

- there are finitely many domains,
- and for each domain there are finitely many choices left, i.e.: assigning meaning to constants and predicate symbols appearing in A.

Proof of the finite model property

Lemma A is valid iff every $\mathcal{I} \in \mathcal{S}$ is a model of A.

Proof. Only-if is immediate.

Let us show the if-part. Let $\mathcal{J}$ be any interpretation.

We define $m : J \to \{0, \ldots, 2^k - 1\}$ as follows:

$$m(d) = \Sigma_{h=0}^{k-1} 2^h \cdot int(\tilde{p}_h(d))$$

where $int(true) = 1$ and $int(false) = 0$

Proof of the finite model property

Intuition 1. $m(d)$ condenses the truth values of d over the predicates of A.

Intuition 2. $m^{-1}(r)$ is the set of $d \in J$ that share the same truth values.

Proof of the finite model property

Consider the interpretation $\mathcal{I} \in \mathcal{S}$ defined as:

- its domain is $\{r \in \{0, \ldots, 2^k - 1\} \mid m^{-1}(r) \neq \emptyset\} = \{m(d) \mid d \in J\}$
- $\tilde{a} = m(\tilde{a}_{\mathcal{J}})$ for a constant in A
- $\tilde{p}(r) = \tilde{p}_{\mathcal{J}}(d)$ where d is any element in $m^{-1}(r)$ and p appearing in A
- $\tilde{p} = \tilde{p}_{\mathcal{J}}$ for 0-ary predicate symbols in A.

Proof of the finite model property

By showing that $\mathcal{I}$ models of A implies $\mathcal{J}$ models of A, we get the conclusion of the Lemma.

Let v be a valuation of $\mathcal{J}$. We show that for $\sigma(x) = m(v(x))$ valuation of $\mathcal{I}$, $v(A) = \sigma(A)$. Since $\mathcal{I}$ is a model of A, we conclude $v(A) = \sigma(A) = true$ and then $\mathcal{J}$ is a model of A.

The proof is by induction on structure of A. We give the base steps.

$$\sigma(p(x)) = \tilde{p}_I(\sigma(x)) = \tilde{p}_I(m(v(x))) = \tilde{p}_J(v(x)) = v(p(x))$$

$$\sigma(p(a)) = \tilde{p}_I(\sigma(a)) = \tilde{p}_I(m(\tilde{a}_J)) = \tilde{p}_J(\tilde{a}_J) = v(p(a))$$

Exercises

1. Discuss whether or not the validity of the class of wffs containing **no quantifier** and no function symbol is decidable.

2. Show that by extending monadic calculus with non-constant function symbols the scheme of the proof for the finite model property is broken somewhere.

Datalog formulas

(i) A infinitely enumerable set of function symbols $\mathcal{F}$ of arity 0, i.e. only constants

(ii) Wffs are of the form $U_1 \wedge \ldots \wedge U_n \rightarrow E$

where U_i is of the form

$\forall(A_1 \wedge \ldots \wedge A_{n_i} \rightarrow A_{n_i+1})$, with A_j atomic and $n_i > 0$, (called rule)

or of the form

A, with A atomic and without variables (called fact)

Datalog formulas

and E is of the form

$\exists(B_1 \wedge \ldots \wedge B_n)$

with A_j atomic and $n \geq 0$.

Sample datalog formulas

$$student(aa\mathsf{001}, john, male)$$

$$\wedge\ student(aa\mathsf{002}, mary, female)$$

$$\wedge\ student(aa\mathsf{003}, ann, female)$$

$$\wedge\ study(aa\mathsf{001}, laws)$$

$$\wedge\ study(aa\mathsf{002}, mathematics)$$

$$\wedge\ study(aa\mathsf{003}, informatics)$$

$$\rightarrow\ \exists XY\, student(X, Y, female), study(X, informatics)$$

Sample datalog formulas

$$student(aa\mathsf{001}, john, male)$$

$$\wedge\ student(aa\mathsf{002}, mary, female)$$

$$\wedge\ student(aa\mathsf{003}, ann, female)$$

$$\wedge\ study(aa\mathsf{001}, laws)$$

$$\wedge\ study(aa\mathsf{002}, mathematics)$$

$$\wedge\ study(aa\mathsf{003}, informatics)$$

$$\wedge\ \forall X\ \ study(X, informatics) \rightarrow study(X, science)$$

$$\wedge\ \forall X\ \ study(X, mathematics) \rightarrow study(X, science)$$

$$\rightarrow\ \ \exists XY\, student(X, Y, female), study(X, science)$$

The finite model property

Datalog formulas have the *finite model property*,

Theorem. For a datalog formula A, there exists a single interpretation $\mathcal{H}$ with finite and computable domain such that:

A is valid iff A is valid in $\mathcal{H}$.

Corollary. Validity of datalog formulas is decidable.

Sketch of proof of the finite model property

Consider interpretation $\mathcal{H}$ (called Herband interpretation) such that:

- the domain is the set of constants appearing in A
- $\tilde{a} = a$
- $\tilde{p}(a_1, \ldots, a_n) = true$ iff $p(a_1, \ldots, a_n) \in M$, where M is the smallest (finite) set such that:
  - every fact is in M
  - if $A_1 \wedge \ldots \wedge A_{n_i} \rightarrow A_{n_i+1}$ is an instance* of a rule, and $A_1, \ldots, A_{n_i}$ are in M then A_{n_i+1} is in M as well.

*By consistently replacing all variables with constants.

Sketch of proof of the finite model property

We show that A valid in $\mathcal{H}$ implies A valid (the other way is trivial).

It is easy to see that the facts and the rules are true in every valuation of $\mathcal{H}$ (by its definition). Therefore, the consequence in A must be true, i.e. for some v we have $v(B_1 \wedge \ldots \wedge B_n) = true$, i.e. $B'_1, \ldots, B'_n \in M$ for some instance $B'_1 \wedge \ldots \wedge B'_n$ of $B_1 \wedge \ldots \wedge B_n$.

Each $B'_i \in M$ can be obtained by a finite sequence of rule/fact applications (definition of M). Therefore, they can be obtained in any valuation of $\mathcal{J}$ satisfying facts and rules by the same sequence of rule/fact applications.

Some history

Church 1929 Soundness of Hilbert calculus

Godel 1931 Completeness of Hilbert calculus

Church 1936 Undecidability of first order logic

Löwenheim 1915 Decidability of monadic predicate calculus

van Emden and Kowalski 1976 Datalog programs

Further reference

E. Börger, E. Grädel, Y. Gurevich.
The Classical Decision Problem.
Springer, 2001.